Weak Solutions for Inviscid SQG with Lorentz Data

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Abstract. We construct global weak solutions of the inviscid surface quasi-geostrophic equation in $\mathbb{R}^2$ and in smooth bounded domains, for arbitrary initial data in the critical Lorentz space $L^{4/3,2}$. The solutions conserve the Hamiltonian $\|\Lambda^{-1/2}\theta(t)\|_{L^2}^2$ for all times. The second Lorentz exponent is determined by the sharp boundedness $\Lambda^{-1/2} : L^{4/3,2} \to L^2$; the corresponding estimate fails for $L^{4/3,q}$ when $q > 2$. We use an approximation scheme that is tailored for Lorentz spaces. It smooths the advecting velocity and the initial data, and preserves order in distribution functions. Removing the approximation is made possible by uniform bounds for the Lorentz norms of high amplitude cutoffs of the solutions.

1. Introduction and main result

The inviscid surface quasi-geostrophic (SQG) equation is an active scalar model in which a transported scalar determines the incompressible velocity through a singular integral of order zero. It arose as a model for front formation in geophysical flow and has long served as a two-dimensional analogue of the three-dimensional incompressible Euler equations [**5, 10**]. The finite time blow up problem from smooth and localized initial data remains a challenging open problem.

In either the plane or in a smooth bounded domain, the equation takes the form

$$\partial_t\theta + u\cdot\nabla\theta = 0, \qquad u = \nabla^\perp\Lambda^{-1}\theta.$$

where $\Lambda = (-\Delta)^{\frac{1}{2}}$. For smooth solutions, divergence-free transport preserves the distribution function of θ, and hence every finite L^p norm. It also preserves the Hamiltonian

$$\mathcal{H}(\theta) = \frac{1}{2}\int_\Omega \theta\Lambda^{-1}\theta\,dx = \frac{1}{2}\|\Lambda^{-1/2}\theta\|_{L^2(\Omega)}^2,$$

because $u = \nabla^\perp\Lambda^{-1}\theta$ is pointwise orthogonal to $\nabla\Lambda^{-1}\theta$.

Both statements may break down for weak solutions. The velocity and scalar may be too rough for the product $u\theta$ to have a direct meaning, while weak convergence in the natural negative Sobolev space need not preserve the quadratic Hamiltonian. Global weak solutions in L^2 were first obtained in the thesis [**20**]. This was extended to L^p-based settings in [**18**]; for the equation with homogeneous Dirichlet boundary condition in bounded domains, global L^2 weak solutions and their Hamiltonian conservation were established in [**11**]. At lower regularity, convex integration constructions showed that weak SQG solutions need not be unique [**3, 16**]. More recently, in [**2**] the authors obtained flexibility and nonuniqueness for weak solutions on $\mathbb{T}^2$ with $\theta \in C([0,1]; L^{\bar{p}})$, where $\bar{p} = 4/3 + 10^{-5}$. This result is complementary to the present theorem, which constructs a Hamiltonian-conserving solution for every initial datum at the Lorentz endpoint.

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Two recent results on $\mathbb{T}^2$ are closely related to this problem. In [13, 14] the authors constructed Hamiltonian-conserving solutions for initial data in $L^{4/3}(\mathbb{T}^2)$ and in Sobolev spaces by vanishing viscosity methods.

In this paper we replace $L^{4/3}$ by the larger Lorentz space $L^{4/3,2}$, consider both the plane and the Dirichlet boundary condition equations on bounded domains, and regularize the velocity advection law without adding viscosity. Since $L^{4/3} = L^{4/3,4/3} \subsetneq L^{4/3,2}$, the Lorentz endpoint allows rougher data than the Lebesgue space.

The natural Lorentz space for the Hamiltonian is given by the two-dimensional bound

$$\|\Lambda^{-1/2} f\|_{L^2} \lesssim \|f\|_{L^{4/3,2}},$$

and the secondary Lorentz exponent 2 is optimal: the estimate fails with $L^{4/3,q}$ for every $q > 2$. Thus $L^{4/3,2}$ is the largest Lorentz space at the fixed principal exponent 4/3 for which this estimate controls the Hamiltonian. We prove the estimate using $L^{p,q}$ convolution inequalities [19].

Bounded sequences in $L^{4/3,2}$ may concentrate, and weak convergence alone does not preserve the Hamiltonian. For our approximations, however, all amplitude distributions come from heat regularizations of the initial datum. We use the high amplitude cutoff modulus $\tau_M(f) = \|(|f| - M)_+\|_{L^{4/3,2}}$. It is contracted by the heat semigroup, preserved by measure-preserving transport, and weakly lower semicontinuous. In the plane, we use also a transport estimate to obtain spatial tightness.

We formulate the notion of weak solution by symmetrizing the nonlinearity. This is an idea that was used before for two dimensional incompressible Euler equations in vorticity formulation [12, 21]. Like SQG, the 2D vorticity equation is an active scalar with quadratic nonlinearity, but it is smoother than SQG whose $\nabla^\perp\theta$ is the analogue of the 3D Euler vorticity, The 2D Biot-Savart kernel has a singularity of order 1 near the diagonal, and symmetrization paired with a test function produces a bounded kernel. For SQG, the kernel has a singularity of order 2, $K_{\mathrm{SQG}}(z) = cz^\perp/|z|^3$, and symmetrization paired with a test function produces a kernel of size $|x-y|^{-1}$. Smallness near the diagonal is obtained from the $L^{4/3,2}$ high amplitude cutoff modulus estimate. In the whole space, the transport estimate prevents travel to spatial infinity.

1.1. The equation and functional setting. Let either $\Omega = \mathbb{R}^2$ or let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain. Throughout the paper, implicit constants may depend on the fixed domain Ω and on explicitly displayed test functions, but not on time or on the approximation parameters unless stated otherwise. We set

$$\Lambda = (-\Delta)^{1/2}.$$

On $\mathbb{R}^2$ this is the Fourier multiplier, $\Lambda = |D|$. On a bounded domain we use spectral calculus of the Laplacian with homogeneous Dirichlet boundary conditions. Thus

$$\Lambda^{-1} f = (-\Delta)^{-1/2} f = c_0 \int_0^\infty s^{-1/2} e^{s\Delta} f \, ds \tag{1}$$

where $e^{s\Delta}$ is the Dirichlet heat semigroup, i.e., $w(s,x) = e^{s\Delta} f(x)$ is the solution of

$$\begin{cases} \partial_s w - \Delta w = 0, & s > 0, \quad x \in \Omega, \\ w = 0, & s > 0, \quad x \in \partial\Omega, \\ w(0,x) = f(x), & x \in \Omega. \end{cases} \tag{2}$$

We consider the inviscid SQG equation

$$\partial_t \theta + u \cdot \nabla\theta = 0, \qquad u = \nabla^\perp \Lambda^{-1}\theta. \tag{3}$$

The corresponding Hamiltonian energy is

$$\|\theta\|^2_{\dot{H}^{-1/2}(\Omega)} := \int_\Omega \theta\, \Lambda^{-1}\theta \, dx = \|\Lambda^{-1/2}\theta\|^2_{L^2(\Omega)}. \tag{4}$$

For a measurable function f we denote its distribution function by

$$d_f(\lambda) = |\{x \in \Omega : \ |f(x)| > \lambda\}|, \qquad \lambda > 0,$$

where $|E|$ is the Lebesgue measure of E. We recall that, for $0 < p < \infty$ and $0 < q < \infty$,

$$\|f\|_{L^{p,q}}^q \sim \int_0^\infty \lambda^{q-1} d_f(\lambda)^{q/p}\, d\lambda. \tag{5}$$

