MATH 317 - Midterm – Solutions

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Please read carefully the following instructions:

- Time: 80 min.
- Write your name on the top of EVERY page.
- Answer ALL parts of ALL questions.
- This is a closed book exam. You may NOT use any reference material (including calculators).
- EXPLAIN EVERY STEP, COMPUTATION AND GRAPH YOU DRAW.

GOOD LUCK!

NAME: Elon Lindenstrauss 2/6

1. Let $f(z)$ be the function $f(z) = \frac{z^3-1}{z^6+1}$,

(a) **(5)** what are the singular points of f? (write them down explicitly, and don't forget to justify your answer)

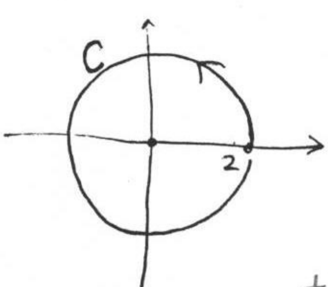
both z^3-1 & z^6+1 are entire, so f is analytic except where the denominator $z^6+1=0$, ie. where

$$z^6 = -1 = e^{i\pi}$$

or $z = e^{i\theta}$ for $\theta = \frac{\pi}{6} + \frac{2n\pi}{6} = \frac{\pi}{6} + n\frac{\pi}{3}$ $n=0 \dots 5$.

note that among these six points are $i = e^{\frac{\pi}{6}+\frac{\pi}{3}}$ & $-i = e^{\frac{\pi}{6}+\frac{4\pi}{3}}$.

at these 6 pts f is not defined, in particular is not analytic, but at every other pt in a sufficiently small nbhd is f is analytic, so these 6 pts are the singular pts of f.



(b) **(27)** Let C be the positively oriented closed contour along the contour $|z| = 2$ beginning and ending at $z = 2$. Show that

$$\left| \int_C f(z)dz \right| < 2.$$

the length of C, a circle of radius 2, is $2\pi r = 4\pi$.

$$\left| \int_C f(z)dz \right| \le \max_{z \text{ on } C} |f(z)| \cdot 4\pi$$

$$\left| \frac{z^3-1}{z^6+1} \right| \le \frac{|z^3|+1}{|z^6|-1} \underset{\text{on } C}{=} \frac{2^3+1}{2^6-1} = \frac{9}{63} = \frac{1}{7}$$

So $\left| \int_C f(z)dz \right| \le \frac{4\pi}{7} < 2$.

NAME: Elon Lindenstrauss

2. **(33)** Let $D \subset \mathbb{C}$ be a domain, and f analytic on D with $f(z) \neq 0$ for all $z \in D$. Show that if $\text{Arg} f(z) = \textit{constant}$ in D then f itself is constant.

Let $\theta = \text{Arg } f(z)$. then

$g(z) = e^{-i\theta} f(z)$ is real valued and analytic in D.

write $g(z) = u(x,y) + iv(x,y)$.

Since $g(z)$ is real valued in fact $v(x,y) \equiv 0$.

since g is analytic in D, it satisfies C-R, hence

$$u_x = v_y = 0 \quad \& \quad u_y = -v_x = 0$$

so $u \equiv \text{const}$ and $v = 0$ on D.

it follows that $g(z)$ is const on D, and so

$f(z) = e^{i\theta} g(z)$ is also constant.

NAME: Elon Lindenstrauss

3. Consider the function $f(z) = e^{e^z}$

   (a) **(18)** what is the image of $D = \{z : -\pi/2 \leq Im(z) \leq \pi/2\}$ under the function f? Sketch D, image of D under e^z and image of D under $f = e^{e^z}$.

D:

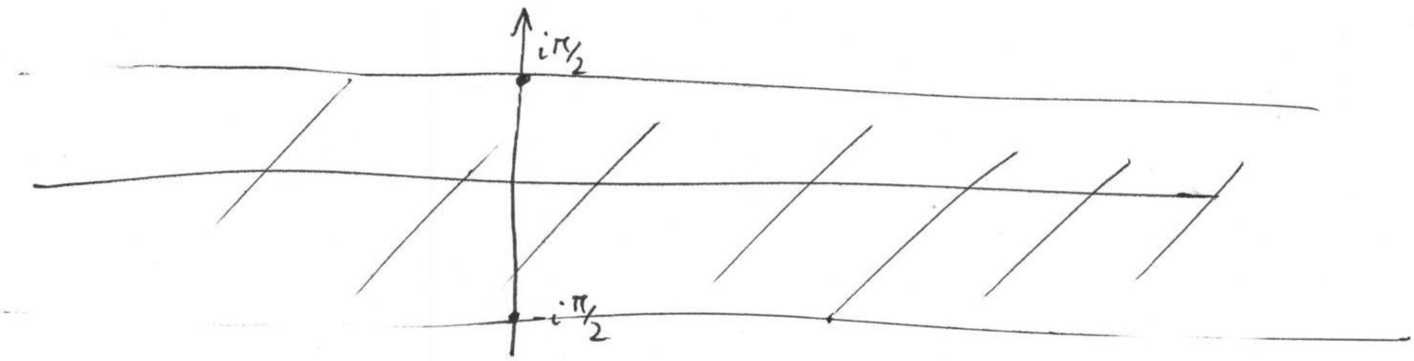
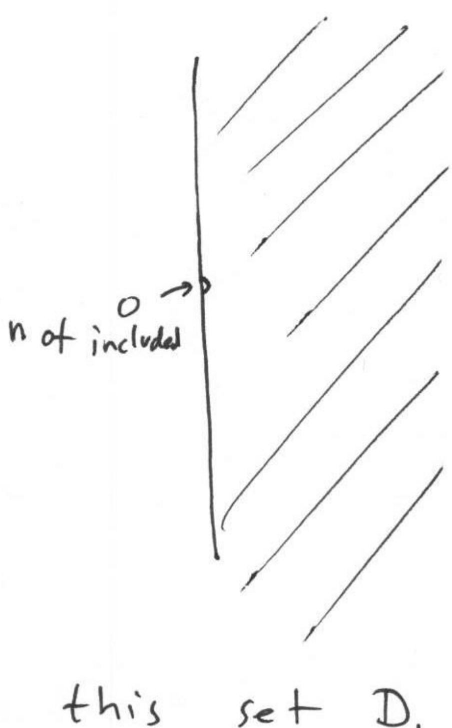


image of D under e^z:

$$\{e^{x+iy} : x \in \mathbb{R}, -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\} = \{re^{i\theta} : r > 0 \ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\}$$

$$= \{x+iy : x \geq 0\} \setminus \{0\}.$$



Let's call this set D_1

Elon Lindenstrauss

image of D under e^{e^z} = image of D_1 under $e^z =: D_2$

$$= \left\{ e^{x+iy} : x \geq 0, \text{ \& if } x=0 \ y \neq 0 \right\} \subset \left\{ e^{x+iy} : x \geq 0 \right\} = \left\{ z : |z| \geq 1 \right\}$$

for every $y \neq 0$, the ray $\left\{ e^{x+iy} : 0 \leq x \right\} = \left\{ re^{iy} : r \geq 1 \right\}$

in in D. & in particular:

$$\left\{ e^{x+iy} : x \geq 0 \quad 4\pi \geq y \geq 2\pi \right\} \subset D_2$$

but any $|z| \geq 1$ can be expressed in this way:

so $\quad D_2 = \{ z : |z| \geq 1 \}$.

note: $z = 1$ is in D_2

NAME: Elon Lindenstrauss

(b) **(16)** find the following limits, or prove none exists

i. $\lim_{z\to 0} e^{e^z}$

ii. $\lim_{z\to\infty} e^{e^z}$

i. e^z is an entire function, in particular continuous at every $z \in \mathbb{C}$.

therefore e^{e^z} is continuous as the composition of continuous functions, hence by def. of continuity

$$\lim_{z\to 0} e^{e^z} = e^{e^0} = e^1 = e.$$

Elon Lindenstrauss

ii. $\lim_{z\to\infty} e^{e^z}$ does not exist.

two proofs:

<u>proof 1:</u> it is not true that $\lim_{z\to\infty} e^{e^z} = c$ for some $c \in \mathbb{C}$.

indeed, if $t \geq \log(\max[1, \log|c|]) + 1 = t_0$

then $e^{e^t} > e^{1+\max[1,\log|c|]} \geq e\cdot\max(e, |c|)$

so for $t > t_0$ $\quad |e^{e^t} - c| \geq |e^{e^t}| - |c| \geq 1$

it is not true that $\lim_{z\to\infty} e^{e^z} = \infty$ since for all real $t<0$

$$e^{e^t} < e.$$

hence $\lim_{z\to\infty} e^{e^z}$ does not exist.

<u>proof 2:</u> we showed in class that for any $R>0$, image of $\{|z|>R\}$ under e^z is $\mathbb{C}\setminus\{0\}$.

it follows that for any $R>0$ image of $\{|z|>R\}$ under e^{e^z} is the image of $\mathbb{C}\setminus\{0\}$ under e^z which is again $\mathbb{C}\setminus\{0\}$.

hence e^{e^z} cannot converge as $|z|\to\infty$.