1. Solve the following problems from the textbook: P 162 ex. 1(de), 8, P 172 ex. 4, 5, 6, 10, P 181 ex. 3, 4

2. Prove the following generalization of Liouville’s theorem: let $f(z)$ be an entire function so that for some $A, B > 0$ the function $f$ satisfies $|f(z)| < A|z|^n + B$. Show that $f(z)$ is a polynomial of degree $\leq n$. Hint: $f(z)$ is a polynomial of degree $\leq n$ if and only if it is entire and its $n$th derivative is identically zero.

3. (a) let $D_1$ be the right half plane $\{z \in \mathbb{C} : \text{re } z > 0\}$. (i) what is the boundary of $D_1$? (ii) show that $f(z) = e^z$ satisfies $|f(z)| = 1$ for every $z$ on the boundary of $D_1$ but that $f(z)$ is not bounded on $D_1$.

(b) Let $D_2$ be the strip $\{-\frac{1}{2} < \text{re }z < \frac{1}{2}\}$. Find a function $f(z)$ analytic on $D_2$ and at every $z$ on the boundary of $D_2$ so that $|f(z)| \leq 1$ for every $z$ on the boundary of $D_2$ but that $f(z)$ is not bounded on $D_2$.

(c) (a) and (b) are not a contradiction to the theorems we discussed in class regarding the maximum modulus principle since both $D_1$ and $D_2$ are not bounded. But something can be salvaged even in this case: this is called the Phragmen-Lindelof theorem or method (there are many variants), and is extremely useful in many applications. For both $D = D_1$ and $D_2$ there is a dichotomy: if a function $f$ is analytic on $D$ as well as all points on the boundary of $D$, satisfying $|f(z)| \leq M$ for every point on the boundary of $D$ then either $|f(z)| \leq M$ for every $z \in D$ or $|f(z)|$ has to increase very rapidly. Here is a precise formulation for $D_1$:
Theorem 0.1. Suppose $f$ is analytic on the half plane $D_1$ as well as all points on the boundary of $D_1$, satisfying $|f(z)| \leq M$ for every point on the boundary of $D_1$. Suppose furthermore that for some $A, B > 0$ and $\alpha < 1$

$$|f(z)| \leq A \exp(Bz^\alpha) \quad \text{for } z \in D_1$$

then in fact $|f(z)| \leq M$ for all $z \in D_1$.

BONUS Ex. [25pts]: prove this theorem. Hint: fix $\alpha < \beta < 1$ and $w \in D_1$. We want to show $|f(w)| \leq M$. To do this, take $\epsilon > 0$ to be arbitrary, and apply the maximum modulus theorem to $g(z) = f(z) \exp(-\epsilon z^\beta)$ for sufficiently large semicircles (how large will depend on $\epsilon$).