In particular,

$$\|f\|_{L^{4/3,2}}^2 \sim \int_0^\infty \lambda\, d_f(\lambda)^{3/2}\, d\lambda. \tag{6}$$

Let f^* be the decreasing rearrangement of $|f|$ and $f^{**}(s) = s^{-1}\int_0^s f^*(r)\, dr$. Following [1, Chapter 2], we use on $X := L^{4/3,2}(\Omega)$ also the rearrangement norm

$$\|f\|_X = \left(\int_0^{|\Omega|} [s^{3/4} f^{**}(s)]^2 \frac{ds}{s}\right)^{1/2}, \tag{7}$$

where the upper endpoint is infinity when $|\Omega| = \infty$. For $M \geq 0$ define the high amplitude cutoff modulus

$$\tau_M(f) := \big\|(|f| - M)_+\big\|_X. \tag{8}$$

Because X has order-continuous norm, $\tau_M(f) \to 0$ as $M \to \infty$ for every $f \in X$. Moreover,

$$\|f \mathbf{1}_{\{|f|>2M\}}\|_X \leq 2\tau_M(f). \tag{9}$$

The functional τ_M is convex and norm-continuous, and is therefore weakly lower semicontinuous.

Let $G_1(x, y)$ denote the kernel of Λ^{-1} and set $K_\Omega(x, y) = \nabla_x^\perp G_1(x, y)$. In the whole-space case this reduces to $K_{\mathbb{R}^2}(x, y) = c(x - y)^\perp/|x - y|^3$. For a spatial test function φ, define the symmetrized transport form

$$\mathcal{N}_\Omega(\theta, \varphi) := \frac{1}{2} \iint_{\Omega\times\Omega} [K_\Omega(x, y) \cdot \nabla\varphi(x) + K_\Omega(y, x) \cdot \nabla\varphi(y)]\, \theta(x)\theta(y)\, dx\, dy. \tag{10}$$

The cancellation displayed in Section 4 makes this form finite for $\theta \in L^{4/3,2}$.

Definition 1.1 (Weak solution). *Let $\theta_0 \in L^{4/3,2}(\Omega)$. A function*

$$\theta \in L^\infty_{\mathrm{loc}}([0, \infty); L^{4/3,2}(\Omega)) \cap C([0, \infty); \mathcal{D}'(\Omega))$$

is a weak solution of (3) with initial datum θ_0 if, for every $\varphi \in C_c^\infty([0, \infty) \times \Omega)$,

$$\int_0^\infty \int_\Omega \theta\, \partial_t\varphi\, dx\, dt + \int_\Omega \theta_0(x)\varphi(0, x)\, dx + \int_0^\infty \mathcal{N}_\Omega(\theta(t), \varphi(t))\, dt = 0. \tag{11}$$

1.2. Main theorem.

Theorem 1.1 (Weak solutions at critical Lorentz regularity). *Let $\theta_0 \in L^{4/3,2}(\Omega)$. Then there exists a global weak solution θ of (3) such that*

$$\theta \in C([0, \infty); X_{\mathrm{w}}) \cap L^\infty_{\mathrm{loc}}([0, \infty); X).$$

Moreover, for every $M \geq 0$ and every $t \geq 0$,

$$\tau_M(\theta(t)) \leq \tau_M(\theta_0). \tag{12}$$

There also exists a nonnegative nondecreasing function $\Phi = \Phi_{\theta_0}$, depending only on the initial datum, such that

$$\Phi(\lambda) \longrightarrow \infty \qquad \text{as } \lambda \longrightarrow \infty, \qquad 1 + \Phi(2\lambda) \leq 2\big(1 + \Phi(\lambda)\big),$$

and, for every $t \geq 0$,

$$\int_0^\infty \lambda\big(1 + \Phi(\lambda)\big) d_{\theta(t)}(\lambda)^{3/2}\, d\lambda \lesssim \int_0^\infty \lambda d_{\theta_0}(\lambda)^{3/2}\, d\lambda \simeq \|\theta_0\|_X^2. \tag{13}$$

Finally, the negative Sobolev energy is conserved:

$$\|\theta(t)\|_{\dot{H}^{-1/2}(\Omega)} = \|\theta_0\|_{\dot{H}^{-1/2}(\Omega)} \qquad \text{for every } t \geq 0. \tag{14}$$

1.3. Main ideas and organization. We truncate the heat semigroup representation of Λ^{-1} and smooth the initial datum. The resulting approximate equation is then a transport equation with a smooth divergence-free velocity. It has global smooth solutions. This approximation was used in [**15**] to construct global regular solutions of critical dissipative SQG in bounded domains. In the present case without dissipation the distribution functions of the solutions are preserved in time, and the self-adjointness of the regularized operator gives an exact Hamiltonian identity.

The main difficulty is to prevent concentration on the diagonal. One of the key compactness ingredients is given by the existence of the uniform factor Φ of Proposition 2.2 which, together with the high amplitude cutoff modulus τ_M allows passage to the limit.

Section 2 contains the endpoint Lorentz and high amplitude cutoff modulus estimates. Section 3 constructs the conservative approximation and establishes its uniform bounds and Hamiltonian identity. The symmetrized formulation in the plane and on bounded domains is developed in Section 4. Section 5 contains the compactness argument and the proof of Theorem 1.1, while Appendix A proves the optimality of the secondary Lorentz exponent.

2. Critical Lorentz and high amplitude cutoff estimates

We first prove the Lorentz estimate for the Hamiltonian and then collect the high amplitude cutoff modulus estimates used in the compactness argument. The relevance of the pair $(4/3, 2)$ can already be read from the kernel: in two dimensions the operator o $\Lambda^{-1/2}$ has kernel of size $|x|^{-3/2}$, which belongs to weak $L^{4/3}$. An $L^{p,q}$ convolution inequality then places the image in $L^{2,2} = L^2$ when the secondary exponent of the datum is at most 2.

Proposition 2.1 (Endpoint Lorentz control). *Let either $\Omega = \mathbb{R}^2$ or let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain, and define*

$$\mathcal{I}_{\Omega,1/2} f := \begin{cases} |D|^{-1/2} f, & \Omega = \mathbb{R}^2, \\ \Lambda^{-1/2} f, & \Omega \text{ bounded.} \end{cases}$$

Then

$$\|\mathcal{I}_{\Omega,1/2} f\|_{L^2(\Omega)} \lesssim \|f\|_{L^{4/3,2}(\Omega)} \lesssim \left(\int_0^\infty \lambda d_f(\lambda)^{3/2} \, d\lambda \right)^{1/2}. \tag{15}$$

Furthermore, on $\mathbb{R}^2$ the estimate

$$\||D|^{-1/2} f\|_{L^2} \lesssim \|f\|_{L^{4/3,q}}$$

fails in general for every $q > 2$.

Proof. We first consider $\Omega = \mathbb{R}^2$. The Riesz-potential representation is

$$|D|^{-1/2} f = I_{1/2} f = c_2 |x|^{-3/2} * f.$$

Since

$$\left|\{x : |x|^{-3/2} > \lambda\}\right| \sim \lambda^{-4/3},$$

the kernel $|x|^{-3/2}$ belongs to $L^{4/3,\infty}(\mathbb{R}^2)$. The convolution inequality [**19**]

$$\|f * g\|_{L^{r,c}} \le \|f\|_{L^{p,a}} \|g\|_{L^{q,b}},$$

where $r^{-1} = p^{-1} + q^{-1} - 1$, for $p, q \in (1, \infty)$, $c^{-1} = a^{-1} + b^{-1}$, for $a, b \in [1, \infty]$, therefore gives

$$\left\||D|^{-1/2} f\right\|_{L^2} \lesssim \|f\|_{L^{4/3,2}},$$

because $L^{4/3,\infty} * L^{4/3,2} \subset L^{2,2} = L^2$. The second inequality in (15) follows from (6).

