HOMEWORK #10

EX PAG 230, N. 2 $\quad C = \{|z|=3\}$

(a) $$\int_C \frac{e^{-z}}{z^2}\,dz = 2\pi i \operatorname{Res}_0\left(e^{-z}/z^2\right) = -2\pi i$$

$$\frac{e^{-z}}{z^2} = \frac{1-z+\frac{z^2}{2}+\dots}{z^2} = \frac{1}{z^2}-\frac{1}{z}+\frac{1}{2}+\dots$$

(otherwise, one could use Cauchy's thm $\int_C \frac{e^{-z}}{z^2} = 2\pi i \frac{d}{dz}\left[e^{-z}\right]_{z=0} = -2\pi i$)

(b) $$\int_C \frac{e^{-z}}{(z-1)^2}\,dz = 2\pi i \operatorname{Res}_1\left[\frac{e^{-z}}{(z-1)^2}\right] = 2\pi i \left[\operatorname{Res}_{z=1} \frac{e^{-(z-1)}\cdot e^{-1}}{(z-1)^2}\right] = \frac{2\pi i}{e}\operatorname{Res}_1\left(\frac{e^{-(z-1)}}{z-1}\right) =$$

as above

$$= -\frac{2\pi i}{e}$$

(c) $$\int_C z^2 e^{1/z}\,dz = 2\pi i \operatorname{Res}_0\left(z^2e^{1/z}\right) = \frac{2\pi i}{6} = \frac{\pi i}{3}$$

$$z^2e^{1/z} = z^2\left(1+\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{6z^3}+\dots\right) = z^2+z+\frac{1}{2}+\frac{1}{6z}+\dots$$

(d) $$\int_C \frac{z+1}{z^2-2z}\,dz = 2\pi i\left[\operatorname{Res}_0 \frac{z+1}{z(z-2)} + \operatorname{Res}_2 \frac{z+1}{z(z-2)}\right] = 2\pi i\left[\frac{0+1}{0-2}+\frac{2+1}{2}\right] =$$

singular. $z=0$ $z=2$

$$= 2\pi i\left[-\frac{1}{2}+\frac{3}{2}\right] = 2\pi i$$

EX. PAG. 230, N. 4

$C = \{|z|=1\}$ counterclockwise

(a) $$\int_C \exp\left(z+\tfrac{1}{z}\right)dz = \int_C e^{1/z}\sum_{n=0}^{\infty}\frac{z^n}{n!}\,dz \overset{\text{thm 1 (§ 59)}}{=}$$

$$= \sum_{n=0}^{\infty}\int_C \frac{z^n}{n!}e^{1/z}\,dz = \sum_{n=0}^{\infty}\frac{1}{n!}\int_C z^n\exp\left(\tfrac{1}{z}\right)dz$$

(b) $$\sum_{n=0}^{\infty}\frac{1}{n!}\int_C z^n\exp\left(\tfrac{1}{z}\right)dz \overset{\text{Residue's thm}}{=} 2\pi i\sum_{n=0}^{\infty}\frac{1}{n!}\underbrace{\operatorname{Res}_0\left[z^n\exp\left(\tfrac{1}{z}\right)\right]}_{=\frac{1}{(n+1)!}} =$$

it is the coeff in the Laurent expansion corresponding to $\frac{1}{z^{n+1}}$

$$= 2\pi i\sum_{n\geq 0}\frac{1}{n!(n+1)!} =$$

$\Longrightarrow$ (a)+(b) $$\int_C e^{\left(z+\frac{1}{z}\right)}dz = 2\pi i\sum_{n=0}^{\infty}\frac{1}{n!(n+1)!}$$

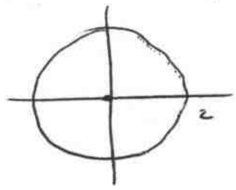
$$= 2\pi i \left[\frac{1}{2} + \frac{(3(3i)^3+2)[3i+1] + (2-(3i)^3 3)(3i-1)}{6i\,(3i-1)(3i+1)} \right]$$

$$= 2\pi i \left[\frac{1}{2} + \frac{2(3i\cancel{+1}+3i\cancel{-1}) + 3^4(-i)[(\cancel{3i}+1)-(\cancel{3i}-1)]}{6i\,(3i-1)(3i+1)} \right] = 2\pi i \left[\frac{1}{2} + \frac{12i - 81i\cdot 2}{6i\,(-9-1)} \right] =$$

$$= 2\pi i \left[\frac{1}{2} + \frac{\overset{5}{\cancel{150}}}{\underset{2}{\cancel{60}}} \right] = 6\pi i$$

$\boxed{\text{EX PAG 238, N. 5}}$

$$\int_C \frac{\cosh \pi z}{z(z^2+1)}\,dz \qquad C = \{|z|=2\}$$



$f(z) = \dfrac{\cosh \pi z}{z(z^2+1)}$ has singular points at $z = 0, \pm i$

- $z=0 \to$ pole of order 1: $\quad Res_0 f = \dfrac{\cosh 0}{1} = 1$
- $z = \pm i \to$ since $\cosh(\pm\pi i) \neq 0 \to$ poles of order 1

$$Res_{\pm i} f(z) = \frac{\cosh(\pm i\pi)}{\pm i\,(\pm 2i)} = \frac{\cosh \pi i}{-2} = \frac{\overset{-1}{\cos \pi}}{-2} = \frac{1}{2}$$

