**HOMWORK 11**

(Bonus exercise — see the end)

Ex. P.36, N.5

\[
\frac{p(z)}{q(z)} = \frac{a_0 + a_1 z + \cdots + a_m z^m}{b_0 + b_1 z + \cdots + b_m z^m} \quad a_m \neq 0
\]

\[
f(z) = \frac{f(z)}{q(z)}
\]

\[
\int f(z) \, dz = \frac{1}{z^2} \left[ z^2 f(z) \right] = 0
\]

\[
f(z) = \frac{f(z)}{q(z)} = \frac{(a_0 z^m + \cdots + a_m) z^m}{(b_0 z^m + \cdots + b_m) z^m}
\]

\[
= \frac{z^{m-m} a_0 z^m + \cdots + a_m}{b_0 z^m + \cdots + b_m}
\]

Thus, function has no singularities at \(z = 0\).

\[
\lim_{z \to 0} \frac{f(z)}{z^m} = \begin{cases} 
0 & \text{if } m > n+2 \\
\infty & \text{if } m = n+2
\end{cases}
\]

\[
\int_{C} f(z) \, dz = \frac{1}{z^2} \left[ z^2 f(z) \right] = 0
\]

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**Ex. P.285, N.1**

1. \(f(z) = z^2\)

\[\Delta \text{arg} f(z) = 2\pi \left[ z_t - \frac{c}{z_t} \right] = 0\pi\]

\(\Delta \text{z}_t\) only one zeta w/ mult = 1

2. \(f(z) = \frac{2z^2 + 2}{z^2}\)

\(\Delta \text{arg} f(z) = 2\pi \left[ 0 - 1 \right] = -2\pi\)

\(\Delta \text{z}_t\) pole at \(z = 0\) (simple pole)

No poles inside curve \(C\)

(\(z_t\)s have modulus \(>\frac{1}{2}\))

3. \(f(z) = \frac{(z^2 + 3)^3}{z^3}\)

\[\Delta \text{arg} f(z) = 2\pi \left[ z_t - 3 \right] = 8\pi\]

\(\Delta \text{z}_t\) pole at \(z = 0\) — multiplicity = 3

\(\Delta \text{z}_t\) zero at \(z = \frac{1}{3}\) w/ multiplicity 3
\[ n = f(c) \]

Analytic over annulus excluding line segment and never zero on C

\[ \Delta_r \arg f(z) = 2\pi \]

Function winds three times around the origin (MLR principle assertion)

\[ \Delta \arg f(z) = 2\pi = Z \]

\[ Z = 3 \]

Function is analytic inside C

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**Ex. 102.6.5**

If analytic inside and on positively oriented simple closed contour C, f \( \neq 0 \) on C. f has roots \( 2x \) (k = 1, m) with multiplicity \( m_k \) (mod C)

\[ \int_{C} \frac{f(z)}{f(z)} \, dz = 2\pi i \sum_{k=1}^{m} \text{Res}_{z_k} \left[ \frac{f(z)}{f(z)} \right] = 2\pi i \sum_{k=1}^{m} m_k \]

If function has a pole at \( z_k \), with multiplicity \( m_k = (m_k - 1) \)

\[ \frac{z_k}{f(z)} = 2 \left( \frac{1}{m_k} + \frac{g(z)}{f(z)} \right) \]

By formula (2)

\[ f(z) \text{ Res}_{z_k} \frac{1}{f(z)} = 2m_k \]

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**Ex. 102.6.6**

a) \( z^5 - 5z^4 + 2z^2 = 0 \)

Let: \( w = \frac{1}{z} \)

\( f(w) = -5w^4 + 2w^2 \)

\( g(w) = w^5 \)

\[ f(2) = -52 \]

\[ g(2) = 2^6 + 2^3 - 22 \]

\[ |f(2)| = 5 \quad \text{on } C \]

\[ |g(2)| = |2^6 + 2^3 - 22| \leq 1 + 1 + 2 = 4 \]

b) \( z^4 - 2z^3 + z^2 - 22 + 1 = 0 \)

\( f(1) = 1 \)

\( g(2) = 2z^4 - 2z^3 + 2z^2 - 22 \)

\[ |f(2)| = 5 \quad \text{on } C \]

\[ g(2) < |g(2)| < 22 = |f(2)| \]

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Function has the same number of roots on C

\( f(z) \text{ Res}_{z_k} \frac{1}{f(z)} \) (counting multiplicity)

If function is analytic inside and on C, then it will not have any poles within C.