For a bounded smooth domain, the same mapping property follows from heat-kernel domination. The parabolic maximum principle gives the comparison

$$0 \le H_D(\rho, x, y) \le (4\pi\rho)^{-1} \exp\left(-\frac{|x-y|^2}{4\rho} \right)$$

(see, for example, [**6, 11**]). The fractional representation

$$\Lambda^{-1/2} f = c \int_0^\infty e^{\rho\Delta} f \, \rho^{-3/4} \, d\rho$$

therefore has an integral kernel bounded by

$$C \int_0^\infty \rho^{-7/4} \exp\left(-\frac{|x-y|^2}{4\rho}\right) d\rho = C|x-y|^{-3/2},$$

uniformly for $x, y \in \Omega$. Extending f by zero outside Ω and applying the preceding Lorentz convolution estimate gives

$$\|\Lambda^{-1/2} f\|_{L^2(\Omega)} \lesssim \|f\|_{L^{4/3,2}(\Omega)}.$$

This proves the upper estimate in both geometries. The failure for $q > 2$ is proved in Appendix A. $\square$

2.1. High amplitude cutoff estimate. We next collect the properties of τ_M used below.

LEMMA 2.1. *Let $X = L^{4/3,2}(\Omega)$ be equipped with the norm (7). Then, for every $M \geq 0$:*

(i) *$f \mapsto \tau_M(f)$ is convex and norm-continuous, hence weakly lower semicontinuous on X;*

(ii) *if T is a positive linear operator satisfying*

$$\|Tg\|_{L^1(\Omega)} \leq \|g\|_{L^1(\Omega)}, \qquad \|Tg\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\Omega)},$$

then $\tau_M(Tf) \leq \tau_M(f)$;

(iii) *if Y is measure preserving, then $\tau_M(f \circ Y) = \tau_M(f)$;*

(iv) *with $T_M f = \operatorname{sgn}(f)(|f| \wedge M)$ and $R_M f = f - T_M f = \operatorname{sgn}(f)(|f| - M)_+$,*

$$\|R_M f\|_X = \tau_M(f), \qquad \|\Lambda^{-1/2} R_M f\|_{L^2} \lesssim \tau_M(f). \tag{16}$$

Moreover, the restriction of f to the superlevel set $\{|f| > 2M\}$ satisfies

$$\|f \mathbf{1}_{\{|f|>2M\}}\|_X \leq 2\tau_M(f), \qquad \|\Lambda^{-1/2}(f \mathbf{1}_{\{|f|>2M\}})\|_{L^2} \lesssim \tau_M(f). \tag{17}$$

In particular, $\tau_M(f) \to 0$ as $M \to \infty$ for every $f \in X$.

PROOF. The scalar function $\vartheta_M(s) = (|s| - M)_+$ is convex and 1-Lipschitz. In particular,

$$|\vartheta_M(f) - \vartheta_M(g)| \leq |f - g| \quad \text{a.e.},$$

so the lattice property gives $|\tau_M(f) - \tau_M(g)| \leq \|f - g\|_X$. The same lattice property and the pointwise convexity of ϑ_M make τ_M convex; hence it is norm-continuous and every continuous convex functional on a Banach space is weakly lower semicontinuous. Positivity gives $|Tf| \leq T|f|$. Since T is positive and $T\mathbf{1} \leq \mathbf{1}$, Jensen's inequality, with the missing mass placed at zero, gives

$$(|Tf| - M)_+ \leq T\big((|f| - M)_+\big).$$

Set $h = (|f| - M)_+$. The L^1 and L^∞ contraction assumptions imply $Th \prec\prec h$. Here $g \prec\prec h$ means that g is submajorized by h, that is,

$$\int_0^s g^*(r)\, dr \leq \int_0^s h^*(r)\, dr \qquad \text{for every } 0 < s < |\Omega|,$$

with the condition required for every $s > 0$ when $|\Omega| = \infty$. Monotonicity of the norm (7) under submajorization therefore gives

$$\tau_M(Tf) \leq \|Th\|_X \leq \|h\|_X = \tau_M(f),$$

which proves (ii). Assertion (iii) follows from equimeasurability. The identity $|R_M f| = (|f| - M)_+$ and Proposition 2.1 give (16). On $\{|f| > 2M\}$, the inequality $|f| \leq 2(|f| - M)_+$ and the lattice property give the stated X estimate. Applying Proposition 2.1 to $f\mathbf{1}_{\{|f|>2M\}}$ gives the corresponding $\dot{H}^{-1/2}$ estimate. Order continuity of X implies $\tau_M(f) \downarrow 0$. $\square$

PROPOSITION 2.2. *Let* $f \in X = L^{4/3,2}(\Omega)$ *and define*

$$\mathcal{A}_f := \{h \in X : \ \tau_M(h) \le \tau_M(f) \text{ for every } M \ge 0\}.$$

Then there exists a nonnegative nondecreasing function $\Phi_f : [0,\infty) \to [0,\infty)$ *such that*

$$\Phi_f(\lambda) \longrightarrow \infty, \qquad 1 + \Phi_f(2\lambda) \le 2\big(1 + \Phi_f(\lambda)\big).$$

In particular, the weight $1 + \Phi_f$ *is doubling. With*

$$\mathcal{M}_{\Phi_f}(h) := \int_0^\infty \lambda\big(1 + \Phi_f(\lambda)\big) d_h(\lambda)^{3/2}\, d\lambda,$$

it holds that

$$\sup_{h\in\mathcal{A}_f} \mathcal{M}_{\Phi_f}(h) \lesssim \mathcal{M}_{\Phi_f}(f) < \infty. \tag{18}$$

Moreover, we normalize Φ_f *so that*

$$\mathcal{M}_{\Phi_f}(f) \le 2 \int_0^\infty \lambda d_f(\lambda)^{3/2}\, d\lambda. \tag{19}$$

PROOF. If $f = 0$, then $\mathcal{A}_f = \{0\}$ and one may take $\Phi_f(\lambda) = \lambda$, so assume $f \ne 0$ and put

$$I_f := \int_0^\infty \lambda d_f(\lambda)^{3/2}\, d\lambda > 0.$$

For $h \in \mathcal{A}_f$, the distribution-function representation of the Lorentz norm and (17) give

$$\begin{aligned}\int_{2M}^\infty \lambda d_h(\lambda)^{3/2}\, d\lambda &\le \int_0^\infty \lambda d_{h\mathbf{1}_{\{|h|>2M\}}}(\lambda)^{3/2}\, d\lambda \\ &\lesssim \|h\mathbf{1}_{\{|h|>2M\}}\|_X^2 \lesssim \tau_M(f)^2.\end{aligned}$$

Since $\tau_M(f) \to 0$ as $M \to \infty$, the preceding estimate yields

$$\sup_{h\in\mathcal{A}_f} \int_{2M}^\infty \lambda d_h(\lambda)^{3/2}\, d\lambda \lesssim \tau_M(f)^2 \longrightarrow 0.$$

