Construction of approximate quasisymmetric equilibria sustained by a small force

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Simons Hour, April 6 2020
Magneto-hydrostatic Equilibria

Let $T \subset \mathbb{R}^3$ be a domain with smooth boundary (e.g. the infinite cylinder or the axisymmetric torus). The Magnetohydrostatic (MHS) equations in $T$ read

\[ J \times B = \nabla P + f, \quad \text{in } T, \]
\[ \nabla \cdot B = 0, \quad \text{in } T, \]
\[ B \cdot \hat{n} = 0, \quad \text{on } \partial T, \]

where $J = \nabla \times B$ is the current, $f$ is an external force and $P$ is the ‘plasma pressure’.
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where $J = \nabla \times B$ is the current, $f$ is an external force and $P$ is the ‘plasma pressure’.

**PROGRAM:** Identify and construct (smooth) magnetohydrostatic equilibria which are effective at confining ions during a nuclear fusion reaction.
Quasisymmetric Equilibrium in Stellarator Geometry

Figure taken from Landreman (2019).
Definitions of Quasisymmetric Equilibria

**Definition**[Rodríguez, Helander, Bhattacharjee 2020 (preprint)]:
Let $\xi$ be a non-vanishing vector field tangent to $\partial T$. We say that $\xi$ is a **weak quasisymmetry** and the field $B$ is weakly quasisymmetric if

\begin{align*}
\text{div } \xi &= 0, \quad (1) \\
\xi \times B &= -\nabla \psi, \quad (2) \\
\xi \cdot \nabla |B| &= 0, \quad (3)
\end{align*}

for some flux function $\psi : T \to \mathbb{R}$. 

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where $J$ is the Jacobian matrix of the transformation.
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By a result of Burby-Kallinikos-MacKay (2019), in strong quasisymmetry $B$ must be of the form

$$B = \frac{1}{|\xi|^2} (C(\psi)\xi + \xi \times \nabla \psi)$$

for a scalar function $C$. 

For $B$ of this form, "weak quasisymmetry" requires only

$$B \cdot \xi \times J - \nabla (B \cdot \xi) = 0.$$  

(6)

If $\xi \cdot \nabla \psi = 0$ then for $B$ of this form, $B \cdot \nabla \psi = 0$ and particles are confined to constant-$\psi$ surfaces to zeroth order.
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If $\xi \cdot \nabla \psi = 0$ then for $B$ of this form, $B \cdot \nabla \psi = 0$ and particles are confined to constant-$\psi$ surfaces to zeroth order.

**QUESTION:** When does the ansatz (5) satisfy (1)–(3) and MHS?
Quasisymmetric Equilibria
The conditions for quasisymmetry are closely related to deformation tensor $\mathcal{L}_\xi \delta$

$$(\mathcal{L}_\xi \delta)(X, Y) = X \cdot (\nabla \xi + (\nabla \xi)^T) \cdot Y.$$
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**Proposition:** Let $\xi$ be a non-vanishing and divergence-free, $\psi$ be such that $\xi \cdot \nabla \psi = 0$ and $|\nabla \psi| > 0$, and $B$ be as in (5). Then:
The field $B$ is divergence-free if and only if

$$(\mathcal{L}_{\xi}\delta)(\xi, \nabla \perp \psi) = -C(\psi)(\mathcal{L}_{\xi}\delta)(\xi, \xi), \quad \nabla \perp = \xi \times \nabla.$$ 

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Condition (3) required for weak quasisymmetry is satisfied if and only if

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- Condition (4) required for strong quasisymmetry is satisfied if and only if
  $$(\mathcal{L}_\xi \delta)(\nabla_{\perp} \psi, \nabla_{\perp} \psi) = C^2(\psi)(\mathcal{L}_\xi \delta)(\xi, \xi), \quad (9)$$
  $$(\mathcal{L}_\xi \delta)(\nabla \psi, \nabla \psi) = -|B|^2(\mathcal{L}_\xi \delta)(\xi, \xi), \quad (10)$$
  $$(\mathcal{L}_\xi \delta)(\nabla \psi, \nabla_{\perp} \psi) = -C(\psi)(\mathcal{L}_\xi \delta)(\nabla \psi, \xi). \quad (11)$$
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If $\xi$ is a Killing field for the Euclidean metric, then $\mathcal{L}_\xi \delta \equiv 0$ and all the conditions (7)–(11) are satisfied independent of the nature of $\psi$ and $C(\psi)$. 

