EX 1.

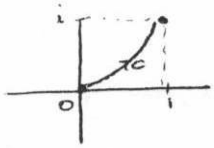
a) $f(z) = \exp(z^6+1)$

f is composition of two entire functions:

- $\exp: \mathbb{C} \longrightarrow \mathbb{C}$, $z \longmapsto e^z$ & $P(z) = z^6+1$ (it's a polynomial)

$\Rightarrow$ f is entire

In particular, it's derivative at every point is: $f'(z) = 6z^5 e^{z^6+1}$.

b) C:  $\{z(t) = \{t + it^2 \quad 0 \le t \le 1\}$

prove that $\left|\int_C f\,dz\right| \le e^{10}$.

In fact: $\text{length}(C) =: L = \int_0^1 |z'(t)|\,dt = \int_0^1 |1+2ti|\,dt = \int_0^1 \sqrt{4t^2+1}\,dt \le \sqrt{5}$ ($0 \le t \le 1$)

hence: $\left|\int_C f\,dz\right| = \left|\int_C e^{1+z^6}\,dz\right| \le \int_C \left|e^{1+z^6}\right|\,|dz| \le$

$$\le \int_C \underbrace{e^{1+|z|^6}}_{\substack{|z| \le \sqrt{2} \\ (\text{since it's in } [0,1]\times[0,1])}} |dz| \le \int_C e^{1+2^3}\,|dz| \le e^9 \cdot \text{length}(C) \le$$

$$\le e^9 \cdot \underset{e}{\underset{\wedge\!\!\!\mid}{\sqrt{5}}} \le e^{10}.$$

EX2

(a) Find all z at which $f(z) = \bar{z}^2 + \sin z$ is differentiable

The function $g(z) = \sin z$ is differentiable at any $z \in \mathbb{C}$, therefore it suffices to study where $h(z) = \bar{z}^2$ is differentiable

(In fact, in all points where they are both differentiable, the sum is diff. as well) and moreover, if f is diff at $z_0 \Rightarrow f - \sin z$ is diff. at z_0 as well and $f - \sin z = \bar{z}^2$!)

$$h(z) = (x-iy)^2 = x^2 - 2ixy + i^2 y = (x^2-y^2) - 2ixy$$

$$\Rightarrow u(x,y) = x^2 - y^2 \qquad \Longrightarrow \begin{cases} u_x = 2x & v_x = -2y \\ u_y = -2y & v_y = -2x \end{cases}$$

$$v(x,y) = -2xy$$

Note that the partial derivatives are all continuous!

By Cauchy Riemann equations: $u_x = v_y \iff x = 0$

$$u_y = -v_x \iff y = 0$$

Hence $\bar{z}^2$ is differentiable only at the origin and $h'(0) = \frac{d}{dz}(\bar{z}^2)\Big|_{z=0} = 0$

$$\Rightarrow f'(z) = 0 + \cos z\Big|_{z=0} = 1$$

(b) f is analytic nowhere! In fact the only point where it is diff., is $z_0 = 0$ but it's not differentiable in any nbhd of the origin.

EX3

Find a harmonic function $p(x,y)$ on all $\mathbb{C}$ and its harmonic conjugate

Consider $f(z)=z^4=(x+iy)^4=\sum_{j=0}^{4} x^j (yi)^{4-j}\binom{4}{4-j} = \cancel{x^4+\binom{4}{3}x^3 yi + x^2(iy)^2}\binom{4}{2}$

$$= x^4+\binom{4}{3}x^3(iy)^1+\binom{4}{2}x^2(yi)^2+\binom{4}{1}x(yi)^3+(iy)^4 =$$

$$= x^4+4ix^3y + 6x^2y^2 - 4ixy^3 + y^4 =$$

$$= (x^4-6x^2y^2+y^4)+i(4x^3y-4xy^3)$$

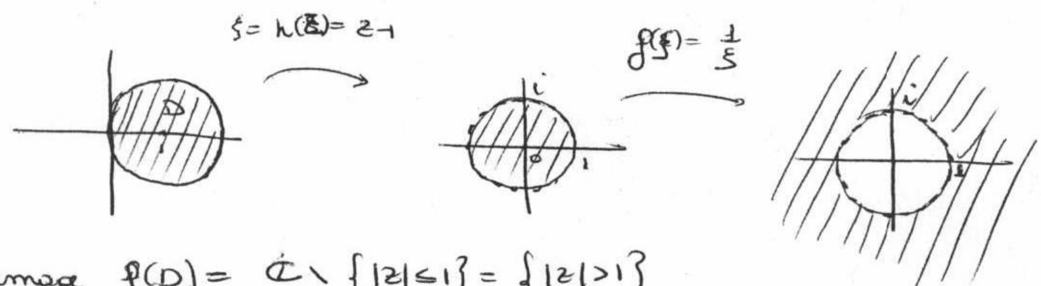
Our polynomial is (for instance) $p=x^4-6x^2y^2+y^4$

and its conjugate $q=4x^3y-4xy^3$

(it follows from the fact that the real and imaginary part of an analytic function, are harmonic and conjugated)

EX4

(a)



the image $f(D) = \mathbb{C}\setminus\{|z|\le 1\} = \{|z|>1\}$

(b)
- it's open
- it's not bounded
- it's connected

(c) consider $f(z)=\sin\frac{1}{z-1}$

Show that the image under f of D, is all $\mathbb{C}$:

Consider $\omega_0 \in \mathbb{C}$; we want to find $z_0 \in D$ st $\omega_0 = f(z_0)$

Using the fact that $\forall \omega \in \mathbb{C}\ \exists\ \xi \in \mathbb{C}$ st $\omega = \sin \xi$, we may conclude that

$\exists\ \xi_0 \in \mathbb{C}: \quad \omega_0 = \sin \xi_0$

Actually $\xi_m = \xi_0 + 2\pi m$ is a solution for any m

$\rightarrow$ I can find N st $|\xi_N| > 1$

$\Rightarrow$ using what we have proved in the previous point, $\exists\ z_0$: $g(z_0) = \frac{1}{1-z_0} = \xi_N$

and thus conclude our proof:

$$\sin \frac{1}{1-z_0} = \sin \xi_N = \omega_0 \quad \square$$