1. Consider the polynomial $p(z) = z^5 - 13z^2 + 1$

(a) find a $R > 0$ so that all roots of $p(z)$ sasisfy $|z| < R$:

$$|p(z)| \geq |z^5| - |13z^2 + 1| \geq |z|^5 - 13|z|^2 - 1$$

if say $|z| > 100$ this is > 0, hence $p(z)$ has no zeros for $|z| > 100$

(b). $p(z) = p_1(z) + p_2(z)$

where
$$p_1(z) = -13z^2$$
$$p_2(t) = z^5 + 1$$

on $|z| = 1$ $\quad |p_1(z)| = 13 > 2 > |p_2(z)|$

so by Rouches thm

# of sols of $p(z) = 0$ in $|z| < 1$

is = to

# of sols of $-13z^2 = 0$ in $|z| < 1$

(counted with multiplicity) - which is 2.

2. hint was not perfect.

should use a different brance of log:

$\mathcal{L}og\ z = \lg r + i\theta$ for $z = re^{i\theta}$

$0 < \theta < 2\pi$

this branch is defined in ~~nezz~~ $\mathbb{C} \setminus \{x : x \geq 0\}$.

by our assumption $\mathrm{Re}\, f(z) < -2005$

$f(z)$ is in domain of def of $\mathcal{L}og$

so $\mathcal{L}og\, f(z)$ is analytic at every z on C & its derivative

is given by chain rule

$$\left[\mathcal{L}og\, f(z)\right]' = \frac{f'(z)}{f(z)}.$$

So $\frac{f'(z)}{f(z)}$ has antiderivative ~~where~~ which

is analytic at every $z \in C \Rightarrow$ hence $\int \frac{f'}{f} = 0$.

note: Cauchy's thm is <u>not</u> relevant

as we know <u>nothing</u> about f inside C.

3 consider $f(z) = \frac{e^{-z^2} \sin(z)}{1+z^2}$

Nominator & denominator are entire (products & compositions of entire functions)

so ~~[illegible]~~ f is analytic except where denominator is 0, i.e. $\pm i$.

taylor series of $f(z)$ converges up until 1'st singularity:

(a) around $z_0=0$, closest singularities at $\pm i$ are 1 away, so taylor series converges for $|z|<1$.

(b) around $z_0=1$, again both singularities $\pm i$ are closest, this time $|z_0-i| = |z_0+i| = \sqrt{2}$, so taylor series converges for $|z-1|<\sqrt{2}$.

4. (a) $f(z)$ is analytic except at zeros of $\sin z$, i.e. $0, \pm\pi, \pm 2\pi, \ldots$.

By Cauchy's thm (in variant form) $f(z)$ has antiderivative on every singly connected domain where it is analytic, as claimed

(b) $\{0 < |z| < 1\}$ is not singly connected.

furthermore, $f(z)$ does <u>not</u> have an antiderivative on this domain as

$$\int_C \frac{1}{\sin z} = 2\pi i \operatorname*{res}_{z=0} \frac{1}{\sin z} = 2\pi i \left. \frac{1}{\cos z} \right|_{z=0} = 2\pi i \neq 0.$$

when C is contour along $|z| = \frac{1}{2}$ (pos. oriented)

note we have used the formula

$$\operatorname*{Res}_{z=z_0} \frac{f(z)}{g(z)} = \frac{f(z_0)}{g'(z_0)}$$ if $g(z)$ has single zero at z_0.

5. if $f(z)$ is entire, so is

$f(z) - \sin(z)$.

if $|f(z) - \sin(z)| < 2005$ it is constant by Liouville's thm.

So only solutions are

$f(z) = \sin z + c$ $\quad |c| < 2005$.

$$f(z) = \tan z + e^z = \frac{\sin z}{\cos z} + e^z$$

e^z is entire, $\frac{\sin z}{\cos z}$ is analytic except when denominator is zero, i.e. when $z = (2n+1)\frac{\pi}{2}$ $n \in \mathbb{Z}$.

at these pts $\cos z$ has a simple zero and $\sin z$ is not zero, so at these pts $\tan z$, hence $f(z)$ have simple pole.

$$\operatorname{Res}_{z=(2n+1)\frac{\pi}{2}} f(z) = \operatorname{Res}_{z=(2n+1)\pi/2} \frac{\sin z}{\cos z} =$$