We may therefore choose an increasing sequence $R_k \to \infty$ with $R_{k+1} \ge 2R_k$ such that

$$\sup_{h\in\mathcal{A}_f} \int_{R_k}^\infty \lambda d_h(\lambda)^{3/2}\, d\lambda \le 2^{-k} I_f, \qquad k \ge 1.$$

Set

$$\Phi_f(\lambda) := \sum_{k=1}^\infty \mathbf{1}_{[R_k,\infty)}(\lambda).$$

This function is nondecreasing and tends to infinity. Because $R_{k+1} \ge 2R_k$, the interval $(\lambda, 2\lambda]$ contains at most one new level R_k; hence $1+\Phi_f$ satisfies the displayed doubling estimate. Taking $M = 0$ in the preceding tail estimate and using $\tau_0(f) = \|f\|_X$ gives the uniform unweighted bound

$$\sup_{h\in\mathcal{A}_f} \int_0^\infty \lambda d_h(\lambda)^{3/2}\, d\lambda \lesssim \|f\|_X^2 \simeq I_f.$$

Finally, Tonelli's theorem yields, uniformly for $h \in \mathcal{A}_f$,

$$\int_0^\infty \lambda\Phi_f(\lambda) d_h(\lambda)^{3/2}\, d\lambda = \sum_{k=1}^\infty \int_{R_k}^\infty \lambda d_h(\lambda)^{3/2}\, d\lambda \le I_f.$$

Together with the preceding unweighted estimate, this proves (18). For $h = f$, the unweighted term equals I_f, while the term containing Φ_f is at most I_f; hence (19) follows with the stated constant. □

Proposition 2.2 gives the weighted estimate in Theorem 1.1.

3. Conservative approximation and uniform estimates

We regularize the velocity law by truncating the heat representation of Λ^{-1} and we independently smooth the initial datum. The first operation gives a smooth velocity, while self-adjointness of the truncated operator preserves the Hamiltonian cancellation.

3.1. Approximation and preservation of distributions. Let $\epsilon_1, \epsilon_2 > 0$ and define the regularized Λ^{-1} operator

$$\Lambda_{\epsilon_1}^{-1} f := c_0 \int_{\epsilon_1}^{\infty} e^{\rho\Delta} f \, \rho^{-1/2} \, d\rho. \tag{20}$$

The operator $\Lambda_{\epsilon_1}^{-1}$ is self-adjoint and nonnegative. As $\epsilon_1 \downarrow 0$, its quadratic form increases to that of Λ^{-1} by monotone convergence in (20). The parameter ϵ_2 is used only to smooth the initial datum.

We solve

$$\begin{cases} \partial_t \theta^{\epsilon_1,\epsilon_2} + u^{\epsilon_1,\epsilon_2} \cdot \nabla \theta^{\epsilon_1,\epsilon_2} = 0, \\ u^{\epsilon_1,\epsilon_2} = \nabla^\perp \Lambda_{\epsilon_1}^{-1} \theta^{\epsilon_1,\epsilon_2}, \\ \theta^{\epsilon_1,\epsilon_2}(0) = e^{\epsilon_2\Delta}\theta_0. \end{cases} \tag{21}$$

For fixed $\epsilon_1, \epsilon_2 > 0$, the initial datum belongs to $X \cap L^\infty$ and is smooth. On $\mathbb{R}^2$, if K_{ϵ_1} denotes the velocity kernel, then

$$K_{\epsilon_1} \in L^{4,2}(\mathbb{R}^2), \qquad \|\nabla K_{\epsilon_1}\|_{L^1(\mathbb{R}^2)} \le C_{\epsilon_1}.$$

Lorentz Hölder and Young inequalities consequently give

$$\|u^{\epsilon_1,\epsilon_2}(t)\|_{L^\infty} \le C_{\epsilon_1} \|\theta^{\epsilon_1,\epsilon_2}(t)\|_X, \qquad \|\nabla u^{\epsilon_1,\epsilon_2}(t)\|_{L^\infty} \le C_{\epsilon_1} \|\theta^{\epsilon_1,\epsilon_2}(t)\|_{L^\infty}.$$

The corresponding bounded-domain estimates follow from the differentiated Dirichlet heat-kernel representation. Higher derivatives satisfy analogous bounds, with constants depending on ϵ_1 and the derivative order. The velocity is therefore smooth, bounded, and divergence-free. In a bounded domain, $\Lambda_{\epsilon_1}^{-1}\theta$ has zero Dirichlet trace, so $u \cdot n = 0$ on $\partial\Omega$. Conservation of the X and L^∞ norms along characteristics then gives global continuation of the classical solution.

More precisely, if

$$\frac{d}{dt} X^{\epsilon_1,\epsilon_2}(t,a) = u^{\epsilon_1,\epsilon_2}\big(t, X^{\epsilon_1,\epsilon_2}(t,a)\big), \qquad X^{\epsilon_1,\epsilon_2}(0,a) = a,$$

then incompressibility gives $\det D_a X^{\epsilon_1,\epsilon_2}(t,a) = 1$ and

$$\theta^{\epsilon_1,\epsilon_2}\big(t, X^{\epsilon_1,\epsilon_2}(t,a)\big) = e^{\epsilon_2\Delta}\theta_0(a).$$

Consequently, for every nonnegative Borel function $\beta : \mathbb{R} \to [0,\infty]$,

$$\int_\Omega \beta(\theta^{\epsilon_1,\epsilon_2}(t,x)) \, dx = \int_\Omega \beta(e^{\epsilon_2\Delta}\theta_0(x)) \, dx. \tag{22}$$

The same identity holds for a signed Borel function whenever $\beta(e^{\epsilon_2\Delta}\theta_0)$ is integrable. Equivalently,

$$d_{\theta^{\epsilon_1,\epsilon_2}(t)}(\lambda) = d_{e^{\epsilon_2\Delta}\theta_0}(\lambda). \tag{23}$$

Thus all amplitude information at positive times is inherited from the single smoothed function $e^{\epsilon_2\Delta}\theta_0$; no new high-amplitude tail is created by the approximate dynamics. Lemma 2.1, applied first to the sub-Markov heat semigroup and then to the measure-preserving flow, gives for every $M \ge 0$

$$\sup_{\epsilon_1,\epsilon_2>0} \sup_{t\ge0} \tau_M(\theta^{\epsilon_1,\epsilon_2}(t)) \le \tau_M(\theta_0). \tag{24}$$

Taking $M = 0$ gives

$$\sup_{\epsilon_1,\epsilon_2>0} \sup_{t\ge0} \|\theta^{\epsilon_1,\epsilon_2}(t)\|_X \le \|\theta_0\|_X, \tag{25}$$

and hence, by Proposition 2.1,

$$\sup_{\epsilon_1,\epsilon_2>0} \sup_{t\ge0} \|\theta^{\epsilon_1,\epsilon_2}(t)\|_{\dot H^{-1/2}} \lesssim \|\theta_0\|_X. \tag{26}$$

The constants in (24)–(26) are independent of time and of both regularization parameters. This uniformity is what permits a diagonal limit on every compact time interval.

