AN EXCURSION INTO ÉTALE COHOMOLOGY

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ABSTRACT. In this expository article, we review the basic constructions and properties of the Étale cohomology groups of general schemes, including their relation to the Čech cohomology groups, and a brief discussion of some of their applications.

I'd love to go on an excursion - why not?...

Into the mountains, of course, where else? - Franz Kafka

1. INTRODUCTION

The original motivation of Étale cohomology was to realize the program set forth by Weil of formulating a topological theory on a variety over a field k which would play a similar role as the singular homology on a general topological space. The problem with Zariski cohomology is that it is often too difficult to compute, and does not resemble singular cohomology.

Grothendieck's solution was to consider, instead of the category of open sets of the space, the category of "étale mappings" to the space. These maps, in a sense, define open sets of finite coverings of the original space, which can be used to define a new cohomological theory. Perhaps most importantly, étale cohomology is superior to Zariski topology in its easy computability through Čech cohomology. Futhermore, unlike singular cohomology, Étale cohomology enables one to obtain results in positive characteristics.

In this note, we undertake an excursion into the basic definitions and constructions behind general Étale groups, as well as a practical means of calculating them through Čech cohomology. We now begin our investigation with the appropriate definitions to make the above construction precise.

2. Definitions

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be schemes. A *morphism of schemes* consists of a pair of maps $(f : X \to Y, f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X)$ which satisfy the obvious compatibility conditions with the restriction maps. By the phrase *f is locally of finite type*, we mean that for any affine neighborhood $V = \text{Spec}B \subset Y$, then there exists an open affine covering $\{U_j = \text{Spec}A_j\}_{j\in J}$ of $f^{-1}(V)$ such that for each j, A_j is a finite B-algebra. If noreover the affine cover $\{U_j\}$ can be chosen to be finite, then f is said to be of *finite type*. A scheme Y along with a morphism $f : Y \to X$ is called a *scheme over* X, and f is called the *structure morphism* of Y.

Definition 1 (Flat morphism)**.** *A ring homomorphism $f : A \to B$ is* flat *if B is flat when viewed as an A-module via f (i.e., the functor $\cdot \otimes_A B$ is exact). A morphism of schemes $f : Y \to X$ is* flat *if for each $y \in Y$, the induced map on stalks $f_y^{\#} : \mathcal{O}_{X,f(y)} \to (f_*\mathcal{O}_Y)_{f(y)} \subseteq O_{Y,y}$ is flat.*

A flat morphsim $f : A \to B$ is called faithfully flat *if it satisfies the following condition: for any nonzero A-module M, $B \otimes_A M \neq 0$ (again we use f to consider B as an A-module). In particular, taking M to be principal in A shows that f is injective.*

Definition 2 (Unramified morphism)**.** *Let $f : Y \to X$ be a morphism of schemes. f is said to be* unramified at *$y \in Y$ if it is locally of finite type and $\mathcal{O}_{Y,y}/m_x\mathcal{O}_{Y,y}$ is a finite separable field extension of the residue field $k(x)$, where here $x = f(y)$ and $m_x = f^{\#}(m_y)$ with m_x being the maimal ideal of $\mathcal{O}_{X,x}$. f is said to be* unramified *if it is unramified at each point $y \in Y$.*

Remark 3. *Note that this implies that in terms of rings,$f : A \to B$ is unramified at $q \in SpecB$ if and only if $p = f^{-1}(q)$ generates the maximal ideal in B_q and $k(q)$ is a finite separable extension of $k(p)$. Thus our definition of* unramified *agrees with the number-theoretic notion.*

Definition 4 (Étale morphism)**.** *Let $f : X \to Y$ be a flat, unramified morphism of schemes. Then f is called* étale*. f being unramified has the geometric interpretation that when f as a flat morphism is viewed as a covering map, then it has no branch points.*

Theorem 5. *Any flat morphism that is locally of finite-type is open.*

Proof. See Milne [5,I.2.12], or Hartshorne [4,III.9.1]. □

The family of étale morphisms $X \to Y$ behaves nicely.

Proposition 6. (1) *Any open immersion is étale.*
(2) *The composition of two étale morphisms is étale.*
(3) *The property of being étale is preserved under base change.*

Proof. Any open immersion is a local isomorphism, which proves (1). (2) immediately follows from that fact that any immersion is unramified (this follows directly from the definition of an immersion). For (3), see Milne [5,I.3.3]. □

Note that by this observation, we have that the class E of all étale morphisms of finite-type satsifies:

- all isomorphisms of schemes are in E
- the composition of any two elements of E is in E
- any base change of any element in E is an element in E

We call an element of E and *E-morphism.*

Note that by Theorem 5 and Proposition 6, any E-morphism is open and any open immersion is an E-morphism. It is in this sense that the E-morphisms will play the part of the open sets in our new topology, the *Grothendieck Topology.*

The general definition of a Grothendieck topology $\mathcal{T}$ on a category $\mathbf{C}$ is defined as follows. By *sieve on an object $c \in \mathbf{C}$*, we mean a subfunctor of $\mathrm{Hom}(-, c)$. Suppose $\mathbf{s}$ is a sieve on an object c. Then for any morphism $f : c' \to c$, we define the *pullback of* $\mathbf{s}$ *along* f by $f^*\mathbf{s}(c'') = \{g : c'' \to c' : f \circ g \in \mathbf{S}(c'')\}$, where the functor $f^*\mathbf{s}$ inherits its action on morphisms from being a subfunctor of $\mathrm{Hom}(-, c')$.

