Homework assignment #5

Math 317

Due Friday, March 11

Solve the following problems from the textbook:

1. P. 94 ex. 6, P. 96 ex. 2

2. P. 116 ex. 7. Show in addition that $P_n$ is a polynomial of degree $n$ in $x$.

3. P. 121 ex. 5, P. 130 ex. 10, P. 134 ex. 6, 7 (in 7(a) prove the two stated inequalities), P. 142 ex. 3

Additional questions:

1. Let $C$ be the closed positively oriented contour along $|z| = 1$. Prove that for any function $g(z)$,
   \[
   \int_C g(z)dz = \int_C \overline{g(z)}z^{-2}dz.
   \]

2. Let $C$ be a contour in $\mathbb{C} \setminus \{0\}$, given by the parametrization $z(s)$, $0 \leq s \leq T$. For every $0 \leq t \leq T$ consider the contour $C_t$ given by $z(s)$, $0 \leq s \leq t$. We define $I(t) = \int_{C_t} \frac{dz}{z}$.
   
   (a) by using P. 121, ex. 5 and the fundamentals theorem of the calculus, show that the function $g(t) = \exp(I(t))/z(t)$ satisfies $g'(t) = 0$, i.e. is a constant.
   
   (b) conclude that for any contour $C$ between a point $z_0$ and a point $z_1 \in \mathbb{C}$ that avoids 0
   \[
   \exp \int_C \frac{dz}{z} = \frac{z_1}{z_0}.
   \]
(c) deduce that if $C$ is a closed contour avoiding 0,
\[
\int_C \frac{dz}{z} = 2\pi i n
\]  
for some integer $n$. This integer is called the index or winding number of $C$ around zero.

(d) (BONUS) try proving (0.1) using antiderivatives. You may need to restrict the type of contours you are considering.