LEMMA 3.1. *Let $\Omega = \mathbb{R}^2$ and $0 < \epsilon_1, \epsilon_2 \le 1$. For every $T > 0$,*

$$\lim_{R\to\infty} \sup_{0<\epsilon_1,\epsilon_2\le 1} \sup_{0\le t\le T} \|\theta^{\epsilon_1,\epsilon_2}(t)\mathbf{1}_{\{|x|>R\}}\|_{L^{4/3,2}} = 0. \tag{27}$$

PROOF. We write $X = L^{4/3,2}(\mathbb{R}^2)$ and abbreviate $\theta^\epsilon = \theta^{\epsilon_1,\epsilon_2}$, $u^\epsilon = u^{\epsilon_1,\epsilon_2}$, and $f^{\epsilon_2} = e^{\epsilon_2\Delta}\theta_0$. The regularized Riesz transforms are uniformly bounded on X, so (25) gives

$$\sup_{\epsilon_1,\epsilon_2,t} \|u^{\epsilon_1,\epsilon_2}(t)\|_X \le U. \tag{28}$$

Indeed, their multipliers are the Riesz multipliers times $q_\epsilon(r) = c\int_{\epsilon r^2}^\infty e^{-s}s^{-1/2}\,ds$. For $j = 0, 1, 2$, we have

$$\sup_{\epsilon>0}\sup_{r>0} r^j|\partial_r^j q_\epsilon(r)| \le C_j.$$

The Mikhlin multiplier theorem and the fact that $L^{p,q}$ is an interpolation space give (28). Strong continuity of the heat semigroup on X makes $\{e^{s\Delta}\theta_0 : 0 \le s \le 1\}$ compact in X. Since X has order-continuous norm,

$$\alpha(R_0) := \sup_{0\le s\le 1} \|e^{s\Delta}\theta_0\mathbf{1}_{\{|x|>R_0\}}\|_X \longrightarrow 0. \tag{29}$$

Indeed, multiplication by $\mathbf{1}_{\{|x|>R_0\}}$ is uniformly bounded on X and converges strongly to zero as $R_0 \to \infty$; this convergence is uniform on compact subsets of X by a finite-net argument. Let $X^\epsilon(t, a)$ be the measure-preserving flow. Rearrangement invariance and the flow formula yield

$$\|\theta^\epsilon(t)\mathbf{1}_{\{|x|>R\}}\|_X = \|f^{\epsilon_2}(a)\mathbf{1}_{\{|X^\epsilon(t,a)|>R\}}\|_X.$$

We fix $M, R_0 > 0$ and split the last function according to $|a| > R_0$, $|f^{\epsilon_2}(a)| > 2M$, and the remaining set

$$E = \{a : |a| \le R_0,\ |f^{\epsilon_2}(a)| \le 2M, |X^\epsilon(t, a)| > R\}.$$

The first two pieces are bounded by $\alpha(R_0)$ and $2\tau_M(\theta_0)$, respectively, using (24). Now every $a \in E$ has moved a distance of at least $R - R_0$. Measure preservation, Lorentz Hölder, and (28) give

$$(R - R_0)|E| \le \int_E |X^\epsilon(t, a) - a|\,da \le \int_0^t \int_{X^\epsilon(s,E)} |u^\epsilon(s, x)|\,dx\,ds \le CTU|E|^{1/4}.$$

Here $X^\epsilon(s, E) = \{X^\epsilon(s, a) : a \in E\}$; it has measure $|E|$ because the flow is measure preserving. We used the duality $(L^{\frac{4}{3},2})' = L^{4,2}$ and the fact that for any measurable set

$$\|\mathbf{1}_A\|_{L^{p,q}} = |A|^{\frac{1}{p}}$$

holds for $1 < p < \infty$, $1 \le q \le \infty$. Thus $|E|^{3/4} \le CTU/D$. Since $\|\mathbf{1}_E\|_X \simeq |E|^{3/4}$, the remaining piece has norm at most $CMTU/(R - R_0)$. We have proved

$$\sup_{0<\epsilon_1,\epsilon_2\le 1} \sup_{0\le t\le T} \|\theta^\epsilon(t)\mathbf{1}_{\{|x|>R\}}\|_X \le \alpha(R_0) + 2\tau_M(\theta_0) + \frac{CMTU}{R - R_0}. \tag{30}$$

We first choose M large, then R_0 large, and finally R large. This proves (27). □

The preceding estimate also controls the far field. If a bilinear kernel is bounded by $C|x - y|^{-1}$, the contribution in which either variable lies outside B_R is at most

$$C\|\theta\mathbf{1}_{B_R^c}\|_{L^{4/3,2}}\|\theta\|_{L^{4/3,2}}$$

by Proposition 2.1. Thus both the transport form and the Hamiltonian can be localized to $B_R \times B_R$ before the diagonal is removed.

3.2. The regularized Hamiltonian identity. For $f \in X$, set

$$\mathcal{H}_{\epsilon_1}(f) := \frac{1}{2}\int_\Omega f \Lambda_{\epsilon_1}^{-1} f \, dx.$$

This is well defined by Proposition 2.1. Moreover,

$$\langle f, \Lambda_{\epsilon_1}^{-1} g\rangle = c_0 \int_{\epsilon_1}^\infty \langle e^{\rho\Delta/2} f, e^{\rho\Delta/2} g\rangle \rho^{-1/2} \, d\rho,$$

so the form is symmetric and nonnegative. Write

$$\theta^\epsilon = \theta^{\epsilon_1,\epsilon_2}, \qquad \psi^\epsilon = \Lambda_{\epsilon_1}^{-1}\theta^\epsilon, \qquad u^\epsilon = \nabla^\perp \psi^\epsilon.$$

Testing the approximate equation with ψ^ϵ and using self-adjointness gives

$$\frac{d}{dt}\mathcal{H}_{\epsilon_1}(\theta^\epsilon) = -\int_\Omega u^\epsilon \cdot \nabla\theta^\epsilon \, \psi^\epsilon \, dx = \int_\Omega \theta^\epsilon \nabla^\perp\psi^\epsilon \cdot \nabla\psi^\epsilon \, dx = 0.$$

On a bounded domain there is no boundary term because $u^\epsilon \cdot n = 0$ on $\partial\Omega$. On $\mathbb{R}^2$, the same calculation follows by inserting a cutoff: for fixed $\epsilon_1, \epsilon_2 > 0$, one has $\theta^\epsilon, u^\epsilon \in L^2$ and $\psi^\epsilon \in L^\infty$, so the cutoff error tends to zero. Therefore

$$\int_\Omega \theta^{\epsilon_1,\epsilon_2}(t)\Lambda_{\epsilon_1}^{-1}\theta^{\epsilon_1,\epsilon_2}(t)\, dx = \int_\Omega e^{\epsilon_2\Delta}\theta_0 \Lambda_{\epsilon_1}^{-1} e^{\epsilon_2\Delta}\theta_0 \, dx. \tag{31}$$

Finally,

$$\int_\Omega e^{\epsilon_2\Delta}\theta_0\Lambda_{\epsilon_1}^{-1} e^{\epsilon_2\Delta}\theta_0 \, dx \longrightarrow \int_\Omega \theta_0 \Lambda^{-1}\theta_0 \, dx \qquad (\epsilon_1, \epsilon_2 \downarrow 0). \tag{32}$$

Indeed, $e^{\epsilon_2\Delta}\theta_0 \to \theta_0$ in $\dot{H}^{-1/2}(\Omega)$, while the nonnegative quadratic forms associated with $\Lambda_{\epsilon_1}^{-1}$ are bounded by the Λ^{-1} form and increase to it. This proves (32).

4. The symmetrized weak formulation

At critical Lorentz regularity the product $u\theta$ need not be defined. After symmetrization, however, the order-two velocity kernel is paired with a difference of test-function gradients. The resulting kernel is bounded by $C|x-y|^{-1}$ and defines a continuous quadratic form on $L^{4/3,2}$. We prove this first in the plane and then on bounded domains.

