MATH 317 - Practice midterm

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Spring 2005

Please read carefully the following instructions:

- Time: 80 min.
- Write your name on the top of EVERY page.
- Answer ALL parts of ALL questions.
- This is a closed book exam. You may NOT use any reference material (including calculators).
- EXPLAIN EVERY STEP, COMPUTATION AND GRAPH YOU DRAW.

GOOD LUCK!

1. Let $f(z)$ be the function $f(z) = \exp(z^6 + 1)$,
   (a) **(7)** Show that f is entire.
   (b) **(26)** Let C be the contour from $z_0 = 0$ to $z_1 = 1 + i$ given by the parametrization $z(t) = t + it^2$ for $0 \leq t \leq 1$. Prove that $\left|\int_C f(z)dz\right| < e^{10}$.

2. (a) **(10)** Find all z at which $f(z) = \bar{z}^2 + \sin z$ is differentiable.
   (b) **(5)** where is $f(z)$ analytic?

3. **(19)** find a harmonic function $p(x, y)$ on the whole plane which is polynomial of degree 4 in x and y (i.e. $p(x, y) = \sum_{i=0}^{4} \sum_{j=0}^{4-i} c_{i,j} x^i y^j$ for some constants $c_{i,j}$, with at least one of the $c_{i,j}$ for $i + j = 4$ nonzero), **and its harmonic conjugate**. Don't forget to explain why the functions you give work!

4. (a) **(5)** Sketch the image under the map $g(z) = \frac{1}{z-1}$ of the domain $D = \{0 < |z - 1| < 1\}$.
   (b) **(3)** Is this image a) open? b) bounded? c) connected? (give only a yes/no answer).
   (c) **(25)** consider the function $f(z) = \sin \frac{1}{z-1}$. Show that the image under f of the domain D is all of $\mathbb{C}$.

   You **may use** without proof the fact that every $w \in \mathbb{C}$ there is a $z \in \mathbb{C}$ so that $w = \sin z$.

Some formulas: (don't look for hints in the formulas)

- Caucy-Riemann equations: $u_x = v_y$ and $u_y = -v_x$ where $f = u + iv$; in polar coordinates: $ru_r = v_\theta$ and $u_\theta = -rv_r$.
- Euler's formula: $\exp(ix) = \cos x + i \sin x$
- $|z|^2 = z\bar{z}$