Definition 7 (Grothendieck Topology)**.** *A* Grothendieck topology *on a category* $\mathbf{C}$ *is a rule* $\mathcal{S}$ *which associates to each* $c \in obj(\mathbf{C})$ *a collection of sieves* $\mathbf{S}(c)$ *on* c*, called the* covering sieves of c*, and which is subject to the following axioms:*

(1) *If* $f : c' \to c$ *is a morphism, and* $\mathbf{s} \in \mathbf{S}(c)$ *then* $f^*\mathbf{s}(c') \in \mathbf{S}(c')$.
(2) *Suppose* $\mathbf{s} \in \mathbf{S}(c)$, $\mathbf{s}'$ *is any sieve on* c*, and for each* $c' \in obj(\mathbf{C})$*, and any* $(f : c' \to c) \in \mathbf{S}(c')$*, that* $f^*\mathbf{s}' \in \mathbf{S}(c')$*. Then* $\mathbf{s}' \in \mathbf{S}(c')$.
(3) *For any* $c \in obj(\mathbf{C})$*, we have* $Hom(-, c) \in \mathbf{S}(c)$.

Now fix a full subcategory $\mathbf{G}/X$ of $\mathbf{Sch}/X$ which is closed under fiber products and which satisfies the following condition: for any $g : Y \to X$ in $\mathbf{G}/X$ and any E-morphism $f : U \to Y$, the composition $f \circ g : U \to X$ is in $\mathbf{G}/X$. We define an *E-covering* of $Y \in \mathrm{obj}(\mathbf{G}/X)$ to be a family $\{f_i : U_i \to Y\}_{i \in I}$ of E-morphisms such that $Y = \bigcup_{i \in I} f_i(X)$. The class of all E-coverings $\mathcal{E}$ is called the *E-topology* on $\mathbf{G}/X$. We then call $(\mathbf{G}/X, \mathcal{E})$ the *E-site over* X, and denote it by $(\mathbf{G}/X)_E$, or when $\mathbf{G}/X$ is clear from the context, X_E.

Remark 8. Note that $(\mathbf{G}/X)_E$ satisfies the following properties:

(1) if $\phi : U \to U$ is an isomorphism in $\mathbf{G}/X$, then it is an E-covering;
(2) if $\{\phi_i : U_i \to U\}_{i \in I}$ is an E-covering, and for each i, $\{\psi_{ij} : V_{ij} \to U_i\}_{j \in J}$ is an E-covering, then $\{\psi_{ij} \circ \phi_i : V_{ij} \to U\}_{i \in I, j \in J}$ is an E-covering;
(3) if $\{\phi_i : U_i \to U\}_{i \in I}$ is a covering, then for any morphism $(f : V \to U) \in \mathrm{Mor}(\mathbf{G}/X)$, after applying base-change with respect to f to each ϕ_i, $\{(\phi_i)_{(V)} : U_i \times_U V \to V\}_{i \in I}$ is an E-covering.

Hence $(\mathbf{G}/X)_E$ is indeed a *Grothendieck topology.*

We have, in effect, replaced the open sets of X with the objects of $\mathbf{G}/X$, and the rigid requirement on a scheme X that a covering consist of subsets of X with the more flexible notion of E-coverings. It turns out that we can define the usual notions of presheaf, sheaf, and associated sheaf analagously in the étale situation.

Remark 9. *Since E contains all open immersions, then an open covering $\{U_i\}_{i \in I}$ in the usual Zariski topoology is a covering $\{id_i : U_i \to U_i\}_{i \in I}$ in the E-topology. Moreover, a scheme $\mathcal{F}$ defines, by restriction, a sheaf $\mathcal{F}_U$ on each $U \in \mathbf{G}/X$. So the E-topology is indeed "stronger" than the Zariski topology.*

Definition 10 (Presheaf)**.** *A* presheaf $\mathcal{F}$*, just as in the normal setting, is a contravariant functor from the category of open sets of $(\mathbf{G}/X)^\circ$ to the category of Abelian groups $\mathbf{Ab}$. That is, for any open set $U \in \mathbf{G}/X$, $\mathcal{F}$ associates to U and abelian group $\mathcal{F}(U)$. Often this is also denoted, as in the standard setting, by $\Gamma(U, \mathcal{F})$, and are called the* sections of $\mathcal{F}$ over U*. To any morphism $f : V \to U$ is associated a homomorphism of abelian groups $\mathcal{F}(f) : \mathcal{F}(U) \to \mathcal{F}(V)$. These correspond to the restriction maps in the standard situation, and so are sometimes denoted in the same way by $\rho_{U'U}$ or $(s \mapsto s|_{U'})$. Unlike in the standard setting over topological spaces however, in general there may be many restrictions $U' \to U$ which need not agree.*

Definition 11 (Morphism of presheaves)**.** *A* morphism of presheaves $\phi : \mathcal{F} \to \mathcal{F}'$ on $(\mathbf{G}/X)_E$ *is a morphism of functors. That is, to any $U \in \mathbf{G}/X$, ϕ associates a morphism of abelian groups $\phi(U) : \mathcal{F}(U) \to \mathcal{F}'(U)$ which is compatible with the restriction maps.*

Remark 12. *The presheaves over $(\mathbf{G}/X)_E$ along with the presheaf morphism form a category $\mathbf{P}((\mathbf{G}/X)_E)$, which just as in the standard setting, inherits most of the properties of $\mathbf{Ab}$. For any two presheaves $\mathcal{F}$ and $\mathcal{F}'$, we can define their direct sum in the obvious way $U \mapsto$*

$\mathcal{F}(U) \oplus \mathcal{F}'(U)$ and $f \mapsto \mathcal{F}(f) \oplus \mathcal{F}'(f)$. Similarly, for the kernel and cokernel of any morphism $\phi : \mathcal{F} \to \mathcal{F}'$, we get the presheaves $ker(\phi)(U) = ker(\phi(U))$ and $coker(\phi)(U) = coker(\phi(U))$. One can also define general direct sums, direct products, direct limits and inverse limits analagrously. Hence $\mathbf{P}((\mathbf{G}/X)_E)$ forms an abelian category.