For polynomials, the degree of the polynomial in C will be equal to the number of roots inside C.
Let \( c = 1 + i \sqrt{e} \)

Consider an equation \( c^m = e^z \)

Consider \( f(z) = e^z \)

\[ g(z) = -c^z \]

At \( C = \{ z : |z| = 1 \} \), \( f \) and \( g \) are both entire.

\[ f(z) = e^z \leq e \quad \text{on} \quad C \]

\[ g(z) = 1 + i > e \]

Thus, \( f \) and \( g \) have the same number of zeros inside \( C \) (counting multiplicities).

\[ |a| = n \quad \text{since} \quad 2^m \text{ has } n \text{ zeros} \]

with multiplicities \( m \).

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**Exercise 2.25, Problem 1**

(2) Show that all zeros of \( P(z) = a_0 + \cdots + a_2 z^m \) \( a_n \neq 0 \) \( n \geq 1 \)

also inside a circle \( |z| = R \) about the origin.

**Solution:** They are all finite zeros and there is a finite number of them.

**Expansion (1)** \( \Gamma(z) = 0 \)

\[ 2n \pi i \cdot N = \int \frac{P(z)}{P'(z)} \, dz = \sum_{z = 0}^{\infty} \text{Res} \left[ \frac{P(z)}{P'(z)} \right] \]

\[ \text{let's evaluate this zeroes:} \quad \frac{1}{2} \left[ \frac{a_0}{2} + \cdots + a_0 z^{m-1} \right] = \frac{1}{2} \left[ \frac{a_0}{2} + \cdots + a_0 z^{m-1} \right] = \frac{1}{2} \left[ \frac{1}{2} \left( \frac{P(z)}{P'(z)} \right) \right] = g(z) = m \]

\[ \rightarrow N = m \]
\[ \Phi(z) = 1 \int_{C} \frac{f'(z) + g'(z)}{f(z) + g(z)} \, dz \quad \text{as } z \in C \]

9) The denominator in the integrand of \( \Phi(z) \) is \( \neq 0 \) \( \forall z \in C \)

Assume \( f(z) + g(z) = 0 \) \( \exists z \in C \) and \( t \in (a, b) \)

\[ f(z_0) \neq g(z_0) \quad \Rightarrow \quad |f(z_0)| = 1 \quad |g(z_0)| \leq |f(z_0)| \]

\[ \text{Combination rule} \]

10) Let \( k, t \in (a, b) \)

\[ |\Phi(t) - \Phi(k)| = \left| \int_{C} \frac{f'(z) + g'(z)}{f(z) + g(z)} \, dz \right| = \int_{C} \left| \frac{f'(z) + g'(z)}{f(z) + g(z)} \right| \, dz \]

\[ = \left| \frac{1}{2\pi i} \int_{C} \frac{f'_g - f'_f}{(f + g)(f + g') - f(f + g')} \, dz \right| \]

\[ \max \left| \frac{f'_g - f'_f}{(f + g)(f + g') - f(f + g')} \right| = \frac{\max \left| f'_g - f'_f \right|}{(f + g)(f + g') - f(f + g')} \]

\[ = \frac{\max \left| f'_g - f'_f \right|}{(1 + |f| - |g|)^2} \]

\[ \text{Thus we obtain} \]

\[ \frac{1}{1 + |f| - |g|} \]

\[ \text{and its domain depend on } \epsilon \]

\[ \Rightarrow \quad \Phi(z) \text{ is continuous in } [a, b] \quad \text{(actually it is a primitive).} \]
From equation (9), assume \( \phi(a) \neq 0 \) or \( \phi(b) \neq 0 \).

