# HOMEWORK 11

(Bonus exercise → see the end)

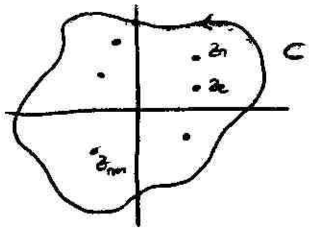
## EX. PAG. 230, N. 5

$P(z) = a_0 + a_1 z + \dots + a_n z^n \quad a_n \neq 0$

$Q(z) = b_0 + b_1 z + \dots + b_m z^m \quad b_m \neq 0 \qquad m \geq n+2$

$f(z) = \dfrac{P(z)}{Q(z)}$

LET $z_1, \dots z_m$ be all zeroes of $Q(z)$
and let C a closed curve (simple) containing all of them in its interior.



$$\int_C f(z)\, dz = \operatorname*{Res}_{z=0} \left[ \frac{1}{z^2} f\left(\frac{1}{z}\right) \right] = 0$$

$$f(z) = \frac{P(z)}{Q(z)} = \frac{(a_0 z^n + \dots + a_n)\, z^m}{(b_0 z^m + \dots + b_m)\, z^n} =$$

$$= z^{m-n} \frac{a_0 z^n + \dots + a_n}{b_0 z^m + \dots + b_m}$$

$$\Rightarrow \frac{1}{z^2} f\left(\frac{1}{z}\right) = z^{m-(n+2)} \frac{a_n + \dots + a_0 z^n}{b_m + \dots + b_0 z^m}$$

this function has no singularity at $z=0$
and $\lim_{z \to 0} \frac{1}{z^2} f(z) = \begin{cases} 0 & \text{if } m > n+2 \\ \frac{a_n}{b_m} \in \mathbb{C} & \text{if } m = n+2 \end{cases}$

$$\Rightarrow \operatorname*{Res}_{z=0} \left[ f\left(\frac{1}{z}\right) \frac{1}{z^2} \right] = 0$$

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## EX. PAG. 285 N. 1

ⓐ $f(z) = z^2 \qquad \Delta_C \arg[f(z)] = 2\pi \cdot [Z_f - P_f] = 4\pi$

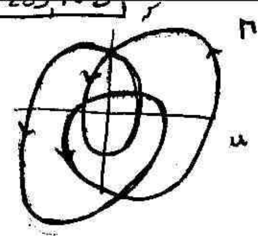
$\uparrow$
only one zero w/ mult = 2

ⓑ $f(z) = \dfrac{z^3 + 2}{z} \qquad \Delta_C \arg f(z) = 2\pi\, [0 - 1] = -2\pi$

1 pole at $z=0$ (simple pole)
no zeros inside inside C
(zeros have modulus $2^{1/3} > 1$)

ⓒ $f(z) = \dfrac{(2z-1)^7}{z^3} \qquad \Delta_C \arg f(z) = 2\pi\, [7 - 3] = 8\pi$

1 pole at $z=0$ → multiplicity $= 3$
1 zero at $z = \frac{1}{2}$ w/ multiplicity 7



$\Gamma = f(C)$

f analytic inside a simple closed
and never zeros on C

- $\Delta_C \arg f(z) = 6\pi$   since Γ winds three times around the origin (in the positive direction)
- $\frac{1}{2\pi}\Delta_C \arg f(z) = Z - P = Z$   $\Rightarrow Z = 3$

$\uparrow$
since P=0
(the function is analytic inside C)

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**EX. PAG 285, N.5**

f analytic inside and on positively oriented simple closed contour C
$f \neq 0$ on C   f has zeros $z_k$ $(k=1 \dots m)$ w/ multiplicity $m_k$ (inside C)

$$\int_C z\frac{f'(z)}{f(z)}\,dz = 2\pi i \sum_{k=1}^{m} \operatorname{Res}_{z=z_k}\left[\frac{z f'(z)}{f(z)}\right] = 2\pi i \sum_{k=1}^{m} m_k z_k$$

