Homework assignment #8

Math 317

Due Friday, April 8

1. Solve the following problems from the textbook: P 162 ex. 1(de), 8, P 172 ex. 4,5,6,10, P 181 ex. 3,4

2. Prove the following generalization of Liouville's theorem: let $f(z)$ be an entire function so that for some $A, B > 0$ the function f satisfies $|f(z)| < A|z|^n + B$. Show that $f(z)$ is a polynomial of degree $\leq n$. *Hint:* $f(z)$ is a polynomial of degree $\leq n$ if and only if it is entire and its nth derivative is identically zero.

3. (a) let D_1 be the right half plane $\{z \in \mathbb{C} : \operatorname{re} z > 0\}$. (i) what is the boundary of D_1? (ii) show that $f(z) = e^z$ satisfies $|f(z)| = 1$ for every z on the boundary of D_1 but that $f(z)$ is not bounded on D_1.

   (b) Let D_2 be the strip $\left\{-\frac{1}{2} < \operatorname{re} z < \frac{1}{2}\right\}$. Find a function $f(z)$ analytic on D_2 and at every z on the boundary of D_2 so that $|f(z)| \leq 1$ for every z on the boundary of D_2 but that $f(z)$ is not bounded on D_2.

   (c) (a) and (b) are not a contradiction to the theorems we discussed in class regarding the maximum modulus principle since both D_1 and D_2 are not bounded. But something can be salvaged even in this case: this is called the Phragmen-Lindelof theorem or method (there are many variants), and is extremely useful in many applications. For both $D = D_1$ and D_2 there is a dichotomy: if a function f is analytic on D as well as all points on the boundary of D, satisfying $|f(z)| \leq M$ for every point on the boundary of D then either $|f(z)| \leq M$ for every $z \in D$ or $|f(z)|$ has to increase very rapidly. Here is a precise formulation for D_1:

Theorem 0.1. *Suppose f is analytic on the half plane D_1 as well as all points on the boundary of D_1, satisfying $|f(z)| \leq M$ for every point on the boundary of D_1. Suppose furthermore that for some $A, B > 0$ and $\alpha < 1$*

$$|f(z)| \leq A \exp(Bz^{\alpha}) \qquad \text{for } z \in D_1$$

then in fact $|f(z)| \leq M$ for all $z \in D_1$.

BONUS Ex. [25pts]: prove this theorem. Hint: fix $\alpha < \beta < 1$ and $w \in D_1$. We want to show $|f(w)| \leq M$. To do this, take $\epsilon > 0$ to be arbitrary, and apply the maximum modulus theorem to $g(z) = f(z)\exp(-\epsilon z^{\beta})$ for sufficiently large semicircles (how large will depend on ϵ).