Suppose there are a total of $N$ primes, $2 = p_1 < \cdots < p_N$, and pick $K$ such that $2^K > (K + 1)^N$. Consider the mapping $f: \{1, \ldots, 2^K\} \rightarrow \{0, \ldots, K\}^N$ defined by $f(x) := (k_1, \ldots, k_N)$, where $x = p_1^{k_1} \cdots p_N^{k_N}$ is the prime factorization of $x$. Here, the fact that $f(x) \in \{0, \ldots, K\}^N$ follows from

$$K \geq \log_2 x = \sum_{n=1}^{N} k_n \log_2 p_n \geq \sum_{n=1}^{N} k_n \geq \max(k_1, \ldots, k_N).$$

Then $f$ is injective by the fundamental theorem of arithmetic, which contradicts the pigeonhole principle.

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