Quasisymmetric MHS Equilibria

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$$
\begin{align*}
\Delta \psi + CC'(\psi) - \frac{1}{|\xi|^2} [\xi \times \text{curl} \xi \cdot \nabla \psi - C(\psi)\xi \cdot \text{curl} \xi] + |\xi|^2 P'(\psi) &= C(\psi) \frac{\langle L_\xi \delta \rangle(\nabla \psi, B)}{|\nabla \psi|^2} - |\xi|^2 \frac{f \cdot \nabla \psi}{|\nabla \psi|^2}, \\
- \frac{|B|^2}{|\xi|^2} C(\psi) \langle L_\xi \delta \rangle(\xi, \xi) &= f \cdot \nabla^\perp \psi, \\
\frac{|B|^2}{|\xi|^2} \langle L_\xi \delta \rangle(\xi, \xi) &= f \cdot \xi.
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Quasisymmetric MHS Equilibria

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\[
\Delta \psi + CC'(\psi) - \frac{1}{|\xi|^2} \left[ \xi \times \text{curl} \xi \cdot \nabla \psi - C(\psi) \xi \cdot \text{curl} \xi \right] \\
+ |\xi|^2 P'(\psi) = |\xi|^2 \frac{f \cdot \nabla \psi}{|\nabla \psi|^2},
\]

\[
f \cdot \nabla_{\perp} \psi = 0, \\
f \cdot \xi = 0.
\]

This generalized Grad-Shafranov (gGS) equation for \( \psi \) was derived by Burby-Kallinikos-MacKay (2019). The condition \( \xi \cdot \nabla \psi = 0 \) is non-trivial!
Constraints on the deformation tensor with no forcing

**Proposition:** If $\xi$ is a **weak** quasisymmetry for $B$ then the deformation tensor takes the form

$$
(L_\xi \delta)_B = \begin{pmatrix}
0 & L_\xi \delta(\nabla \psi, \nabla^\perp \psi) & (L_\xi \delta)(\nabla \psi, \hat{\xi}) \\
L_\xi \delta(\nabla \psi, \nabla^\perp \psi) & 0 & 0 \\
(L_\xi \delta)(\nabla \psi, \hat{\xi}) & 0 & 0 
\end{pmatrix}
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where the matrix $(L_\xi \delta)$ is represented in the orthonormal basis $B := \{\nabla \psi, \nabla^\perp \psi, \hat{\xi}\}$.
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If $\xi$ is a **strong** quasisymmetry for $B$ then the deformation tensor takes the form

\[
(L_\xi \delta)_B = (L_\xi \delta)(\hat{\nabla} \psi, \hat{\xi}) \begin{pmatrix}
0 & -\frac{c(\psi)}{|\nabla \psi|} & 1 \\
-\frac{c(\psi)}{|\nabla \psi|} & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}_{B} (13)
\]
QUESTION: Given the many constraints on $(\xi, B)$ (conditions (7)–(11) and a “$\xi$-independent” solution of gGS), are there any examples?
Examples of Quasisymmetry: helical symmetry in an infinite cylinder

Consider the helical vector field defined by
\[ \xi_0 = \ell e_z - mre_\theta, \]
whose integral curves generate the infinite cylinder
\[ T_0 = \{ (r, \theta, z) \in (0, 1] \times \mathbb{T} \times \mathbb{R} \}. \]

Then \( \xi_0 \) is a Killing field so all the conditions for quasisymmetry are satisfied.
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Then \( \xi_0 \) is a Killing field so all the conditions for quasisymmetry are satisfied. The flux function is determined by the helical Grad-Shafranov
\[
\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r}{\ell^2 + m^2 r^2} \frac{\partial}{\partial r} \psi \right) + \frac{1}{r^2} \frac{\partial^2}{\partial u^2} \psi + P'(\psi) + \frac{C C'(\psi)}{\ell^2 + m^2 r^2} - \frac{2m\ell C(\psi)}{(\ell^2 + m^2 r^2)^2} = 0,
\]
with helical coordinate \( u = \ell \theta + mz \). Since the coefficients of this equation are independent of \( v = \ell z - m\theta \), it admits solutions with \( \xi_0 \cdot \nabla \psi = 0 \).