4.1. Whole space. On $\mathbb{R}^2$,

$$u(x) = K * \theta(x), \qquad K(x) = c\frac{x^\perp}{|x|^3}.$$

We first perform the calculation for smooth θ with sufficient decay. The estimate obtained below then defines the quadratic term for general $L^{4/3,2}$ data by density and continuity. Let $\varphi \in C_c^\infty(\mathbb{R}^2)$. Integrating the transport term by parts and inserting the kernel representation of u gives

$$\int_{\mathbb{R}^2} u \cdot \nabla\theta \, \varphi \, dx = -\iint_{\mathbb{R}^2\times\mathbb{R}^2} K(x-y)\cdot\nabla\varphi(x)\theta(x)\theta(y)\, dx\, dy.$$

We now interchange x and y in the double integral and average the two representations. Since $K(y-x) = -K(x-y)$, this produces

$$\int_{\mathbb{R}^2} u\cdot\nabla\theta\,\varphi\,dx = -\frac{1}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} K(x-y)\cdot(\nabla\varphi(x) - \nabla\varphi(y))\theta(x)\theta(y)\,dx\,dy. \tag{33}$$

Hence the singularity of the symmetrized kernel is only of order one:

$$\left|\int_{\mathbb{R}^2} u\cdot\nabla\theta\,\varphi\,dx\right| \lesssim \|\nabla^2\varphi\|_{L^\infty}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\frac{|\theta(x)||\theta(y)|}{|x-y|}\,dx\,dy. \tag{34}$$

The remaining double integral is the Riesz-potential quadratic form of $|\theta|$ and is controlled by Proposition 2.1:

$$\iint \frac{|\theta(x)||\theta(y)|}{|x-y|}\,dx\,dy \lesssim \||D|^{-1/2}|\theta|\|_{L^2}^2 \lesssim \|\theta\|_{L^{4/3,2}}^2.$$

Therefore

$$\left|\int_{\mathbb{R}^2} u\cdot\nabla\theta\,\varphi\,dx\right| \lesssim \|\nabla^2\varphi\|_{L^\infty}\|\theta\|^2_{L^{4/3,2}}. \tag{35}$$

More generally, the same calculation and Cauchy–Schwarz for Riesz potentials give

$$\left|\iint_{\mathbb{R}^2\times\mathbb{R}^2} K(x-y)\cdot(\nabla\varphi(x)-\nabla\varphi(y))f(x)g(y)\,dx\,dy\right| \lesssim \|\nabla^2\varphi\|_{L^\infty}\|f\|_{L^{4/3,2}}\|g\|_{L^{4/3,2}}.$$

Consequently, the right-hand side of (33) extends continuously from smooth functions to $L^{4/3,2}$. We use this extension as the nonlinear term.

We use the corresponding estimate for the truncated kernel.

Lemma 4.1. *Let*

$$H(\rho,z)=\frac{1}{4\pi\rho}e^{-|z|^2/(4\rho)},\qquad G_\epsilon(z)=c_0\int_\epsilon^\infty H(\rho,z)\rho^{-1/2}\,d\rho,\qquad K_\epsilon=\nabla^\perp G_\epsilon$$

for $\epsilon\ge 0$ and $z\neq 0$. Then G_ϵ is even, K_ϵ is odd, and, uniformly for $\epsilon\ge 0$,

$$|G_\epsilon(z)|\le\frac{C}{|z|},\qquad |K_\epsilon(z)|\le\frac{C}{|z|^2}. \tag{36}$$

Consequently, for every $\varphi\in C_c^\infty(\mathbb{R}^2)$,

$$\left|K_\epsilon(x-y)\cdot\big(\nabla\varphi(x)-\nabla\varphi(y)\big)\right|\le\frac{C\|\nabla^2\varphi\|_{L^\infty}}{|x-y|}. \tag{37}$$

For every $\eta>0$,

$$\sup_{|z|\ge\eta}\big(|G_\epsilon(z)-G_0(z)|+|K_\epsilon(z)-K_0(z)|\big)\longrightarrow 0\qquad(\epsilon\downarrow 0). \tag{38}$$

Proof. Parity follows from the radial heat kernel. The Gaussian estimates

$$|H(\rho,z)|\le C\rho^{-1}e^{-|z|^2/(C\rho)},\qquad |\nabla H(\rho,z)|\le C|z|\rho^{-2}e^{-|z|^2/(C\rho)}$$

give

$$|G_\epsilon(z)|\le C\int_0^\infty \rho^{-3/2}e^{-|z|^2/(C\rho)}\,d\rho\le C|z|^{-1}$$

and

$$|K_\epsilon(z)|\le C|z|\int_0^\infty \rho^{-5/2}e^{-|z|^2/(C\rho)}\,d\rho\le C|z|^{-2}.$$

The mean-value theorem now proves (37). Finally, G_0-G_ϵ and K_0-K_ϵ are given by the same integrals over $0<\rho<\epsilon$. When $|z|\ge\eta$, their integrands are bounded by an integrable power of ρ times $e^{-c\eta^2/\rho}$, uniformly in z (polynomial factors in $|z|/\sqrt{\rho}$ are absorbed into a slightly weaker Gaussian). The resulting integrals tend to zero as $\epsilon\downarrow 0$, which proves (38). □

4.2. Bounded domains. We assume throughout this subsection that $\Omega\subset\mathbb{R}^2$ is a bounded domain with C^5 boundary, and let $H_D(t,x,y)$ denote its Dirichlet heat kernel. For $\epsilon\ge 0$ we set

$$G_{1,\epsilon}(x,y)=c_0\int_\epsilon^\infty H_D(t,x,y)t^{-1/2}\,dt,\qquad K_\epsilon(x,y)=\nabla_x^\perp G_{1,\epsilon}(x,y). \tag{39}$$

Self-adjointness of the Dirichlet heat semigroup gives $H_D(t,x,y)=H_D(t,y,x)$ and hence $G_{1,\epsilon}(x,y)=G_{1,\epsilon}(y,x)$. In particular, $G_{1,0}=G_1$ is the kernel of Λ^{-1}; we write $K=K_0$. For smooth θ, the velocity therefore has the representation

$$u(x)=\int_\Omega K(x,y)\theta(y)\,dy.$$

If $\varphi\in C_c^\infty(\Omega)$, incompressibility and tangency of u to $\partial\Omega$ give

$$\int_\Omega u\cdot\nabla\theta\,\varphi\,dx=-\iint_{\Omega\times\Omega}K(x,y)\cdot\nabla\varphi(x)\theta(x)\theta(y)\,dx\,dy.$$

Unlike the whole-space kernel, $K(x,y)$ is not translation invariant and is not simply antisymmetric under exchange of its arguments. The correct replacement for oddness is obtained by averaging the expression with its x–y transpose, which gives

$$\int_\Omega u\cdot\nabla\theta\,\varphi\,dx = -\frac{1}{2}\iint_{\Omega\times\Omega}\left[K(x,y)\cdot\nabla\varphi(x)+K(y,x)\cdot\nabla\varphi(y)\right]\theta(x)\theta(y)\,dx\,dy. \tag{40}$$

We write

$$\delta(x)=\operatorname{dist}(x,\partial\Omega).$$

Gaussian and gradient estimates (see, for example, [**6, 11**]) state that, for $0<t\le t_0$,

$$0\le H_D(t,x,y)\le Ct^{-1}\min\left\{1,\frac{\delta(x)}{\sqrt{t}}\right\}\min\left\{1,\frac{\delta(y)}{\sqrt{t}}\right\}\exp\left(-\frac{|x-y|^2}{Ct}\right), \tag{41}$$

$$|\nabla_x H_D(t,x,y)|+|\nabla_y H_D(t,x,y)|\le Ct^{-3/2}\exp\left(-\frac{|x-y|^2}{Ct}\right). \tag{42}$$

We also use the short-time estimate

$$|(\nabla_x+\nabla_y)H_D(t,x,y)|\le Ct^{-3/2}\exp\left(-\frac{\delta(x)^2}{Kt}\right),\qquad 0<t\le c\delta(x)^2. \tag{43}$$

This follows from the pointwise estimates [**7**, (24)–(27)]. By symmetry, the same estimate holds with $\delta(y)$ in place of $\delta(x)$. For $t\ge t_0$, the kernel and its first derivatives decay exponentially by the spectral expansion. Integrating these estimates in (39) gives the global kernel bounds needed below. Recall that ϵ is a heat-time parameter, so its spatial scale is $\sqrt{\epsilon}$.