∘

$\uparrow$
this function has a pole
at $z_k$ w/ multiplicity $\cancel{m_k}=(m_k-1) = 1$

$$\Rightarrow \frac{z f'(z)}{\boxed{f(z)}} = z\left(\frac{m_k}{(z-z_k)} + \frac{g'(z)}{g(z)}\right)$$

$\uparrow$
proposition (7)
§ 79

$$\Rightarrow \operatorname{Res}_{z=z_k} \frac{z f'(z)}{f(z)} = z_k m_k$$

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**EX PAG 285, N.6**

a) $z^6 - 5z^4 + z^3 - 2z = 0$

let's use Rouché's thm   $f(z) = -5z^4$
$g(z) = z^6 + z^3 - 2z$

$|f(z)| = 5$   on C

$|g(z)| = |z^6 + z^3 - 2z| \underset{\text{on } C}{\leq} 1+1+2 = 4$

$\rightarrow$ f(z) and f(z)+g(z)
have the same number of zeros in C
f(z) has $\boxed{4 \text{ zeros}}$ (counting multip.)

ⓑ $2z^4 - 2z^3 + 2z^2 - 2z + 9 = 0$

$f(z) = 9$
$g(z) = 2z^4 - 2z^3 + 2z^2 - 2z$   $\Longmapsto$ Rouché: our polym f+g doesn't vanish inside C !

on C   $|g(z)| < 8 < 9 = |f(z)|$

$c \in \mathbb{C}$ $\quad |c| > e$

consider equation $cz^n = e^z$

consider $f(z) = e^z$

$g(z) = -cz^m$

at $C = \{|z|=1\}$ · f and g are both entire

· $|f(z)| = e^x \le e$ on C

$|g(z)| = |c| > e$ $\quad \models\!\Rightarrow$ g and f+g have the same # of zeros inside C (counting multiplicities)

$\models\!\Rightarrow = n$ since $z^m$ has 1 zero $z=0$ w/ multiplicity m

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EX- PAG225, N.11

(5) §49 $\models\!\Rightarrow$ all zeros of $P(z) = a_0 + \cdots + a_n z^m$ $\quad (a_n \neq 0)$ $\quad n \geq 1$

lies inside a circle $|z|=R$ about the origin

moreover: they are all finite order and there is a finite number $\overset{N}{}$ of them ↙ counted w/ multiplicity

expression (9) §79 $\models\!\Rightarrow$

$$2\pi i N = \int_{|z|=R} \frac{P'(z)}{P(z)}\,dz \underset{\substack{\uparrow \\ \text{section 64}}}{=} 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2}\left[\frac{P'(1/z)}{P(1/z)}\right]\right]$$

let's evaluate this residue:

$$\frac{1}{z^2}\cdot\frac{P'(1/z)}{P(1/z)} = \frac{1}{z^2}\,\frac{\frac{1}{z^{n-1}}\left[a_m m + \cdots + a_0 z^{n-1}\right]}{\frac{1}{z^m}\left[a_n + \cdots + a_0 z^m\right]} =$$

$$= \frac{1}{z}\,\underbrace{\frac{a_n m + \cdots + a_0 z^{n-1}}{a_m + \cdots + a_0 z^m}}_{g(z)} \underset{\substack{\uparrow \\ g \text{ is analytic at} \\ z=0}}{\Longrightarrow} \operatorname*{Res}_{z=0} \frac{1}{z^2}\left[\frac{P'(1/z)}{P(1/z)}\right] = g(0) = m$$

$\Rightarrow N = m$ $\square$

EX. PAG 285, N 12

④

- $f, g$ analytic inside and on a simple closed contour $C$ (positive oriented)
- $|f(z)| > |g(z)|$ on $C$