Definition 13 (Continuous morphism)**.** *Let $(\mathbf{G}'/X')_{E'}$ and $(\mathbf{G}/X)_E$ be sites. Then a morphism $\pi : X' \to X$ of schemes defines a* morphism of sites $(\mathbf{G}'/X')_{E'}) \to (\mathbf{G}/X)_E$ *if:*

(1) *for any $Y \in obj(\mathbf{G}/X)$, the base change $Y_{(X')}$ is in $obj(\mathbf{G}'/X')$;*
(2) *for any E-morphism $(U \to Y) \in Mor(\mathbf{G}/X)$, the morphism $U_{(X')} \to Y_{(X')}$, obtained by the base change $\pi : X' \to X$, is an E'-morphism in $Mor(\mathbf{G}'/X')$.*

Note that since the base change of a surjective family of morphisms remains surjective, then π defines a functor

$$\tilde{\pi} : \mathbf{G}/X \to \mathbf{G}'/X'$$

$$\tilde{\pi}(Y) = Y_{(X')}$$

and so colloquially, we refer to such a π as a continuous morphism $\pi : X_E \to X'_{E'}$.

We now translate the notion of a sheaf from the standard setting to the étale situation. The key is to replace the intersection of two open sets with their fibered product. This indeed defines the intersection in the Grothendieck topology.

Definition 14 (Sheaf)**.** *A* sheaf $\mathbf{F}$ *is a presheaf which satisfies the following axioms*

(1) local identity: *if $s \in \mathcal{F}(U)$ and there is a covering $\{U_i \to U\}_{i\in I}$ of U such that $s|_{U_i} = 0$ for all i, then $s = 0$*
(2) gluing axiom: *if $\{U_i \to U\}_{i\in I}$ is a covering, and $\{s_i\}_{i\in I}$, $s_i \in \mathcal{F}(U_i)$ is a family of sections such that $s_i|_{U_i\times_U U_k} = s_j|_{U_i\times_U U_j}$ for all $i, j \in I$, then there exists a section $s \in \mathcal{F}(U)$ with $s|_{U_i} = s_i$ for all $i \in I$.*

As usual, axiom (1) implies that the section s formed by gluing together the sections s_i in axiom (2) is uniquely determined by the s_i.

Definition 15 (Direct image and Inverse image)**.** *Suppose $\pi : X'_{E'} \to X_E$ is continuous. To any presheaf P' on $X'_{E'}$, define $\pi_p(P') = P' \circ \pi'$ on X_E. More precisely, $\pi_p(P')$ is the presheaf on X_E such that $\Gamma(U, \pi_p(P')) = \Gamma(U_{(X')}, P')$. We call the presheaf $\pi_p(P')$ the* direct image *of P'.*

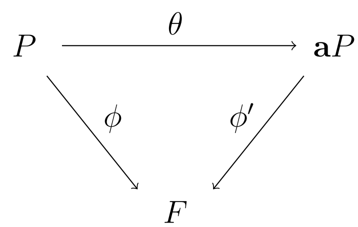
Now it is easy to see that π_p *defines a functor* $\mathbf{P}(X'_{E'}) \to \mathbf{P}(X_E)$, *and we define the* inverse image *functor* $\pi^p : \mathbf{P}(X_E) \to \mathbf{P}(X'_{E'})$ *to be the left adjoint of* π_p*; so* pi^p *is the functor such that*

$$Hom_{\mathbf{P}(X')}(\pi^p P, P') \cong Hom_{\mathbf{P}(X)}(P, \pi_p P')$$

whose existence is guaranteed by a well-known result in category theory (see, for example, Milne, [5,II.2.2]*).*

As in sheaf theory over topological spaces, we have the notion of an *associated sheaf to a presheaf*, whose existence is guaranteed by the following theorem.

Theorem 16. *For any presheaf* P *on* X_E*, there is a sheaf* $\mathbf{a}P$ *on* X_E *and a morphism* $\theta : P \to \mathbf{a}P$ *such that any morphism* ϕ *from* P *into a sheaf* F *factors uniquely as in the following commutative diagram:*



Proof. See Milne [5,II.2.11].

□

Of course it is easy to see that uniqueness is a formal consequence of the universal property.

Definition 17 (Associated sheaf to a presheaf)**.** *We for a presheaf* P*, we call the sheaf* $\mathbf{a}P$ *as in Theorem 16 the* associated sheaf *to* P*.*

Remark 18. *Note that* $P \mapsto \mathbf{a}P$ *is a functor. It is in fact the left adjoint of the inclusion functor* $\mathbf{S}(X_E) \hookrightarrow \mathbf{P}(X_E)$ *(see* Milne, [5,II.2.14]*).*

We are now ready to define étale cohomology.

3. Construction of the Étale cohomology groups

The construction will follow very much the same path as the construction of ordinary sheaf cohomology. Let $\mathbf{C}$ be an abelian category. As usual, we call $I \in \text{obj}(\mathbf{C})$ *injective* if the functor $\mathbf{C} \to \mathbf{Ab}; M \mapsto \text{Hom}_{\mathbf{C}}(M, I)$ is exact. $\mathbf{C}$ is said to *have enough injectives* if for every $M \in \text{obj}(\mathbf{C})$, M can be embedded in an injective object (there is an injective object I and an injective morphism $M \hookrightarrow I$).

If $\mathbf{C}$ has enough injectives, then for any left exact functor $\mathbf{F} : \mathbf{C} \to \mathbf{C}'$ where $\mathbf{C}'$ is another abelian category, there is a sequence, unique up to homotopy, of functors $R^i\mathbf{F} : \mathbf{C} \to \mathbf{C}', i \geq 0$, called the *right derived functors of* $\mathbf{F}$ which has the following properties:

(1) $R^0\mathbf{F} = \mathbf{F}$;
(2) if $I \in \mathrm{obj}(\mathbf{C})$ is injective, then $R^i\mathbf{F}(I) = 0$ for each $i \geq 0$;
(3) for any short exact sequence $0 \to M' \to M \to M'' \to 0$ of objects in $\mathbf{C}$, there are morphisms $\partial^i : R^i\mathbf{F}(M'') \to R^{i+1}\mathbf{F}(M'), i \geq 0$ (called *connecting homomorphisms*) such that the sequence

$$... \to R^i\mathbf{F}(M) \to R^i\mathbf{F}(M'') \xrightarrow{\partial^i} R^{i+1}\mathbf{F}(M') \to R^{i+1}\mathbf{F}(M) \to ...$$

is exact;
(4) the association above in (c) of the short exact sequence in $\mathbf{C}$ to the long exact sequence in $\mathbf{C}'$ is functorial.