If \( \phi \in \mathbb{C} \) and \( \phi \) is continuous, then \( \phi \) must be constant.

\[ \phi(a) = \phi(b) \]

\[ \Rightarrow \quad \text{residue of } \phi(z) \]

\[ \Rightarrow \quad \text{Raoul's Theorem!} \]

\( \text{Ex. 363 N.1} \)

\[ u(x,y) = x^3 - 3xy^2 \]

\[ F = u_x - i u_y = (3x^2 - 3y^2) - i(-6xy) = 3(x^2 - y^2 + 2ixy) = 3z^2 \]

\[ \Rightarrow \quad f(z) = 2^3 \]

(\( u = \text{Re} f \))

\[ 2^3 = (x + iy)^3 = (x^3 - 3xy^2) + i(-3x^2y + 3y^3) \]

\[ \Rightarrow \quad \text{harmonic conjugate} \quad \text{[Im} f \text{]} = 3x^2y - y^3 \]

(\( \text{Ex. Prs. 366 N.2} \))

\[ u(x,y) \text{ harmonic in } D \text{ (simply connected)} \]

\[ \Rightarrow \quad \text{Re} [f(z)] \quad \text{if } D \rightarrow \mathbb{C} \text{ analytic} \]

\[ \Rightarrow \quad \text{we} \in \mathbb{C} \]
$$\omega = f(\phi) = u(x,y) + i\phi(x,y)$$

$$D_2 \longrightarrow D_w \quad \text{(maps } D_2 \text{ onto } D_w)$$

At \( h(x,y) \) function \( C^2(D_0) \) and \( \omega \),

$$Hb(x,y) = h(u(x,y), \phi(x,y))$$

$$\Delta H(x,y) = H_{xx}(x,y) + H_{yy}(x,y) = \frac{\partial^2}{\partial x^2} \left[ \frac{\partial^2}{\partial x^2} u_x + \frac{\partial^2}{\partial y^2} \phi_x \right] + \frac{\partial^2}{\partial y^2} \left[ \frac{\partial^2}{\partial x^2} u_y + \frac{\partial^2}{\partial y^2} \phi_y \right]$$

$$= \frac{\partial^2}{\partial x^2} u_x + \frac{\partial^2}{\partial y^2} \phi_x + \frac{\partial^2}{\partial x^2} u_y + \frac{\partial^2}{\partial y^2} \phi_y$$

$$= \frac{\partial^2}{\partial x^2} u_x + \frac{\partial^2}{\partial y^2} \phi_x + \frac{\partial^2}{\partial x^2} u_y + \frac{\partial^2}{\partial y^2} \phi_y$$

$$= \cdots \quad \Delta H(x,y) = \left( \frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} \phi \right)$$

Using \( \text{Re} \) parts

\( \Delta u = \Delta \phi \neq 0 \)

$$\Rightarrow H \text{ is harmonic in } D_2 \quad \text{if} \quad \phi \text{ is harmonic in } D_w.$$
Assume $f$ is analytic everywhere in $C$ except for a finite # of angular points $z_i$. Let $C$ be a positively oriented simple closed curve in $C$ except $z_i$.

Consider $C' = \arg \theta^2$ under the map $\omega = \frac{1}{2}$. Let $\omega'$ be the orientation, since $\omega' = -\frac{1}{2}$. Then $\omega'$ will be negatively oriented.

Consider a parameterization $\tilde{C} = \{ \theta(t) \mid t \in [a, b] \}$

$$C = \{ \frac{i}{\theta(t)} \mid t \in [a, b] \}$$

$$\Rightarrow \int_{\tilde{C}} f(z) \, dz = \int_{a}^{b} f(\frac{1}{\theta(t)}) \left( \frac{\theta'(t)}{(\theta(t))^2} \right) \, dt = \int_{\tilde{C}} f(\frac{1}{z}) \, dz = \int_{\tilde{C}} f(\frac{1}{z}) \, dz$$

Consider the region $g(z) = \frac{1}{z^2} f(z)$ which is analytic in $C \setminus \{ \frac{1}{2}, \ldots, \frac{1}{2n}, 0 \}$ if one of these $z_i = 0$ doesn't coincide.

$$\Rightarrow \int_{\tilde{C}} f\left( \frac{1}{z} \right) \, dz = -\int_{\tilde{C}} f\left( \frac{1}{z} \right) \, dz$$

The only residue made $\tilde{C}$ is $z = 0$.

by Residue Theorem:

$$\int_{\tilde{C}} f(\frac{1}{z}) \, dz = \int_{\tilde{C}} \frac{1}{2\pi i} \frac{f(\frac{1}{z})}{\theta(t)} \, dt = 2\pi i \, \text{Re} \left[ \frac{1}{2\pi i} f\left( \frac{1}{z} \right) \right]$$