Define $\Phi(t) = \frac{1}{2\pi i} \int_C \frac{f'(z) + tg'(z)}{f(z) + tg(z)} dz \qquad 0 \le t \le 1$

a) the denominator in the integrand of $\Phi(t)$ is $\neq 0 \quad \forall z \in C$

suppose $f(z_0) + t_0 g(z_0) = 0 \quad \exists z_0 \in C$ and $t_0 \in [0,1]$

$\Rightarrow f(z_0) = -t_0 g(z_0) \Longrightarrow |f(z_0)| = |t_0|\,|g(z_0)| \le |g(z_0)| \qquad \not\ast$

contradiction since $z_0 \in C$

b) let $t, t_0 \in [0,1]$

$$|\Phi(t) - \Phi(t_0)| = \left| \frac{1}{2\pi i} \left( \int_C \frac{f' + tg'}{f + tg} - \int_C \frac{f' + t_0 g'}{f + t_0 g} \right) \right| = \left| \frac{1}{2\pi i} \int_C \frac{\overset{f't_0g + tg'f - f'tg - t_0g'f}{}}{(f+tg)(f+t_0g)} dz \right| =$$

$$= \frac{|t - t_0|}{2\pi} \left| \int_C \frac{fg' - f'g}{(f+tg)(f+t_0g)} dz \right| \le$$

moreover

$$\frac{|fg' - f'g|}{|(f+tg)(f+t_0g)|} \underset{\uparrow}{\le} \frac{|fg' - f'g|}{(|f| - |g|)^2}$$

$|f+tg| \ge ||f| - |g||$ ;

$|f+t_0g| \ge ||f| - |g||$

$$\Rightarrow |\Phi(t) - \Phi(t_0)| \le \frac{|t-t_0|}{2\pi} \underbrace{\overset{\overset{A2\pi}{|||}}{\int_C \frac{|fg' - f'g|}{(|f| - |g|)^2} |dz|}}_{} = A|t - t_0|$$

$\uparrow$

this is well defined

b/c $|f| > |g(z)|$

and it doesn't depend on $t$

$\Longrightarrow \Phi(t)$ is continuous in $[0,1]$ (actually it's Lipschitz).

ⓒ From equation (9), section 79

It follows that $\forall\ 0 \le t \le 1$ $\phi(t) \in \mathbb{N}$ representing the number of zeros of $f+tg$ inside $C$

$\phi(t) \in \mathbb{N}$ and it's continuous $\Longrightarrow$ $\phi(t)$ must be constant

$$\Longrightarrow \underset{\substack{\uparrow \\ \text{zeros of } f(z)}}{\phi(0)} = \phi(1) \ \nwarrow \text{zeros of } f(z)+g(z)$$

$\Longrightarrow$ Rouché's theorem! $\square$

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$\boxed{\text{EX. 358 N.1}}$

$u(x,y) = x^3 - 3xy^2$

$F = u_x - iu_y = (3x^2 - 3y^2) - i(-6xy^2) = \quad 3(x^2 - y^2 + 2iyx) = \quad 3z^2$

$\Longrightarrow \quad f(z) = z^3 \qquad (u = \operatorname{Re} f) \qquad z^3 = (x+iy)^3 = (x^3 - 3xy^2) + i(-y^3 + 3x^2y)$

$\Longrightarrow$ harmonic conjugate $\boxed{v = \operatorname{Im} f} = \boxed{3x^2y - y^3}$

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$\boxed{\text{EX. PAG. 358 N.2}}$

$u(x,y)$ harmonic in $D$ (simply connected)

$\Longrightarrow \quad u(x,y) = \operatorname{Re}[f(z)] \qquad \exists f: D \to \mathbb{C}$ analytic

$\Longrightarrow \quad u \in C^\infty(D)$

EX. PAG 358, N 8

$\omega = f(z) = u(x,y) + i v(x,y)$ $\quad - \; D_z \longmapsto D_\omega$ (maps $D_z$ onto $D_\omega$)