Now henceforth fix a subcategory $\mathbf{G}/X \subseteq \mathbf{Sch}/X$ as above, denote the E-site over X with respect to $\mathbf{G}/X$ by X_E. Denote the category of sheaves on X_E by $\mathbf{S}(X_E)$.

Proposition 19. *The category* $\mathbf{S}(X_E)$ *has enough injectives.*

This will follow from the next lemma.

Lemma 20. (1) *A product of injectives is injective.*
(2) *If a functor* $\mathbf{F} : \mathbf{A} \to \mathbf{B}$ *has an exact left adjoint* $\mathbf{G} : \mathbf{B} \to \mathbf{A}$*, then* $\mathbf{F}$ *preserves injectives.*

Proof. (1) is clear. (2) Suppose $I \in \mathrm{obj}(\mathbf{A})$ is injective. Since $\mathbf{G}$ is the left adjoint of $\mathbf{F}$, then the functor $M \mapsto \mathrm{Hom}_{\mathbf{B}}(M, \mathbf{F}(I))$ is isomorphic to $M \mapsto \mathrm{Hom}_{\mathbf{A}}(\mathbf{G}(M), I)$, and the latter is exact since it is a composition $M \mapsto \mathbf{G}(M)$, which is exact by assumption, and $M \mapsto \mathrm{Hom}_{\mathbf{A}}(M, I)$, which is exact since I is injective in $\mathbf{A}$. Hence the former functor $M \mapsto \mathrm{Hom}_{\mathbf{B}}(M, \mathbf{F}(I))$ is exact, and so $\mathbf{F}(I)$ is injective in $\mathbf{B}$. □

Now we prove Proposition 19. Let $\phi_x : \bar{x} \to X$ be a geometric point (k-rational point where k is separably closed) of X. Then $\mathbf{S}(\bar{x}_E)$ is isomorphic to $\mathbf{Ab}$, and so has enough injectives. Now choose any $M \in \mathrm{obj}(\mathbf{S}(X_E))$, and now for each $x \in X$, choose an embedding $\phi_x^* M \to M'_x$ into an injective object of $\mathbf{S}(\bar{x}_E)$. Then define $M^* = \prod_{x \in X} \phi_{x*}\phi_x^* M$ and $M^{**} = \prod_{x \in X} \phi_{x*} M'_x$. The canonical maps $M \to M^*$ and $M \to M^{**}$ are monomorphisms, and by Lemma 20, M^{**} is injective. □

Definition 21 (Étale cohomology groups)**.**

(1) As in the ordinary situation over topological spaces, the global sections functor

$$\Gamma(X, \cdot) : \mathbf{S}(X_E) \to \mathbf{Ab}$$

$$\Gamma(X, F) = F(X)$$

is left exact, and we denote its right derived functors by

$$R^i\Gamma(X, \cdot) = H^i(X, \cdot) = H^i(X_E, \cdot)$$

and the group $H^i(X_E, F)$ is called the *ith cohomology group of X_E with values in F.*

(2) For any $U \in \text{obj}(X_E)$, we can define analagously the right derived functors of the functor $\mathbf{S}(X_E) \to \mathbf{Ab}, F \mapsto \Gamma(U, F) = F(U)$, and they are denoted by $H^i(U, F)$. It is not hard to see that $H^i(U, F|_U) = H^i(U, F)$.

(3) The inclusion functor $\mathbf{S}(X_E) \to \mathbf{P}(X_E)$ of sheaves on X_E into presheaves on X_E is exact, and its right derived functors are denoted by $\underline{H}^i(X_e, F)$ or $\underline{H}^i(F)$.

(4) For any fixed sheaf F_0 on X_E, the functor $F \mapsto \text{Hom}_{\mathbf{S}}(F_0, F)$ is left exact and its right dervied functors are denoted by $R^i\text{Hom}_{\mathbf{S}}(F_0, \cdot) = \text{Ext}^i_{\mathbf{S}}(F_0, \cdot)$.

(5) For any sheaves F and F' on X_E, we denote the sheaf (one easily checks the sheaf axioms) $U \mapsto \text{Hom}(F|_U, F'|_U)$ by $\underline{\text{Hom}}(F, F')$. If one fixes a sheaf F_0, then the functor $\mathbf{S}(X_E) \to \mathbf{S}(X_E), F \mapsto \underline{\text{Hom}}(F_0, F)$ is left exact, and its right derived functors are denoted by $\underline{\text{Ext}}^i(F_0, F)$.

(6) For any continuous morphism (see Definition 11) $\pi : X'_{E'} \to X_E$, we can define the right derived functors $R^i\pi_*$ of the functor $\pi_* : \mathbf{S}(X'_{E'}) \to \mathbf{S}(X_E)$. The sheaves $R^i\pi_*F$ are called the *higher direct images of F.*

We now investigate an analagously defined version of Čech cohomology.