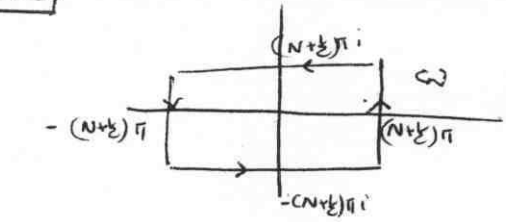
$$\Rightarrow \int_C \frac{\cosh \pi z}{z(z^2+1)}\,dz = 2\pi i \left[1 + \frac{1}{2} + \frac{1}{2}\right] = 4\pi i$$

$\boxed{\text{EX PAG. 245, N. 4}}$ $\qquad C = \{|z|=2\}$

(a) $\displaystyle\int_C \tan z\,dz.$

$f(z) = \tan z = \dfrac{\sin z}{\cos z} \Rightarrow$ it's analytic in $\mathbb{C} \setminus \{\frac{\pi}{2} + n\pi \,/\, n \in \mathbb{Z}\}$

$$\Rightarrow \int_C \tan z\,dz = 2\pi i \left[Res_{\frac{\pi}{2}} f + Res_{-\frac{\pi}{2}} f \right]$$



$$\int_{C_N} \frac{1}{z^2 \sin z} = ?$$

$$f(z) = \frac{1}{z^2 \sin z} \quad \to \text{poles at } z = m\pi \quad m \in \mathbb{Z}$$

poles inside C_N: $\quad z = 0, \pm\pi, \dots \pm N\pi$

in particular:

$\boxed{z=0}$: is a pole of order 3: $\cancel{z^2 \sin z} \;\; \cancel{z^2(z+O(z^3))} \;\; \cancel{z^3(1+O(z^2))}$

$$\frac{1}{z^2\sin z} = \frac{1}{z^2\left(z - \frac{z^3}{6} + O(z^5)\right)} = \frac{1}{z^3\left(1 - \frac{z^2}{6} + O(z^4)\right)} = \frac{1}{z^3}\left(1 + \frac{z^2}{6} + O(z^4)\right) =$$

$$= \frac{1}{z^3} + \frac{1}{6z} + O(z)$$

$$\Rightarrow \mathrm{Res}_0 f = \frac{1}{6}$$

$\boxed{z = m\pi}$ $\quad m \in ([-N,N] \cap \mathbb{Z}) \setminus \{0\}$

it's a simple pole $\quad \Longrightarrow \quad \mathrm{Res}_{m\pi} f = ?$

$$\frac{1}{z^2 \sin z} = \frac{1}{(z - m\pi + m\pi)^2 \sin(z - m\pi + m\pi)} = \frac{1}{\left[(z-m\pi)^2 + m^2\pi^2 + 2m\pi(z-m\pi)\right](-1)^m \sin(z-m\pi)} =$$

$$\underset{\zeta = (z-m\pi)}{=} \frac{(-1)^m}{\left[\zeta^2 + m^2\pi^2 + 2m\pi\zeta\right]\left(\zeta - \frac{\zeta^3}{3!} + O(\zeta^5)\right)} = \frac{(-1)^m}{\zeta(n^2\pi^2)\left[1 + \frac{2\zeta}{m\pi} + \frac{\zeta^2}{n^2\pi^2}\right]\left[1 - \frac{\zeta^2}{6} + O(\zeta^4)\right]} =$$

$$= \frac{(-1)^m}{n^2\pi^2\zeta}\left(1 - \frac{2\zeta}{n\pi} + O(\zeta^2)\right)\left(1 + O(\zeta^2)\right) = \frac{(-1)^m}{n^2\pi^2\zeta} + O(1)$$

$$\Rightarrow \underset{\substack{n\pi \\ (n \neq 0)}}{\mathrm{Res}} f = \frac{(-1)^m}{n^2\pi^2}$$

EXERCISE #2

g continuous real valued function of a real variable

s.t. $\int_{\mathbb{R}} |g(t)|\, dt < \infty$ and assume $g \not\equiv 0$

$\rightarrow$ Define $\hat{g}(s) = \int_{\mathbb{R}} e^{-ist} g(t)\, dt \quad \forall s \in \mathbb{R}$ (Fourier transform)

- Suppose g is compactly supported $\rightarrow \exists\ [a,b]$ finite interval st $g|_{[a,b]^c} \equiv 0$

(I) Since g is compactly supported $h(s) = \int_{\mathbb{R}} e^{-ist} g(t)\, dt$ makes sense $\forall s \in \mathbb{C}$.
and $h|_{\mathbb{R}} = \hat{g}$

(II) h is entire

(III) If $\hat{g}$ were compactly supported $\Longrightarrow$ h would have a non-discrete set of zeros
$\Longrightarrow h \equiv 0$ on $\mathbb{C}$ $\Longrightarrow \hat{g} \equiv 0$ on $\mathbb{R}$

(IV) $\hat{g} \equiv 0$ on $\mathbb{R}$ $\Longrightarrow$ (g continuous) $g \equiv 0$ on $\mathbb{R}$.

$\divideontimes$ (contradiction)