

FROM TEXTBOOK:

①

P. 11

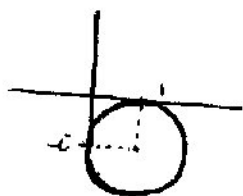
EX. 3: VERIFY THAT  $\sqrt{2}|z| \geq |\operatorname{Re} z| + |\operatorname{Im} z|$ LET  $z = x + iy \Rightarrow$  THE INEQ. BECOMES  $\sqrt{2} \sqrt{x^2 + y^2} \geq |x| + |y|$ 

$$\Leftrightarrow 2(x^2 + y^2) \geq (|x| + |y|)^2 = x^2 + y^2 + 2|x||y|$$

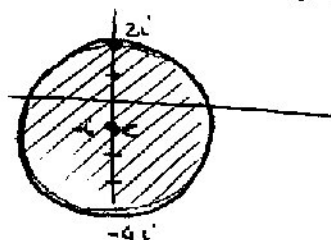
$$\Leftrightarrow x^2 + y^2 - 2|x||y| \geq 0 \quad \Leftrightarrow (|x| - |y|)^2 \geq 0 \quad \square$$

EX. 4 SKETCH THE SET OF POINT DETERMINED BY THE GIVEN CONDITION:

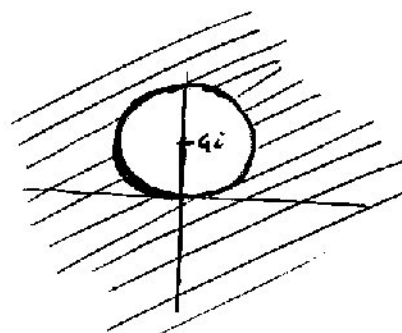
②  $|z - 1 + i| = 1$

IT IS A CIRCLE W/ CENTER  $C = 1 - i$   
AND RADIUS  $R = 1$ 

③  $|z + i| \leq 3$

IT IS A DISK W/ CENTER  $C = -i$   
AND RADIUS  $R = 3$ 

④  $|z - 4i| \geq 4$

IT IS THE SET OF POINTS  
OUTSIDE THE DISK OF RADIUS  
 $R = 4$  AND CENTER  $C = 4i$ 

EX. 7 SHOW THAT WHEN  $|z_3| \neq |z_4|$ :  $\left| \frac{z_1+z_2}{z_3+z_4} \right| \leq \frac{|z_1|+|z_2|}{||z_3|-|z_4||}$

PROOF:  $\left| \frac{z_1+z_2}{z_3+z_4} \right| = \frac{|z_1+z_2|}{|z_3+z_4|} \leq \frac{|z_1|+|z_2|}{||z_3|-|z_4||}$

• TRIANGLE INEQ:  $|z_1+z_2| \leq |z_1|+|z_2|$

• INEQ. PAG 10, #8:  $|z_3 \pm z_4| \geq ||z_3|-|z_4||$

EX. 10 FACTORIZING  $z^4 - 4z^2 + 3$  INTO TWO QUADRATIC FACTORS, WE GET:

$$z^4 - 4z^2 + 3 = (z^2 - 1)(z^2 - 3)$$

$\Rightarrow$  ON  $|z|=2$   $\left| \frac{1}{z^4 - 4z^2 + 3} \right| = \frac{1}{|z^4 - 4z^2 + 3|} = \frac{1}{|(z^2 - 1)(z^2 - 3)|} = \frac{1}{|z^2 - 1| \cdot |z^2 - 3|} \leq$

$$\leq \frac{1}{||z|^2 - 1| \cdot ||z|^2 - 3|} = \frac{1}{(4-1) \cdot (4-3)} = \frac{1}{3}$$

PAG. 21

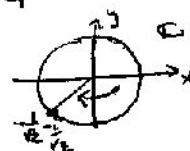
EX. 10

FIND THE PRINCIPAL ARGUMENTS OF:

(a)  $z = \frac{i}{-2-2i} = -\frac{1}{2} \cdot \frac{i}{1+i} \cdot \frac{(1-i)}{(1-i)} = -\frac{1}{2} \cdot \frac{i-i^2}{1-i^2} = -\frac{1}{4}(1+i) =$

$$= \frac{\sqrt{2}}{4} \left( -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \right) = \frac{\sqrt{2}}{4} e^{-\frac{3}{4}\pi i}$$

$$\Rightarrow \boxed{\text{Arg } z = -\frac{3\pi}{4}}$$



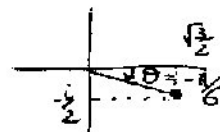
(b)  $z = (\sqrt{3} - i)^6$

NOTE  $\xi = \sqrt{3} - i \Rightarrow |\xi| = \sqrt{3+1} = 2 \Rightarrow \xi = 2 \left( \frac{\sqrt{3}}{2} - \frac{i}{2} \right) = 2 e^{-\frac{\pi}{6}i}$

$\Rightarrow z = \xi^6 = (2 e^{-\frac{\pi}{6}i})^6 = 2^6 e^{-\pi i} = 2^6 e^{\pi i}$

$\uparrow$   
 $-\pi = \pi + 2\pi$   
 $\& e^{2\pi i} = 1$

$$\Rightarrow \boxed{\text{Arg } z = \pi}$$



EX. 5. (a)  $(e^{i\theta})^3 = e^{3i\theta} \Leftrightarrow (\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$  (3)

$$\Leftrightarrow \cos^3\theta + i^3\sin^3\theta + 3\cos^2\theta(i\sin\theta) - 3\cos\theta\sin^2\theta = \cos 3\theta + i\sin 3\theta$$

$$\Leftrightarrow (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta) = \cos 3\theta + i\sin 3\theta \quad (*)$$

TAKING THE REAL PART OF (\*):  $\boxed{\cos^3\theta - 3\cos\theta\sin^2\theta = \cos 3\theta}$

(b) TAKING THE IMAGINARY PART OF (\*):  $\boxed{3\cos^2\theta\sin\theta - \sin^3\theta = \sin 3\theta}$

EX. 10 • ESTABLISH THE FIRST IDENTITY:

Denote:  $S = 1 + z + \dots + z^n \Rightarrow S - zS = (1 + \dots + z^n) - z(1 + \dots + z^n) = 1 - z^{n+1}$

$$\Rightarrow S = \frac{1 - z^{n+1}}{1 - z} \quad \square$$

• TO PROVE THE TRIGONOMETRIC ONE; LET  $z = e^{i\theta}$  AND USE THE PREVIOUS IDENTITY:

$$S = \sum_{k=0}^n (\underbrace{\cos k\theta + i\sin k\theta}_{e^{ik\theta} = (e^{i\theta})^k}) = \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}$$

(#)

$$\left( \sum_{k=0}^n \cos k\theta \right) + i \left( \sum_{k=0}^n \sin k\theta \right)$$

TAKING THE REAL PART OF (#):

$$\sum_{k=0}^n \cos k\theta = \operatorname{Re} \left[ \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} \right] = \operatorname{Re} \left[ \frac{1 - e^{i(n+1)\theta}}{e^{i\theta/2} (e^{-i\theta/2} - e^{i\theta/2})} \right] = \operatorname{Re} \left[ \frac{e^{-i\theta/2} - e^{i(n+1/2)\theta}}{-2i\sin\theta/2} \right]$$

$$= \frac{\operatorname{Re} [i(e^{-i\theta/2} - e^{i(n+1/2)\theta})]}{2\sin\theta/2} = \frac{-\operatorname{Im}(e^{-i\theta/2} - e^{i(n+1/2)\theta})}{2\sin\theta/2} = \frac{\sin(-\theta/2) + \sin(n+1/2)\theta}{2\sin\theta/2}$$

$$= \frac{1}{2} + \frac{\sin(n+1)\theta/2}{2\sin\theta/2} \quad \square$$

① • PROVE  $\text{cl}(S) = S \cup \{\text{boundary of } S\}$  is closed.