4. Čech Cohomology

Let $\mathcal{U} = \{\phi_i : U_i \to X\}_{i \in I}$ be a covering of X in the E-topology. Then for any $(p+1)$-tuple $(i_0, ..., i_p)$ where each $i_j \in I$, and put $U_{i_0 \ldots i_p} = U_{i_0} \times_X ... \times_X U_{i_p}$. Now let P be any presheaf on X_E. The canonivcal projection map

$$U_{i_0 \ldots i_p} \to U_{i_0 \ldots \hat{i_j} \ldots i_p} = U_{i_0} \times_X ... \times_X U_{i_{j-1}} \times_X U_{i_{j+1}} \times_X ... \times_X U_{i_p}$$

which induces a "restriction morphism"

$$P(U_{i_0\ldots\hat{i_j}\ldots i_p}) \to P(U_{i_0\ldots i_p})$$

which we denote (unambiguously) by res_j. Now we define a complex

$$C^{\cdot}(\mathcal{U}, P) = (C^p(\mathcal{U}, P), d^p)_p$$

in the following way:

$$C^p(\mathcal{U}, P) = \prod_{I^{p+1}} P(U_{i_0\ldots i_p}); \quad d^p : C^p(\mathcal{U}, P) \to C^{p+1}(\mathcal{U}, P)$$

where d^p is the homomorphism defined as follows: if $s = (s_{i_0\ldots i_p}) \in C^p(\mathcal{U}, P)$, then

$$(d^p s)_{i_0\ldots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \mathrm{res}_j(s_{i_0\ldots\hat{i_j}\ldots i_{p+1}})$$

. The standard pairing argument shows that $d^{p+1}d^p = 0$, and so the above group defines a complex. The cohomology groups of $(C^p(\mathcal{U}, P), d^p)$ are called the *Čech cohomology groups*, $\check{H}^p(\mathcal{U}, P)$ of P *with respect to the covering* $\mathcal{U}$ of X.

We immediately note that there is a canonical map

$$\check{H}^p(\mathcal{U}, P) = \ker(\prod P(U_i) \to \prod P(U_{ij}))$$

and hence there is a canonical map $P(X) \to \check{H}^0(\mathcal{U}, P)$ which in particular is an isomorphism if P is a sheaf.

A second covering $\mathcal{V} = \{\psi_j : V_j \to X\}_{j\in J}$ is called a *refinement* of $\mathcal{U}$ if there is a map $\tau : J \to I$ such that for each j, ψ_j factors through $\phi_{\tau j}$; i.e., $\psi_j = \phi_{\tau j}\lambda_j$ for some $\lambda_j : V_j \to U_{\tau j}$. This τ along with the family $\{\lambda_j\}_{j\in J}$ induces a map $\tau^p : C^p(\mathcal{U}, P) \to C^p(\mathcal{V}, P)$ in the following way: if $s = (s_{i_0\ldots i_p}) \in C^p(\mathcal{U}, P)$, then

$$(\tau^p s)_{j_0\ldots j_p} = \mathrm{res}_{\lambda_{j_0}\times\ldots\times\lambda_{j_p}}(s_{\tau j_0\ldots\tau j_p}).$$

It is easy to see that the maps τ^p commute with the differential d and hence they induce maps on cohomology,

$$\rho(\mathcal{V}, \mathcal{U}, \tau) : \check{H}^p(\mathcal{U}, P) \to \check{H}^p(\mathcal{V}, P).$$

Of course we need the standard housekeeping lemma.

Lemma 22. *The map* $\rho(\mathcal{V}, \mathcal{U}, \tau)$ *does not depend on* τ *or* $\{\lambda_j\}_{j\in J}$.

Proof. Choose another $\tau' : J \to I$ and another family $\{\lambda'_j\}_{j\in J}$ such that $\psi_j = \phi_{\tau' j}\lambda'_j$ for all $j \in J$. Then for $s \in C^p(\mathcal{U}, P)$, define a morphism h by

$$(h^p s)_{j_0 \dots j_{p-1}} = \sum_r (-1)^r \mathrm{res}_{\lambda_{j_0} \times \dots \times (\lambda_{j_r}\lambda'_{j_r}) \times \dots \times \lambda'_{j_{p-1}}}(s_{\tau j_0 \dots \tau j_r \tau' j_r \dots \tau' j_{p-1}}).$$

Then h^p is a homomorphism $C^p(\mathcal{U}, P) \to C^{p-1}(\mathcal{V}, P)$ which satisfies the chain homotopy condition

$$d^{p-1}h^p + h^{p+1}d^p = \tau'_p - \tau_p.$$

Then passing to cohomology, this becomes

$$\rho(\mathcal{V}, \mathcal{U}, \tau') - \rho(\mathcal{V}, \mathcal{U}, \tau) = 0.$$

□

Thus, if $\mathcal{V}$ is a refinement of $\mathcal{U}$, we get a homomorphism $\rho(\mathcal{V}, \mathcal{U}) : \check{H}^p(\mathcal{U}, P) \to \check{H}^p(\mathcal{V}, P)$ which depends only on $\mathcal{V}$ and $\mathcal{U}$.Hence, if $\mathcal{U}, \mathcal{V}$, and $\mathcal{W}$ are three coverings of X such that $\mathcal{W}$ is a refinement of $\mathcal{V}$ and $\mathcal{V}$ is a refinement of $\mathcal{U}$, then $\rho(\mathcal{W}, \mathcal{U}) = \rho(\mathcal{W}, \mathcal{V})\rho(\mathcal{V}, \mathcal{U})$. So we can define the *Čech cohomology groups* of P over X to be $\check{H}^p(X_E, P) = \varinjlim \check{H}^p(\mathcal{U}, P)$ where the limit is taken over all coverings $\mathcal{U}$ of X.

Definition 23 (Čech cohomology groups)**.** *We define the* Čech cohomology groups *of P over X to be $\check{H}^p(X_E, P) = \varinjlim \check{H}^p(\mathcal{U}, P)$ where the limit is taken over all coverings $\mathcal{U}$ of X.*

Remark 24.

