MATH 317 - Practice Final

Prof. Elon Lindenstrauss

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Student ID: _____

Signature: _____

Read carefully the following instructions:

- Print your name and student ID number and write your signature to indicate that you accept the honor code.
- You may use YOUR (undecorated) books as well as a basic scientific calculator (you may not use any 'advanced' features such as graphing etc. beyond what is found on a basic scientific calculator).
- Read each question carefully, **Explain every step, computation and graph you draw**. Put a box around your final answer to each question. Cross out anything you do not wish to be graded (e.g. drafts, false starts, etc.)
- Answer ALL eight questions. The maximum grade is 100.
- You have 3 hours to do all the problems.

1. **(12.5)** consider the polynomial $p(z) = z^5 - 13z^2 + 1$.
   (a) find a $R > 0$ so that all roots of $p(z)$ satisfy $|z| < R$.
   (b) show that $p(z)$ has precisely two roots in the circle $|z| < 1$.

2. **(12.5)** Let C be a simple closed contour, and $f(z)$ a function which is analytic at every point of the contour C (notice that we do not assume $f(z)$ is analytic inside this contour!). Assume that
$$Re(f(z)) < -2005 \tag{0.1}$$
for all z on C. Prove that
$$\int_C \frac{f'(z)}{f(z)} dz = 0.$$
Where did you use (0.1)? *Hint*: consider $Log(f(z))$.

3. **(12.5)** Let $f(z)$ be the function
$$f(z) = \frac{e^{-z^2}\sin(z)}{1+z^2}.$$
What is the circle of convergence of the Taylor series for $f(z)$ around z_0 when
   (a) $z_0 = 0$
   (b) $z_0 = 1$.

4. **(12.5)** Consider the function $f(z) = \frac{1}{\sin z}$.
   (a) Show that f has an antiderivative on every simply connected domain which does not include any of the points $0, \pm\pi, \pm 2\pi, \ldots$.
   (b) Does f have an antiderivative in the domain $\{0 < |z| < 1\}$? Is this domain simply connected?

5. **(12.5)** Find all the entire functions $f(z)$ that satisfy
$$|f(z) - \sin(z)| < 2005$$
for all $z \in \mathbb{C}$.

6. **(12.5)** Classify the singular points and compute the residue at these point for
$$f(z) = \tan(z) + e^z$$

7. **(12.5)** Use contour integration to evaluate the following indefinite integral:
$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+2)} dx$$

8. **(12.5)** Compute
$$\int_{\Gamma} \frac{\mathrm{Log}(z^2+1)}{z^8+1} dz,$$
where Γ is the positively oriented circle $|z| = 1/2$, and $\mathrm{Log}(z)$ is the principal branch of $\log(z)$.