For any solution of the helical Grad-Shafranov equation, \( B_0 \) defined by
\[
B_0 = \frac{1}{|\xi_0|^2} \left( C(\psi)\xi_0 + \xi_0 \times \nabla \psi \right),
\]
is automatically a quasisymmetric MHS equilibrium on the straight cylinder \( T_0 \).
Examples of Quasisymmetry: axisymmetry in solid torus

Consider the Killing vector field defined by

$$\xi_0 = R e_\phi$$

whose integral curves are periodic and generate the axisymmetric torus with axis

$$R = R_0,$$

$$T_0 = \{(R, Z, \phi)| R = R_0 + r \cos \theta, Z = r \sin \theta, r \in [0, 1], \theta \in [0, 2\pi], \phi \in [0, 2\pi]\}.$$  

The flux function is determined by the toroidal Grad-Shafranov equation

$$\partial_r^2 \psi + \frac{1}{r^2} \partial_\theta^2 \psi + \frac{1}{r} \partial_r \psi - \frac{1}{R} \left( \cos \theta \partial_r \psi - \frac{\sin \theta}{r} \partial_\theta \psi \right) + R^2 P'(\psi) + C C'(\psi) = 0,$$

with $$R = R_0 + r \cos \theta.$$ Since the coefficients of this equation are independent of $$\phi,$$ it admits solutions with $$\xi_0 \cdot \nabla \psi = 0.$$
Harold Grad (1967) conjectured that there are no examples of smooth steady states with good confinement properties outside of these explicit examples with symmetry.
Nonexistence outside of symmetry? Grad’s conjecture

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“no additional exceptions have arisen since 1967, when it was conjectured that toroidal existence...of smooth solutions with simple nested surfaces admits only these . . . exceptions. . . . The proper formulation of the nonexistence statement is that, other than stated symmetric exceptions, there are no families of solutions depending smoothly on a parameter.” (Grad, 1985)
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We formalize a version of this statement as a rigidity property of equilibria

**Conjecture** (Grad, 1967): Any non-isolated and non-vanishing smooth MHS equilibrium on a domain $T \subset \mathbb{R}^3$ (diffeomorphic to the solid cylinder or torus) which has a pressure $p$ possessing nested level sets which foliate $T$ is either axially or helically symmetric.
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This conjecture remains open. However a natural question is

**QUESTION:** If one relaxes some of the requirements of quasisymmetry, is it possible to construct non-symmetric equilibrium states of plasma?
Main Results: Approximate quasisymmetry on cylindrical domain
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**Theorem** (C-D-G, in prep): There exists $\xi$ on $\mathbb{R}^3$ which is close to axisymmetric (i.e. with $|\xi - e_z| = O(\varepsilon)$ for $\varepsilon$ small) whose integral curves are periodic and generate a domain $T$ close to the straight cylinder, with the property that there is a vector field $B : T \rightarrow \mathbb{R}^3$ solving

$$J \times B = \nabla P + f, \quad \text{in } T,$$
$$\nabla \cdot B = 0, \quad \text{in } T,$$
$$B \cdot \hat{n} = 0, \quad \text{on } \partial T,$$

with an explicit force $f$ which is $O(\varepsilon)$. Moreover, $\xi$ is an approximate quasisymmetry in the sense that

$$\text{div } \xi = 0, \quad \text{in } T,$$
$$\text{curl}(\xi \times B) = 0, \quad \text{in } T,$$
$$\xi \times J = \nabla (B \cdot \xi) + O(\varepsilon), \quad \text{in } T,$$

and there exists a flux function $\psi$ with nested level surfaces such that

$$B = \frac{1}{|\xi|^2} \left( C(\psi)\xi + \xi \times \nabla \psi \right).$$
Main Results: Approximate quasisymmetry on a toroidal domain