(1) *Note that the category over which we take the above limit is not cofiltered. We can resolve this inconvenience as follows. We mod out the set of coverings of X by the equivalence relation $\mathcal{U} \sim \mathcal{V}$ if and only if $\mathcal{U}$ and $\mathcal{V}$ are each a refinement of the other, and call the set of equivalence classes I_X. Note that refinement induces a partial ordering on I_x which is filtered because any two coverings $\mathcal{U} = \{U_i\}, \mathcal{V} = \{V_j\}$ have a common refinement $\mathcal{U} \times \mathcal{V} = \{U_i \times V_j\}$. By Lemma 22, the functor $\mathcal{U} \mapsto \check{H}^p(\mathcal{U}, P)$ factors through I_x, and the limit can thus be taken over I_X.*

(2) *If $U \to X$ is in obj(X_E) and P is a presheaf on X_E, then we may analagously define cohomology groups $\check{H}^p(\mathcal{U}/U, P)$ and*

$$\check{H}^p(U, P) = \varinjlim \check{H}^p(\mathcal{U}/U, P)$$

where the limit is taken over all coverings $\mathcal{U}$ of U. It is clear that since $\check{H}^p(U, P)$ is defined intrinsically in terms of P, then

$\check{H}^p(U,P)$ is the same thing as $\check{H}^p(U,P|_U)$. Moreover, the mapping $U \mapsto \check{H}^p(U,P)$ extends to a functor $(\mathbf{G}/X) \to \mathbf{Ab}$, i.e., a presheaf on $(\mathbf{G}/X)_E = X_E$, which we denote by $\underline{\check{H}}^p(X_E,P)$ or $\underline{\check{H}}^p(P)$.

Note that for any short exact sequence of *presheaves* $0 \to P' \to P \to P'' \to 0$ and any covering $\mathcal{U}$ of X, we get another short exact sequence

$$0 \to C^p(\mathcal{U},P') \to C^p(\mathcal{U},P) \to C^p(\mathcal{U},P'') \to 0.$$

This sequence is exact because it is a direct product of exact sequences of abelian groups. Hence, we in turn get a short exact sequence of complexes

$$0 \to C^\cdot(\mathcal{U},P') \to C^\cdot(\mathcal{U},P) \to C^\cdot(\mathcal{U},P'') \to 0$$

and as purely formal consequence, we get an associated long exact sequence of cohomology

$$0 \to \check{H}^0(\mathcal{U},P') \to ... \to \check{H}^p(\mathcal{U},P) \to \check{H}^p(\mathcal{U},P'') \to \check{H}^{p+1}(\mathcal{U},P') \to ...$$

Then since exactness is preserved upon passing to the direct limit over all coverings of X (or better yet over I_X, see Remark 24), then we get a long exact sequence of Čech cohomology groups

$$0 \to \check{H}^0(U,P') \to ... \to \check{H}^p(U,P) \to \check{H}^p(U,P'') \to \check{H}^{p+1}(U,P') \to ...$$

So to any short exact sequence of *presheaves* we associate a long exact sequence of Čech cohomology groups. It is clear by the above that this association is functorial. However, if we had started with a short exact sequence of *sheaves*, that is a sequence exact only in $\mathbf{S}(X_E)$ and not $\mathbf{P}(X_E)$, then the maps $C^p(\mathcal{U},P) \to C^p(\mathcal{U},P'')$ need not be surjective, and so we do not get an associated short exact sequence of Čech complexes. We will denote this

As in the situation over topological spaces, Čech cohomology is useful for practical purposes; namely, it provides a means of calculating Étale cohomology in certain situations.

For derived functor cohomology, the theorem we will prove is

Theorem 25 ((Partial) Equivalence of Čech cohomology and Derived functor cohomology)**.** *Čech cohomology and derived functor cohomology agree, that is, $\check{H}^\cdot(X,-) \cong H^\cdot(X,-)$ if and only if to every short exact sequence of sheaves there is functorially associated a long exact sequence of Čech cohomology groups.*

This theorem will be deduced immediately from the following lemma.

Lemma 26. *If $P \in obj(X_E)$ is injective, then $\check{H}^p(\mathcal{U}/U,P) = 0$, for all $p > 0$.*

Proof. (Following Milne, [5,II.2.4]) This is equivalent to the cochain complex

$$\prod P(U_i) \xrightarrow{d^1} \prod P(U_{i_0 i_1}) \xrightarrow{d^2} \prod P(U_{i_0 i_1 i_2}) \to ...$$

being exact. Now for any $W \to U$ in $\mathrm{obj}(X_E)$, the constant presheaf on W, $\mathbb{Z}_W$ has the property that $\mathrm{Hom}(\mathbb{Z}_W, P) = P(W)$ for any presheaf P on X, and

$$\mathbb{Z}_W(V) = \bigoplus_{\mathrm{Hom}_X(V,W)} \mathbb{Z}.$$

Hence the above complex can be rewritten as

$$\prod \mathrm{Hom}(\mathbb{Z}_{U_{i_0}}, P) \to \prod \mathrm{Hom}(\mathbb{Z}_{U_{i_0 i_1}}, P) \to ...$$

and again as

$$\mathrm{Hom}(\bigoplus \mathbb{Z}_{U_{i_0}}, P) \to \mathrm{Hom}(\bigoplus \mathbb{Z}_{U_{i_0 i_1}}, P) \to ...$$

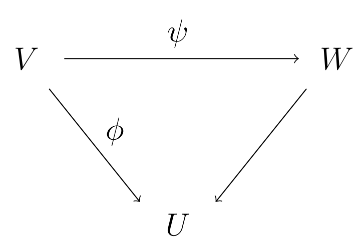
Tehn since P is injective, it is enough to show

$$\bigoplus \mathbb{Z}_{U_{i_0}} \leftarrow \bigoplus \mathbb{Z}_{U_{i_0 i_1}} \leftarrow \bigoplus \mathbb{Z}_{U_{i_0 i_1 i_2}} \leftarrow ...$$

is exact in the category of presheaves $\mathbf{P}(X)$. That is, for any $V \in \mathrm{obj}(X_E)$,

$$\bigoplus \mathbb{Z}_{U_{i_0}}(V) \leftarrow \bigoplus \mathbb{Z}_{U_{i_0 i_1}}(V) \leftarrow \bigoplus \mathbb{Z}_{U_{i_0 i_1 i_2}}(V) \leftarrow ...$$