Corollary 4.1. *Let $r=|x-y|$. Uniformly for $\epsilon\ge 0$ and $x\ne y$ in Ω,*

$$|G_{1,\epsilon}(x,y)|\le\frac{C_\Omega}{r+\sqrt{\epsilon}}, \tag{44}$$

$$|\nabla_x G_{1,\epsilon}(x,y)|+|\nabla_y G_{1,\epsilon}(x,y)|\le\frac{C_\Omega}{(r+\sqrt{\epsilon})^2}. \tag{45}$$

Moreover, for every $\eta>0$,

$$\sup_{\substack{x,y\in\Omega\\ |x-y|\ge\eta}}\Big(|G_{1,\epsilon}(x,y)-G_1(x,y)|+|\nabla_x(G_{1,\epsilon}-G_1)(x,y)| + |\nabla_y(G_{1,\epsilon}-G_1)(x,y)|\Big)\longrightarrow 0 \tag{46}$$

as $\epsilon\downarrow 0$. For every $\eta>0$ and $\varphi\in W^{2,\infty}(\Omega)$, one also has

$$|K_\epsilon(x,y)\cdot\nabla\varphi(x)+K_\epsilon(y,x)\cdot\nabla\varphi(y)|\le\frac{C_{\Omega,\eta}\|\varphi\|_{W^{2,\infty}(\Omega)}}{|x-y|} \tag{47}$$

whenever $x\ne y$ and $\delta(x)+\delta(y)>\eta$, uniformly in $\epsilon\ge 0$.

Proof. Gaussian integration gives

$$\int_\epsilon^{t_0} t^{-3/2}e^{-r^2/(Ct)}\,dt\le\frac{C}{r+\sqrt{\epsilon}},$$
$$\int_\epsilon^{t_0} t^{-2}e^{-r^2/(Ct)}\,dt\le\frac{C}{(r+\sqrt{\epsilon})^2}.$$

The large-time parts are smooth and exponentially decreasing. Integrating (41) and (42) proves (44) and (45). Each difference in (46) is an integral over $0<t<\epsilon$. On $r\ge\eta$, its integrand is bounded by a power of t^{-1} times $\exp(-c\eta^2/t)$, so uniform dominated convergence proves the convergence assertion.

It remains to prove (47). We set $r=|x-y|$. If $r\ge\eta/4$, then (45) gives

$$|K_\epsilon(x,y)\cdot\nabla\varphi(x)+K_\epsilon(y,x)\cdot\nabla\varphi(y)|\le\frac{C_{\Omega,\eta}}{r}\|\nabla\varphi\|_{L^\infty}.$$

Suppose now that $r < \eta/4$. Since the distance function is 1-Lipschitz,

$$\min\{\delta(x), \delta(y)\} = \frac{\delta(x) + \delta(y) - |\delta(x) - \delta(y)|}{2} > \frac{\eta - r}{2} > \frac{3\eta}{8}.$$

The segment joining x and y lies in Ω, and hence

$$|\nabla\varphi(x) - \nabla\varphi(y)| \le r\|\nabla^2\varphi\|_{L^\infty}.$$

Using the symmetry of H_D, we write the expression on the left of (47) as

$$c_0 \int_\epsilon^\infty \Big\{ \nabla_x^\perp H_D(t,x,y) \cdot \big(\nabla\varphi(x) - \nabla\varphi(y)\big) \\ +(\nabla_x^\perp + \nabla_y^\perp) H_D(t,x,y) \cdot \nabla\varphi(y) \Big\}\, t^{-1/2}\, dt.$$

The first term is bounded by $Cr^{-1}\|\nabla^2\varphi\|_{L^\infty}$ using (42). For the second term, (43) controls the short-time integral, while the ordinary gradient bound and the spectral decay control the remaining times. Since $\delta(x), \delta(y) > 3\eta/8$, this gives

$$C_{\Omega,\eta}\|\nabla\varphi\|_{L^\infty} \le \frac{C_{\Omega,\eta}}{r}\|\nabla\varphi\|_{L^\infty},$$

after changing the constant, since $r \le \operatorname{diam}(\Omega)$. This proves (47). □

COROLLARY 4.2. *Let $\varphi \in C_c^\infty(\Omega)$. Then, uniformly for $\epsilon \ge 0$ and $x \ne y$ in Ω,*

$$|K_\epsilon(x,y) \cdot \nabla\varphi(x) + K_\epsilon(y,x) \cdot \nabla\varphi(y)| \le \frac{C_{\Omega,\varphi}}{|x-y|}. \tag{48}$$

PROOF. If $\nabla\varphi \equiv 0$, the assertion is immediate. Otherwise set

$$d_\varphi = \operatorname{dist}(\operatorname{supp}\nabla\varphi, \partial\Omega) > 0, \qquad \eta_\varphi = \frac{d_\varphi}{2}.$$

If $\delta(x)+\delta(y) \le \eta_\varphi$, then $\nabla\varphi(x) = \nabla\varphi(y) = 0$, so the left-hand side vanishes. If $\delta(x)+\delta(y) > \eta_\varphi$, apply Corollary 4.1. This proves (48), with $C_{\Omega,\varphi} = C_{\Omega,\eta_\varphi}\|\varphi\|_{W^{2,\infty}}$. □

Taking $\epsilon = 0$ in Corollary 4.2 proves the asserted bound for (40). Consequently,

$$\left| \int_\Omega u \cdot \nabla\theta\, \varphi\, dx \right| \lesssim_{\Omega,\varphi} \|\theta\|^2_{L^{4/3,2}}. \tag{49}$$

Thus both geometries give the same critical estimate for each fixed compactly supported test function. For $\epsilon \ge 0$, define

$$A_{\varphi,\epsilon}(x,y) := \frac{1}{2}\begin{cases} K_\epsilon(x-y) \cdot \big(\nabla\varphi(x) - \nabla\varphi(y)\big), & \Omega = \mathbb{R}^2, \\ K_\epsilon(x,y) \cdot \nabla\varphi(x) + K_\epsilon(y,x) \cdot \nabla\varphi(y), & \Omega \text{ bounded}, \end{cases}$$

and let

$$B_{\varphi,\epsilon}(f,g) := \iint_{\Omega\times\Omega} A_{\varphi,\epsilon}(x,y) f(x) g(y)\, dx\, dy. \tag{50}$$

The kernel $A_{\varphi,\epsilon}$ is symmetric in (x,y), and the estimates proved above give, uniformly for $0 \le \epsilon \le 1$,

$$|A_{\varphi,\epsilon}(x,y)| \le \frac{C_{\Omega,\varphi}}{|x-y|}, \qquad x \ne y. \tag{51}$$

Moreover, $A_{\varphi,\epsilon} \to A_{\varphi,0}$ uniformly on every set $\{|x-y| \ge \eta\}$, after the natural spatial localization in the whole-space case. Proposition 2.1 therefore shows that $B_{\varphi,\epsilon}$ is a bounded symmetric bilinear form on $X \times X$, with norm independent of ϵ.