Theorem (C-D-G, in prep): There exists $\xi$ on $\mathbb{R}^3$ which is close to axisymmetric (i.e. with $L_\xi \delta = O(\varepsilon)$ for $\varepsilon$ small) whose flow generates a domain $T$ close to the axisymmetric torus with large aspect ratio (i.e. with $(\text{min radius})/(\max radius) = O(1/R)$ for $R$ large) such that there is a vector field $B: T \to \mathbb{R}^3$ solving

$$J \times B = \nabla P + f,$$

$$\nabla \cdot B = 0,$$

$$B \cdot \hat{n} = 0,$$

on $\partial T$, with an explicit force $f$ which is $O(\max\{\varepsilon, 1/R\})$. Moreover, $\xi$ is an approximate quasisymmetry in the sense that $\nabla \cdot \xi = 0$, $\nabla \cdot (\xi \times B) = 0$, $\xi \times J = \nabla (B \cdot \xi) + O(\max\{\varepsilon, 1/R\})$, and there exists a flux function $\psi$ with nested level surfaces such that $B = 1/|\xi|^2 C(\psi) \xi + \xi \times \nabla \psi$. 
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and there exists a flux function $\psi$ with nested level surfaces such that

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B = \frac{1}{|\xi|^2} \left( C(\psi) \xi + \xi \times \nabla \psi \right).
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Ideas of the proof:

Basic technique due to Vanneste-Wirosoetisno (2005).
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**Problem:** Given a solution \( u_0 = \nabla^\perp \psi_0 \) of the steady 2D Euler equations with vorticity \( \omega_0 \),

\[
\Delta \psi_0 = \omega_0(\psi_0) \tag{14}
\]

on a domain \( D_0 \) and a “nearby” domain \( D \), find a solution \( u = \nabla^\perp \psi \) with possibly different vorticity \( \omega(\psi) \).
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on a domain $D_0$ and a “nearby” domain $D$, find a solution $u = \nabla^\perp \psi$ with possibly different vorticity $\omega(\psi)$.

Idea: look for a solution of the form $\psi = \psi_0 \circ \gamma^{-1}$ for a diffeomorphism $\gamma : D_0 \rightarrow D$. 

![Diagram showing transformation from $D_0$ to $D$ via $\gamma$]
Ideas of the proof:

Writing $\gamma = \text{Id} + \nabla \eta + \nabla^\perp \phi$, the requirement that

$$\Delta \psi = \omega(\psi_0),$$

becomes an equation of the form

$$\Delta \phi = F(\partial^2 \phi, \partial^2 \eta),$$

(16)

The function $\eta$ is determined from the requirement that $\text{Vol}(D) = \text{Vol}(D_0)$, since

$$1 = \det \nabla \gamma = 1 + \Delta \eta + G(\partial^2 \phi, \partial^2 \eta).$$

(17)
Ideas of the proof:

Then $\gamma$ can be found by solving a nonlinear system of elliptic equations

\begin{align*}
\Delta \phi &= F(\partial^2 \phi, \partial^2 \eta) \quad (18) \\
\Delta \eta &= G(\partial^2 \phi, \partial^2 \eta) \quad (19)
\end{align*}

with appropriate boundary conditions.
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with appropriate boundary conditions.

Observation: Trivial modification to allow for $\text{Vol}(D) \sim \text{Vol}(D_0)$ by picking a function $\rho$ with $\int_{D_0} \rho = \text{Vol}(D)$ and solving

\[
\Delta \eta = 1 - \rho + G(\partial^2 \phi, \partial^2 \eta). \quad (20)
\]
Ideas of the proof:

We will construct a quasisymmetric solution which is nearby an axisymmetric one as follows.

Step 1: Take a solution defined on the cylinder $T_{0}$ which is a function only of $r$, i.e., $\psi_{0} := \psi_{0}(r)$, having the property $C(\psi_{0}) \psi_{0}' \approx \epsilon_{0}$.

Step 2: Let $\xi$ be a vector field with a periodic flow which is nearly axisymmetric and generates a 'nearby' non-axisymmetric cylinder $T$. We require a special relationship between components of its deformation tensor $|\mathcal{L}_{\xi} \delta |(\xi,\xi) / |\mathcal{L}_{\xi} \delta |(\xi,\xi \times e_{r}) \approx \epsilon_{0}$.
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$$\left| \frac{C(\psi_0)}{\psi'}_0 \right| \approx \frac{1}{\epsilon_0}. \tag{21}$$

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$$\frac{|(L_{\xi} \delta)(\xi, \xi)|}{|(L_{\xi} \delta)(\xi, \xi \times e_r)|} \approx \epsilon_0 > 0. \tag{22}$$
Ideas of the proof:

**Step 3:** Fix coordinates \((x_1, x_2, x_3)\) on \(\mathbb{R}^3\) so that \(\xi \cdot \nabla = \frac{\partial}{\partial x_3}\) and a disk \(D\) in the \(x_3 = 0\) plane. We will construct a solution in the cylinder \(T\) generated from the integral curves of \(\xi\) starting from \(D\).
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**Step 4:** In these coordinates (gGS) takes the form

\[
\Delta \psi + G(x_1, x_2, x_3, \psi) = 0
\]

for an explicit function \(G\).
Ideas of the proof:

**Step 3:** Fix coordinates \((x_1, x_2, x_3)\) on \(\mathbb{R}^3\) so that \(\xi \cdot \nabla = \frac{\partial}{\partial x_3}\) and a disk \(D\) in the \(x_3 = 0\) plane. We will construct a solution in the cylinder \(T\) generated from the integral curves of \(\xi\) starting from \(D\).

**Step 4:** In these coordinates \((gGS)\) takes the form

\[
\Delta \psi + G(x_1, x_2, x_3, \psi) = 0
\]

for an explicit function \(G\). Freeze coefficients and write as

\[
\Delta_{2d} \psi + G_1(x_1, x_2, \psi) + L \psi + R(x_1, x_2, x_3, \psi) = 0
\]  

(23)

where \(\Delta_{2d}\) is the part of the Laplacian only involving derivatives in the \(x_1, x_2\) direction, where \(L = L(\partial_{x_3})\) and the remainder \(R\) satisfies

\[
|R| \lesssim |L_{\xi} \delta| = O(\varepsilon).
\]  

(24)
Ideas of the proof:

**Step 5:** Construct a streamfunction $\psi$ with nested level sets foliating the domain and enjoying the property that $\xi \cdot \nabla \psi = 0$ (i.e. $\psi = \psi(x_1, x_2)$ in this coordinate system), satisfying

$$\Delta_{2d}\psi + G_1(x_1, x_2, \psi) = 0.$$  \hspace{1cm} (25)

This $\psi$ nearly satisfies (gGS) in the sense that

$$|\Delta \psi + G| = O(\varepsilon).$$ \hspace{1cm} (26)

This, and the assumption $|\xi - e_2| = O(\varepsilon)$, guarantees that $B$ satisfies MHS with small force $f = O(\varepsilon)$. 
Ideas of the proof:

**Step 5 (cont.):** We do this by looking for $\psi$ of the form $\psi = \psi_0 \circ \gamma^{-1}$ for a diffeomorphism $\gamma = Id + \nabla \eta + \nabla^\perp \phi$ for small $\eta, \phi$ to be determined. Such a solution automatically has nested level sets foliating the cylinder. The requirement that $\psi$ satisfy (25) becomes a nonlinear elliptic equation for $\phi$ of the form

$$\Delta \phi = N(\partial^2 \phi, \partial^2 \eta),$$

(27)

for a given nonlinearity $N$. 

Ideas of the proof:

**Step 5 (cont.):** We do this by looking for $\psi$ of the form $\psi = \psi_0 \circ \gamma^{-1}$ for a diffeomorphism $\gamma = \text{Id} + \nabla \eta + \nabla \perp \phi$ for small $\eta, \phi$ to be determined. Such a solution automatically has nested level sets foliating the cylinder. The requirement that $\psi$ satisfy (25) becomes a nonlinear elliptic equation for $\phi$ of the form

$$\Delta \phi = N(\partial^2 \phi, \partial^2 \eta),$$

(27)

for a given nonlinearity $N$.

**Step 6:** We must find a solution $\psi$ consistent with $\text{div} \, B = 0$, i.e. so that

$$C(\psi)(\mathcal{L}_\xi \delta)(\xi, \xi) + (\mathcal{L}_\xi \delta)(\xi, \xi \times \nabla \psi) = 0.$$  

(28)

We emphasize that any deviation from (28) holding exactly cannot be compensated directly by a force as the condition $\text{div} \, B = 0$ sees the form of $B$ alone and can be altered only through changing $\psi$. 
Ideas of the proof:

**Step 5 (cont.):** We do this by looking for $\psi$ of the form $\psi = \psi_0 \circ \gamma^{-1}$ for a diffeomorphism $\gamma = \text{Id} + \nabla \eta + \nabla^\perp \phi$ for small $\eta, \phi$ to be determined. Such a solution automatically has nested level sets foliating the cylinder. The requirement that $\psi$ satisfy (25) becomes a nonlinear elliptic equation for $\phi$ of the form

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We emphasize that any deviation from (28) holding exactly cannot be compensated directly by a force as the condition $\text{div} \, B = 0$ sees the form of $B$ alone and can be altered only through changing $\psi$.