Now for any U-scheme W and $\phi \in \mathrm{Hom}_X(V, U)$, then denote by $\mathrm{Hom}_\phi(V, W)$ the set of morphisms $\psi : V \to W$ such that



commutes. Then

$$\mathrm{Hom}_X(V, U_{i_0 i_1 ...}) = \coprod_{\mathrm{Hom}_X(V,U)} \mathrm{Hom}_\phi(V, U_{i_0 i_1 ...})$$

$$= \coprod_{\mathrm{Hom}_X(V,U)} (\mathrm{Hom}_\phi(V, U_{i_0}) \times \mathrm{Hom}_\phi(V, U_{i_1}) \times ...)$$

Now put $S(\phi) = \coprod_i \mathrm{Hom}_\phi(V, U_i)$, and

$$\bigcup_{i_0, ..., i_p} \mathrm{Hom}_X(V, U_{i_0 ... i_p}) = \bigcup_{\mathrm{Hom}_X(V,U)} (S(\phi) \times ... \times S(\phi))$$

(i.e., $p+1$ copies of $S(\phi)$). Then $\bigoplus \mathbb{Z}_{U_{i_0\dots i_p}}(V)$ is the free abelian group on $\bigcup_{\mathrm{Hom}_X(V,U)}(S(\phi)\times\dots\times S(\phi))$, and so the complex can be rewritten as

$$\bigoplus_{\mathrm{Hom}_X(V,U)}\left(\bigoplus_{S(\phi)}\mathbb{Z}\leftarrow\bigoplus_{S(\phi)\times S(\phi)}\mathbb{Z}\leftarrow\bigoplus_{S(\phi)\times S(\phi)\times S(\phi)}\mathbb{Z}\leftarrow\dots\right)$$

The complex inside the parentheses is the standard complex associated with $\bigoplus_{S(\phi)}\mathbb{Z}$, which is exact. One can check that $k^p(s)_{i_0\dots i_{p-1}} = s_{1 i_0\dots i_{p-1}}$, where 1 is a fixed element of $S(\phi)$ and because an injective sheaf is injective as a presheaf

$$s = (s_{i_0\dots i_p}) \in \bigoplus_{S(\phi)^{p+1}}\mathbb{Z}$$

□

Now we can prove Theorem 25. For sufficiency, we note that $H^0(X,F) = \check{H}^0(X,F)$, and the assertion follows from Lemma 26 and the fact that the associated sheaf functor **a** is right adjoint to the inclusion functor $\mathbf{S}(X_E)\hookrightarrow\mathbf{P}(X_E)$ (see Remark 18). Necessity is clear. □

Note that if, for example, for every surjection $F\to F''$ of sheaves, the map $\varinjlim(\prod F(U_{i_0\dots i_p}))\to\prod F''(U_{i_0\dots i_p})$ is also surjective, where the limit is taken over all coverings of X, then the hypotheses of the theorem are satisfied, and so we have equivalence of the two cohomology theories.

The étale case proves to be nicer, as the following important theorem illustrates.

Theorem 27 (Equivalence of Čech cohomology and Étale cohomology)**.** *Suppose X is a quasi-compact (every open cover of X has a finite subcover) scheme such that every finite subset is contained in an affine open set (as is the case when, for example, X is quasi-projective over an affine scheme). Let F be a sheaf on X_E. Then there is a canonical isomoprhism $\check{H}(X_E,F)\xrightarrow{\cong}H^p(X_E,F)$ for each p.*

For the proof, we follow Milne [5,III.2.17]. We will cite the following two lemmas; the first comes purely from commutative algebra.

Recall that a local ring R with maximal ideal m is called *Henselian* if for any monic polynomial $P\in R[x]$, then the factorization of its image $\overline{P}\in(R/m)[x]$ into a product of coprime monic polynomials can be lifted up to a factorization in $R[x]$. It is a fact that the completion $\hat{R}$ of any local ring R is Henselian, and so R is the a subring of a Henselian ring. Then we call the smallest Henselian ring which contains R the *Henselization* of R.

Lemma 28. *Suppose A is a ring, $\mathbf{p_1}, ...\mathbf{p_r}$ are prime ideal of A, and $A^{sh}_{\mathbf{p_1}}, ...A^{sh}_{\mathbf{p_r}}$ are strict Henselizations of the local rings $A_{\mathbf{p_1}}, ..., A_{\mathbf{p_r}}$. Then $B = A^{sh}_{\mathbf{p_1}} \otimes ... \otimes A^{sh}_{\mathbf{p_r}}$ has the property that any faithfully flat étale map $B \to C$ has a section $C \to B$.*

Proof. See Artin [1,3.4(iii)]. □

Now for any morphism $U \to X$, let $U^0 = X$ and $U^n = U \times_X ... \times_X U$ be the n-fold fibered product. Note that if $(\overline{x}) = (\overline{x}_1, ..., \overline{x}_r)$ is an r-tuple of geometric points, then the base change $X_{(\overline{x})} = X$ if $r = 0$, and

$$X_{(\overline{x})} = \text{spec}(\mathcal{O}_{\overline{x}_1}) \times_X ... \times_X \text{spec}(\mathcal{O}_{\overline{x}_r})$$

if $r \neq 0$.