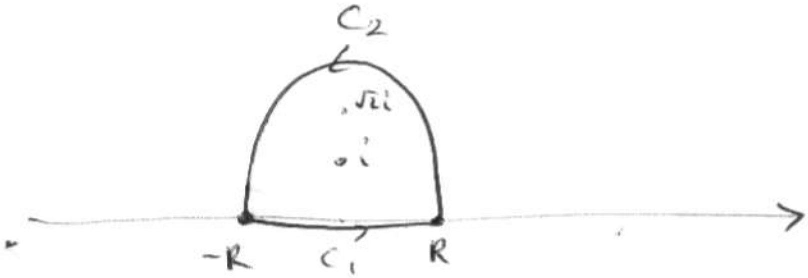
$$= \left.\frac{(\sin z)}{(\cos z)'}\right|_{z=(2n+1)\frac{\pi}{2}} = -1.$$

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+2)}$$

hot integrand is even, so enough to find P.V. $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+2)}$

$\frac{1}{(z^2+1)(z^2+2)}$ analytic except at $\pm i$ $\pm\sqrt{2}i$

where it has simple pole. by Cauchy's residue thm applied to $C = C_1 + C_2$



$R > \sqrt{2}$

$$\int_{C_1} \frac{dz}{(z^2+1)(z^2+2)} + \int_{C_2} \frac{dz}{(z^2+1)(z^2+2)} = 2\pi i \left[\operatorname*{Res}_{z=i} \frac{1}{(z^2+1)(z^2+2)} + \operatorname*{Res}_{z=\sqrt{2}i} \text{ -- '' -- } \right]$$

$$\int_{C_1} \frac{dz}{(z^2+1)(z^2+2)} = \int_{-R}^{R} \frac{dx}{(x^2+1)(x^2+2)}$$

$$\int_{C_2} \text{ -- '' -- } \leq \text{length } C_2 \times \max_{\text{on } C_2} \frac{1}{(z^2+1)(z^2+2)} \leq \pi R \cdot \frac{1}{(R^2-1)(R^2-2)} \to 0.$$

7 (cont)

$$\operatorname*{Res}_{z=i} \frac{1}{(z^2+1)(z^2+2)} = \operatorname*{Res}_{z=i} \frac{\frac{1}{z^2+2}}{z^2+1} = \left.\frac{\frac{1}{z^2+2}}{(z^2+1)'}\right|_{z=i}$$

both nominator & denominator analytic at $z=i$, denominator has simple zero there

$$\frac{\frac{1}{i^2+2}}{2i} = \frac{1}{2i} = -\frac{i}{2}.$$

$$\operatorname*{Res}_{z=\sqrt{2}i} \frac{1}{(z^2+1)(z^2+2)} = \operatorname*{Res}_{z=\sqrt{2}i} \frac{\frac{1}{z^2+1}}{z^2+2} = \frac{\frac{1}{-2+1}}{2\cdot\sqrt{2}i} = +\frac{i}{2\sqrt{2}}$$

so $\displaystyle\int_{-R}^{R} \frac{dx}{(x^2+1)(x^2+2)} - 2\pi i\left[\frac{i}{2\sqrt{2}} - \frac{i}{2}\right] \longrightarrow 0$ as $R\to\infty$

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+2)} = 2\pi i\left[\frac{i}{2\sqrt{2}} - \frac{i}{2}\right] = 2\pi\left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right)$$

thankfully, this is >0 as it should!

8 Compute

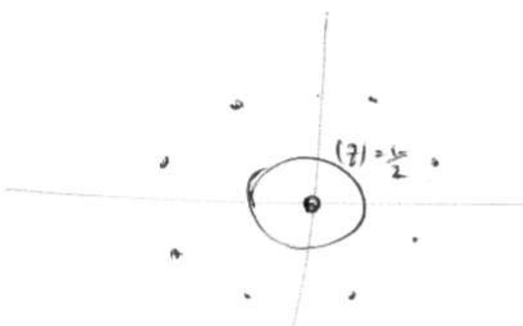
$$\int_{\Gamma} \frac{\mathrm{Log}\,(z^2+1)}{z^8+1}\,dz$$

Γ positively oriented circle $|z| = 1/2$.

$\mathrm{re}\,(z^2+1) \geq 1 - |z|^2 \geq \frac{3}{4}$ for $|z| \leq 1/2$

so nominator is analytic on & inside C.

denominator has 8 zeros at $\{e^{\frac{\pi i}{8}(1+2n)}; n = 0, 1, \ldots, 7\}$



in particular all zeros have $|z| = 1$ so are outside C.

hence integrand is analytic on & inside C, hence by Cauchy the integral $= 0$.