WE WANT TO SHOW THAT  $\{\text{boundary pts of } \text{cl}(S)\} \subseteq \text{cl}(S)$

ASSUME  $\exists z_0$  boundary point of  $\text{cl}(S)$ , s.t.  $z_0 \notin \text{cl}(S) \Rightarrow \begin{cases} z_0 \notin S \\ \& \\ z_0 \notin \{\text{boundary of } S\} \end{cases}$

$\Rightarrow z_0$  IS AN EXTERIOR POINT, i.e.  $\exists$  <sup>for S</sup>  $N$  open nbhd of  $z_0$  st  $N \cap S = \emptyset$

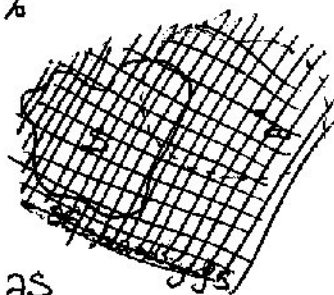
BUT  $z_0$  IS A BOUNDARY POINT OF  $\text{cl}(S) \Rightarrow N \cap \text{cl}(S) \neq \emptyset$  (by definition of boundary pt.)

$\Rightarrow N \cap \{\text{boundary point of } S\} \neq \emptyset$  (b/c  $N \cap S = \emptyset$ )

LET'S CALL  $z_1$  A POINT IN THIS INTERSECTION:  $z_1 \in N \cap \{\text{boundary pt of } S\}$

$\Rightarrow$  SINCE  $N$  IS OPEN AND  $z_1$  IS A BOUNDARY POINT OF  $S$ ,  $\exists U \subset N$  st  $U \cap S \neq \emptyset$

$\Rightarrow N \cap S \supset U \cap S \neq \emptyset$  : CONTRADICTION !!



• PROVE THAT: ANY CLOSED SET  $A \supset S$ , CONTAINS  $\text{cl}(S)$

proof:

If  $z_0 \in \text{cl}(S) \Rightarrow z_0 \in S$  OR  $z_0 \in \{\text{boundary of } S\} = \partial S$

• IF  $z_0 \in S \Rightarrow z_0 \in A$

• IF  $z_0 \in \partial S \Rightarrow \forall N$  nbhd of  $z_0$   $N \cap A \supset N \cap S \neq \emptyset$

~~$\Rightarrow z_0$  cannot be an exterior point of  $A$   
 $\Rightarrow z_0$  is an interior point of  $A$  OR  $z_0$  is a boundary point of  $A$   
 (in either cases,  $z_0 \in A$ , since  $A$  is closed)~~

$\Rightarrow z_0$  cannot be an exterior point of  $A \Rightarrow z_0$  is either an interior point or a boundary point of  $A$

$\Rightarrow$  in both cases  $z_0 \in A$  (since  $A$  is closed)

$\Rightarrow A \supseteq \text{cl}(S)$   $\square$

② ① Show  $|z| = \max_{-\pi < \theta \leq \pi} \operatorname{Re}(ze^{i\theta})$

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$$z = e^{i\varphi} |z| \quad \exists \varphi \in (-\pi, \pi] \quad \Rightarrow \quad \operatorname{Re}(ze^{i\theta}) = \operatorname{Re}(|z| e^{i(\theta+\varphi)}) = |z| \cos(\theta+\varphi)$$

$$\Rightarrow \max_{-\pi < \theta \leq \pi} (|z| \cos(\theta+\varphi)) = |z|$$

$$\uparrow \text{ TAKE } \begin{cases} \theta = -\varphi & \text{if } \varphi \in (-\pi, \pi) \\ \theta = \pi & \text{if } \varphi = \pi \end{cases}$$

$$\textcircled{b} \quad |z_1 + z_2| = \max_{-\pi < \theta \leq \pi} \operatorname{Re}((z_1 + z_2)e^{i\theta}) = \max_{-\pi < \theta \leq \pi} [\operatorname{Re}(z_1 e^{i\theta} + z_2 e^{i\theta})] =$$

$$= \max_{-\pi < \theta \leq \pi} [\operatorname{Re}(z_1 e^{i\theta}) + \operatorname{Re}(z_2 e^{i\theta})] \leq$$

$$\leq \max_{-\pi < \theta \leq \pi} [\operatorname{Re}(z_1 e^{i\theta})] + \max_{-\pi < \theta \leq \pi} [\operatorname{Re}(z_2 e^{i\theta})] = |z_1| + |z_2|$$