Lemma 29. *Let X be as in the statement of Theorem 27, and let $U \to X$ be étale of finite-type, and let $(\overline{x}) = (\overline{x}_1, ..., \overline{x}_r)$ be a collection of geometric points of X. Also let $\tilde{U} \to U^n \times X_{(\overline{x})}$ be an étale surjective morphism. Then if $n > 0$ or $r > 0$, there is an étale surjective morphism $U' \to U$ such that the induced X-morphism $U'^n \times X_{(\overline{x})} \to U^n \times X_{(\overline{x})}$ factors through $\tilde{U}$.*

Proof. The proof relies on induction on n. For the base case $n = 0$, the assertion follows from Lemma 28. The other special case $n = 1, r = 0$ is obvious. The induction step involves in extending an étale map of finite type $V \to U$ which is not surjective but whose induced map on $V^n \times X_{(\overline{x})} \to U^n \times X_{(\overline{x})}$ factors through W, and extending it in finitely many steps to a surjective map. See Milne, [5,III.2.19]. □

Now we can prove Theorem 27. We will use Theorem 25. Suppose $F \to F''$ is a surjective map of sheaves, and suppose V is in E/X. Then since X and V are quasi-compact, all coverings $U \to V$ which consts of a single morphism are cofinal in the set of coverings of V. Now suppose $s'' \in F''(U^n)$. Then there is an étale covering $\tilde{U} \to U^n$ and $s \in F(\tilde{U})$ such that $s \mapsto s'''|_{\tilde{U}}$ Then by Lemma 29, there is a covering $U' \to U$ such that $U'^n \to U^n$ factors through $\tilde{U}$, and so $s|_{U'^n} \mapsto s''|_{U'^n}$, and we are done. □

5. A Brief Discussion of Applications

In this section, we give a brief description of some applications of the computability of Étale cohomology through Čech cohomology.

Over an algebraically closed field, we denote by $\mathbb{Z}/k(1) = \mu_k$ the (inveritible) étale sheaf of kth roots of unity, and then define $\mathbb{Z}/k(r) = \mu_k^{\otimes r}$. Étale cohomology with coefficients in $\mathbb{Z}/n(r)$ in fact behaves much like singular homology. For example, it satisfies a kind of "Poincaré duality" and a "Lefschetz trace formula", which we will describe later below.

One can take the homotopy inverse limit of the sheaves $\varprojlim \mathbb{Z}/\ell^n(r)$ over n (as opposed to simply taking the inverse limit of the cohomology groups) to get *ℓ-adic cohomology*, that is, étale cohomology with coefficients in $\mathbb{Z}_\ell(r)$. If we work over quasi-projective varieties however, it is alright to define this as the inverse limit of cohomology groups, as there is usually no $\lim^1$ group.

Now we state (without justification, see Milne [5,VI] for proofs) two important theorems regarding ℓ-adic cohomology.

First, we define *cohomology with compact support* of a variety X with coefficients in a sheaf F are defined to be

$$H_c^i(X_E, F) = H^i(Y_E, j_! F)$$

where $j : X \to Y$ is an open immersion of X into a proper variety Y, and $j_! F$ is the ordinary extension by zero of the sheaf F. Of course, one can show that the H_c^i are independent of the chosen immersion j.

By *constructible sheaf* F on X, we mean X is the union of a finite number of locally closed subsets on each of which F is a locally constant sheaf.

Theorem 30 (Poincaré duality for smooth separated varieties)**.** *(See* Milne, [5,VI.11.1]*) Suppose X is a smooth separated variety of dimension d over a separably closed field k. Consider the sheaf $\mathbb{Z}/\ell\mathbb{Z}$ where ℓ is coprime with Char(k) (note that ℓ does not have to be prime). Then:*

1. *There exists a unique map $\lambda(X) : H_c^{2d}(X_E, \mathbb{Z}/\ell\mathbb{Z}(d)) \to \mathbb{Z}/\ell\mathbb{Z}$ which is in fact an isomorphism.*
2. *For any constructible sheaf F or $\mathbb{Z}/l\mathbb{Z}$-modules on X, the canonical pairings*

$$H_c^i(X_E, F) \times Ext_X^{2d-i}(F, \mathbb{Z}/\ell\mathbb{Z}(d)) \to H_c^{2d}(X_E, \mathbb{Z}/\ell\mathbb{Z}(d)) \xrightarrow[\cong]{\lambda(X)} \mathbb{Z}/\ell\mathbb{Z}$$

are nondegenerate.

We also have the famous *Lefschetz Trace Formula*. Let $H_c^i(X_E, \mathbb{Q}_\ell) = H_c^i(X_E, \mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell$.

Theorem 31 (Lefschetz Trace Formula)**.** *(See* Milne, [5,VI.12.3] *for a more general formuation.) Suppose* X_0 *is a smooth projective variety over a finite field* k *of characteristic coprime with* ℓ*. Denote the algebraic closure of* k *by* $\overline{k}$*, and let* $X = X_0 \times_k \overline{k}$*. Let Frob denote the Frobenius correspondence on* X*, and let* $K|k$ *be a degree* $n < \infty$ *extension of* k*. Then*

$$\#X_0(K) = \sum_{i=0}^{2d} (-1)^i Tr(Frob^n | H_c^i(X_E, \mathbb{Q}_\ell))$$

where $X_0(K)$ *denotes the set of fixed points of* X_0 *with coefficients in* K *under Frob.*

Of course one of the crowning achievements of Étale cohomology is its applications in the proofs by Grothendieck and Deligne of the Weil conjectures; roughly speaking, these results count points on smooth projective varieties where the Lefschetz trace formula is applied to Galois actions. It should be noted however that additional methods were needed in these proofs, for example in Deligne's proof of the analogue of the Riemann hypothesis (see [2] and [3]).

References

[1] E. Artin, On the joins of Hensel rings. Advances in Mathematics, 7 (1971)
[2] P. Deligne, La conjecture de Weil: I, Publications Mathématiques de l'IHÉS, 43, (1974)
[3] P. Deligne, La conjecture de Weil : II. Publications Mathématiques de l'IHÉS, 52, (1980)
[4] R. Hartshorne: Algebraic Geometry. New York, NY: Springer Science+Business Media, Inc., 1977. Print.
[5] J.S. Milne: Étale Cohomology. Princeton, NJ: Princeton University Press, 1980. Print.

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