A calculation shows that (28) is

$$C(\psi)(\mathcal{L}_\xi \delta)(\xi, \xi) + \psi_0'(\det \nabla \gamma)^{-1} ((\mathcal{L}_\xi \delta)(\xi, \xi \times e_r) + (\mathcal{L}_\xi \delta)(\xi, \xi_{\phi \eta}))$$

(29)

with $\xi_{\phi \eta} \sim (\partial^2 \phi, \partial^2 \eta, 0)$. 
Ideas of the proof:

**Step 6 (cont.):** Therefore $\text{div} \, B = 0$ provided we can choose $\gamma$ satisfying

$$
(\det \nabla \gamma)^{-1} = \frac{C(\psi_0 \circ \gamma^{-1})}{\psi'_0 \circ \gamma^{-1}} \frac{(\mathcal{L}_{\xi} \delta)(\xi, \xi)}{(\mathcal{L}_{\xi} \delta)(\xi, \xi \times e_r) + (\mathcal{L}_{\xi} \delta)(\xi, \xi_\phi \eta)},
$$

(30)
Ideas of the proof:

**Step 6 (cont.):** Therefore $\text{div } B = 0$ provided we can choose $\gamma$ satisfying

\[
(d\nabla \gamma)^{-1} = \frac{C(\psi_0 \circ \gamma^{-1})}{\psi_0' \circ \gamma^{-1}} \frac{(L_\xi \delta)(\xi, \xi)}{(L_\xi \delta)(\xi, \xi \times e_r) + (L_\xi \delta)(\xi, \xi_\phi \eta)},
\]  

(30)

In general we have

\[
\Delta \eta = (d\nabla \gamma)^{-1} - 1 + N(\partial^2 \eta, \partial^2 \phi),
\]  

(31)

and by our assumptions

\[
(d\nabla \gamma)^{-1} - 1 \sim \epsilon_0(\partial^2 \phi + \partial^2 \eta).
\]  

(32)
Ideas of the proof:

**Step 6 (cont.):** Therefore $\text{div } B = 0$ provided we can choose $\gamma$ satisfying

$$
(d \nabla \gamma)^{-1} = \frac{C(\psi_0 \circ \gamma^{-1})}{\psi'_0 \circ \gamma^{-1}} \frac{(\mathcal{L}_\xi \delta)(\xi, \xi)}{(\mathcal{L}_\xi \delta)(\xi, \xi \times e_r) + (\mathcal{L}_\xi \delta)(\xi, \xi \phi \eta)},
$$

(30)

In general we have

$$
\Delta \eta = (d \nabla \gamma)^{-1} - 1 + N(\partial^2 \eta, \partial^2 \phi),
$$

(31)

and by our assumptions

$$
(d \nabla \gamma)^{-1} - 1 \sim \epsilon_0(\partial^2 \phi + \partial^2 \eta).
$$

(32)

**Step 7:** We then need to solve a system of the form

$$
\Delta \eta = \epsilon_0(\partial^2 \phi + \partial^2 \eta) + F(\partial^2 \eta, \partial^2 \phi),
$$

(33)

$$
\Delta \phi = F(\partial^2 \phi, \partial^2 \eta).
$$

(34)

This can be solved by iteration:

$$
\Delta \eta^{N+1} = \epsilon_0(\partial^2 \phi^N + \partial^2 \eta^N) + F(\partial^2 \eta^N, \partial^2 \phi^N),
$$

(35)

$$
\Delta \phi^{N+1} = F(\partial^2 \phi^N, \partial^2 \eta^N).
$$

(36)
Some final remarks

• The proof is constructive and provides an algorithm which, in principle, can be used to generate these equilibria on the computer.

• The technique is robust to small perturbations, allowing steady states occupying a given domain to be deformed to fit nearby ones for a variety of model equations including 2d Euler, Boussinesq, as well as MHS.

Acknowledgements

The research of PC and DG was partially supported by the Simons Center for Hidden Symmetries and Fusion Energy. Research of TD was partially supported by NSF grant DMS-1703997. We thank A. Cerfon for insightful discussions.
Thanks for your attention!