Let $h(u,v)$ function $C^2(D_\omega)$ and assume $H(x,y) = h(u(x,y), v(x,y))$

$$\Rightarrow \quad \Delta H(x,y) = H_{xx}(x,y) + H_{yy}(x,y) = \partial_x \left[ \partial_x h \cdot u_x + \partial_y h \, v_x \right] + \partial_y \left[ \partial_x h \, u_y + \partial_y h \, v_y \right]$$

$$= \partial^2_{xx} h \; u_x^2 + \partial^2_{xy} h \, \cancel{u_x v_y} + \partial_x h \, \cancel{u_{xx}} +$$

$$+ \partial^2_{xy} h \, \cancel{u_x \overset{-u_y}{v_x}} + \partial^2_{yy} h \, v_x^2 + \partial_y h \, \cancel{v_{xx}} +$$

$$+ \partial^2_{xx} h \; u_y^2 + \partial^2_{yy} h \, \cancel{u_y \overset{u_x}{v_y}} + \partial_x h \, \cancel{u_{yy}} +$$

$$+ \partial^2_{yy} h \; v_y^2 + \partial^2_{yx} h \, \cancel{u_x} v_y + \partial_y h \, \cancel{v_{yy}} =$$

$$\underset{\uparrow}{=} \quad \ldots \quad (\Delta h)(u_x^2 + v_x^2) = \Delta h(u,v) \, |f'(z)|^2$$

using C-R eqns

& $\Delta u = \Delta v = 0$

$\Rightarrow$ $H$ is harmonic in $D_z$ if $h$ is harmonic in $D_\omega$.

$\boxed{\text{Bonus exercise}}$ #1

Assume f is analytic everywhere in $\mathbb{C}$ except for a finite $\overset{\uparrow}{\underset{z_1 \dots z_N}{\#}}$ of singular points interior to a positively oriented simple closed curve $C$

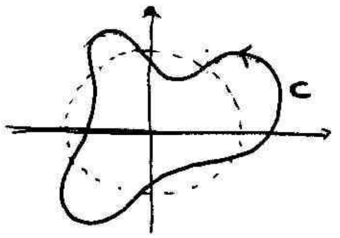
Consider $\tilde{C}$ = image of $C$ under the map $\omega = \frac{1}{z}$

- the map $\omega = \frac{1}{z}$ change the orientation, since $\omega' = -\frac{1}{z^2}$

  $\Rightarrow \tilde{C}$ will be negatively oriented

consider a parametrization of $\tilde{C} = \{\gamma(t) \mid t \in [a,b]\}$

$\Rightarrow C = \left\{\frac{1}{\gamma(t)} \mid t \in [a,b]\right\}$



$$\Rightarrow \int_C f(z)\,dz = \int_a^b f\left(\frac{1}{\gamma(t)}\right)\left(-\frac{\gamma'(t)}{\gamma(t)^2}\right)dt =$$

$$= -\int_{\tilde{C}} \frac{1}{z^2} f\left(\frac{1}{z}\right) dz = \int_{\underset{\nearrow \text{ positive orientation}}{+\tilde{C}}} \frac{1}{z^2} f\left(\frac{1}{z}\right) dz$$

consider the function $g(z) = \frac{1}{z^2} f\left(\frac{1}{z}\right)$ it's analytic in $\mathbb{C} \setminus \left\{\frac{1}{z_1}, \dots \underset{\uparrow}{\frac{1}{z_N}}, 0\right\}$

if one of these $z_i = 0$ don't consider it

- $\frac{1}{z_k}$ is outside $\tilde{C}$, since the map $\omega = \frac{1}{z}$ maps the interior of $C$ onto the exterior of $\tilde{C}$

$\Rightarrow$ the only residue inside $\tilde{C}$ is $z=0$

$\Rightarrow$ by <u>Residue's theorem</u>.

$$\boxed{\int_C f(z)\,dz = \int_{\tilde{C}} \frac{1}{z^2} f\left(\frac{1}{z}\right) dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]}$$