Please read carefully the following instructions:

• Time: 80 min.

• Write your name on the top of EVERY page.

• Answer ALL parts of ALL questions.

• This is a closed book exam. You may NOT use any reference material (including calculators).

• EXPLAIN EVERY STEP, COMPUTATION AND GRAPH YOU DRAW.

GOOD LUCK!
1. Let $f(z)$ be the function $f(z) = \exp(z^6 + 1)$,

(a) (7) Show that $f$ is entire.
(b) (26) Let $C$ be the contour from $z_0 = 0$ to $z_1 = 1 + i$ given by the parametrization $z(t) = t + it^2$ for $0 \leq t \leq 1$. Prove that $|\int_C f(z)dz| < e^{10}$.

2. (a) (10) Find all $z$ at which $f(z) = \bar{z}^2 + \sin z$ is differentiable.
(b) (5) where is $f(z)$ analytic?

3. (19) find a harmonic function $p(x, y)$ on the whole plane which is polynomial of degree 4 in $x$ and $y$ (i.e. $p(x, y) = \sum_{i=0}^{4} \sum_{j=0}^{4-i} c_{i,j} x^i y^j$ for some constants $c_{i,j}$, with at least one of the $c_{i,j}$ for $i + j = 4$ nonzero), and its harmonic conjugate. Don’t forget to explain why the functions you give work!

4. (a) (5) Sketch the image under the map $g(z) = \frac{1}{z-1}$ of the domain $D = \{0 < |z-1| < 1\}$.
(b) (3) Is this image a) open? b) bounded? c) connected? (give only a yes/no answer).
(c) (25) consider the function $f(z) = \sin \frac{1}{z-1}$. Show that the image under $f$ of the domain $D$ is all of $\mathbb{C}$. You may use without proof the fact that every $w \in \mathbb{C}$ there is a $z \in \mathbb{C}$ so that $w = \sin z$. 
Some formulas: (don’t look for hints in the formulas)

- Cauchy-Riemann equations: $u_x = v_y$ and $u_y = -v_x$ where $f = u + iv$; in polar coordinates: $ru_r = v_\theta$ and $u_\theta = -rv_r$.

- Euler’s formula: $\exp(ix) = \cos x + i \sin x$

- $|z|^2 = z\bar{z}$