③ 1. denote  $\mathbb{Z}[i] = \{\text{set of the Gaussian integers}\} = \{m + im' \mid m, m' \in \mathbb{Z}\}$

$$\bullet \text{ if } m + im, m' + im' \in \mathbb{Z}[i] \Rightarrow (m + im) + (m' + im') = \underbrace{(m + m')}_{\in \mathbb{Z}} + i \underbrace{(m + m')}_{\in \mathbb{Z}} \in \mathbb{Z}[i]$$

$$\bullet \text{ moreover: } (m + im) \cdot (m' + im') = \underbrace{(mm' - mm'm)}_{\in \mathbb{Z}} + i \underbrace{(mm'm + m'm)}_{\in \mathbb{Z}} \in \mathbb{Z}[i]$$

• WHICH GAUSSIAN INTEGERS HAVE AN INVERSE?

$$\text{LET } \omega = m + im. \text{ Assume } \exists \alpha \in \mathbb{Z}[i] \text{ st } \underbrace{\alpha}_{m' + im'} \cdot \omega = \omega \cdot \alpha = 1$$

$$\Leftrightarrow (mm' - mm'm) + i(mm'm + m'm) = 1 \quad \Leftrightarrow \begin{cases} mm'm + m'm = 0 & (*) \\ mm' - mm'm = 1 & (**) \end{cases}$$

Multiply (\*) by  $m$  and (\*\*) by  $m$ , and sum up together:

$$(mm'm/m + m'm^2) + (m^2m' - mm'm/m) = m \quad \Leftrightarrow \quad m = m'(m^2 + m^2)$$

$$\bullet \text{ if } m' \neq 0 \Rightarrow |m| \geq m^2 + m^2 \geq m^2 \Rightarrow \begin{cases} m = 0 \\ m = \pm 1 \end{cases} \Rightarrow \omega = \pm 1$$

$$\bullet \text{ if } m' = 0 \Rightarrow m = 0 \Rightarrow (*) \text{ becomes } -mm'm = 1 \Rightarrow m = \pm i \Rightarrow \omega = \pm i$$

Therefore  $\omega$  invertible  $\Rightarrow \omega \in \{ \pm 1, \pm i \} =: \mathcal{U}$

obviously each element in  $\mathcal{U}$  has an inverse in  $\mathbb{Z}[i]$   $\Rightarrow$  the only invertible elements in  $\mathbb{Z}[i]$  are  $\omega = \pm 1, \pm i$ .

2. LET'S SHOW:  $m, m$  are sums of two squares  $\Rightarrow m \cdot m$  is sum of two squares

observe: IF  $m = a^2 + b^2 \Rightarrow \exists \omega_1 \in \mathbb{Z}[i]$  st.  $|\omega_1|^2 = m$ , namely  $\omega_1 = a + ib$   
 $m = c^2 + d^2 \Rightarrow \exists \omega_2 \in \mathbb{Z}[i]$  st.  $|\omega_2|^2 = m$ , "  $\omega_2 = c + id$   
 let  $\omega_1, \omega_2 \in \mathbb{Z}[i]$  st.  $|\omega_1|^2 = m$  &  $|\omega_2|^2 = m$

$$\Rightarrow m \cdot m = |\omega_1|^2 \cdot |\omega_2|^2 = (|\omega_1| \cdot |\omega_2|)^2 = |\omega_1 \cdot \omega_2|^2$$

$\omega_1 \cdot \omega_2 \in \mathbb{Z}[i]$ , since the gaussian integers are closed under  $\cdot$ .

$$\Rightarrow \omega_1 \cdot \omega_2 = \alpha + i\beta \in \mathbb{Z}[i]$$

$$\Rightarrow m \cdot m = |\omega_1 \cdot \omega_2|^2 = \alpha^2 + \beta^2 \Rightarrow m \cdot m \text{ is sum of two squares } \square$$