At $\epsilon = 0$, (50) agrees with the transport form introduced in (10):

$$B_{\varphi,0}(\theta,\theta) = \mathcal{N}_\Omega(\theta,\varphi), \qquad \langle u \cdot \nabla\theta, \varphi\rangle = -B_{\varphi,0}(\theta,\theta).$$

For time-dependent tests these identities are applied at almost every time and then integrated. Bounds (35) and (49) ensure the required local integrability.

5. Compactness and proof of the main theorem

We now pass to the limit in the approximate solutions. The symmetrized estimate gives time compactness. We then split the quadratic kernels into near- and far-diagonal parts. The tail estimate controls the near part, and the same argument applies to the Hamiltonian.

Fix a sequence

$$(\epsilon_{1,n}, \epsilon_{2,n}) \longrightarrow (0,0), \qquad \theta_n := \theta^{\epsilon_{1,n},\epsilon_{2,n}}.$$

All estimates below are uniform on a fixed interval $[0,T]$, and a diagonal extraction in T is used for $T \to \infty$.

5.1. Compactness and inherited localization. From (35), (49), and the uniform bound (25), every fixed $\zeta \in C_c^\infty(\Omega)$ satisfies

$$\left| \frac{d}{dt} \langle \theta_n(t), \zeta \rangle \right| \le C_\zeta \qquad \text{for a.e. } t, \tag{52}$$

uniformly in n. Hence the scalar pairings are equibounded and equi-Lipschitz. Arzelà–Ascoli, applied to a countable dense family of test functions, and a diagonal extraction give

$$\theta_n \rightharpoonup^* \theta \quad \text{in } L^\infty(0,T;X), \qquad \theta_n \longrightarrow \theta \quad \text{in } C([0,T];\mathcal{D}'(\Omega)). \tag{53}$$

Since X is reflexive, convergence in distributions at a fixed time, together with the uniform X bound, implies

$$\theta_n(t) \rightharpoonup \theta(t) \qquad \text{weakly in } X$$

for every $t \in [0,T]$. In fact, this convergence is uniform in the weak topology. Indeed, $C_c^\infty(\Omega)$ is dense in $X^* = L^{4,2}(\Omega)$, and therefore (53) and the uniform X bound imply

$$\sup_{0\le t\le T} |\langle \theta_n(t) - \theta(t), g \rangle| \longrightarrow 0 \qquad \text{for every } g \in X^*. \tag{54}$$

Because $e^{\epsilon_{2,n}\Delta}\theta_0 \to \theta_0$ in X, the convergence at $t=0$ gives the prescribed initial trace $\theta(0) = \theta_0$.

Weak lower semicontinuity of the convex high amplitude cutoff modulus and (24) yield, for every $M \ge 0$ and $t \in [0,T]$,

$$\tau_M(\theta(t)) \le \liminf_{n\to\infty} \tau_M(\theta_n(t)) \le \tau_M(\theta_0). \tag{55}$$

This proves (12). Applying Proposition 2.2 with $f = \theta_0$ and $h = \theta(t)$ gives (13) with one weight Φ_{θ_0}, independent of time. This step does not require weak lower semicontinuity of the weighted functional.

We also record the spatial localization inherited by the limit. If $\Omega = \mathbb{R}^2$, Lemma 3.1 and weak lower semicontinuity imply

$$\lim_{R\to\infty} \left[\sup_n \sup_{0\le t\le T} \|\theta_n(t)\mathbf{1}_{B_R^c}\|_X + \sup_{0\le t\le T} \|\theta(t)\mathbf{1}_{B_R^c}\|_X \right] = 0. \tag{56}$$

If Ω is bounded, let $E_\rho = \{x \in \Omega : \delta(x) < \rho\}$. For $f = \theta_n(t)$ or $f = \theta(t)$, the decomposition $f = T_M f + R_M f$ gives

$$\|f\mathbf{1}_{E_\rho}\|_X \le CM|E_\rho|^{3/4} + \tau_M(\theta_0). \tag{57}$$

Since $|E_\rho| \to 0$, choosing first M and then ρ makes the right-hand side arbitrarily small, uniformly in n and $t \in [0,T]$.

5.2. Compactness of the quadratic forms. We use the following lemma for both the transport term and the Hamiltonian.

Lemma 5.1. *Let $f_n, f \in L^\infty(0,T;X)$ satisfy*

$$f_n \longrightarrow f \quad \textit{in } C([0,T];X_{\mathrm{w}}), \qquad \sup_n \sup_{0\le t\le T} \|f_n(t)\|_X + \sup_{0\le t\le T} \|f(t)\|_X \le A,$$

and suppose that, for some function $\omega(M) \downarrow 0$,

$$\sup_n \sup_{0\le t\le T} \tau_M(f_n(t)) + \sup_{0\le t\le T} \tau_M(f(t)) \le \omega(M).$$

Fix a compact exhaustion $K_1 \Subset K_2 \Subset \cdots \Subset \Omega$ *with* $\bigcup_m K_m = \Omega$*; when* $\Omega = \mathbb{R}^2$*, we take* $K_m = \overline{B_m}$*. Assume in addition the spatial localization*

$$\lim_{m\to\infty}\left[\sup_n \sup_{0\le t\le T}\|f_n(t)\mathbf{1}_{\Omega\setminus K_m}\|_X + \sup_{0\le t\le T}\|f(t)\mathbf{1}_{\Omega\setminus K_m}\|_X\right] = 0. \tag{58}$$

Let J_n, J *be symmetric kernels that are jointly continuous in* (t,x,y) *away from* $x=y$*, such that*

$$|J_n(t,x,y)| + |J(t,x,y)| \le \frac{C}{|x-y|}, \tag{59}$$

and, for every $K \Subset \Omega$ *and* $\eta > 0$*,*

$$J_n \longrightarrow J \quad \text{uniformly on } [0,T]\times\{(x,y)\in K^2 : |x-y|\ge\eta\}.$$

Then

$$\sup_{0\le t\le T}\left|\iint J_n(t,x,y)f_n(t,x)f_n(t,y)\,dx\,dy - \iint J(t,x,y)f(t,x)f(t,y)\,dx\,dy\right| \longrightarrow 0. \tag{60}$$

Proof. We choose $\chi \in C_c^\infty([0,\infty))$ with $\chi = 1$ on $[0,1]$ and $\chi = 0$ on $[2,\infty)$. We split the kernels using $\chi(|x-y|/\eta)$ and write

$$Q^{\text{near}}_{L,\eta}(h) := \iint \chi\left(\frac{|x-y|}{\eta}\right) L(t,x,y)h(x)h(y)\,dx\,dy,$$

$$Q^{\text{far}}_{L,\eta}(h) := \iint \left[1-\chi\left(\frac{|x-y|}{\eta}\right)\right] L(t,x,y)h(x)h(y)\,dx\,dy.$$

We begin with the far part. If L satisfies (59), then Proposition 2.1 gives

$$\left|Q^{\text{far}}_{L,\eta}(h) - Q^{\text{far}}_{L,\eta}(h\mathbf{1}_K)\right| \le C\|h\|_X\|h\mathbf{1}_{\Omega\setminus K}\|_X.$$

By (58), we may therefore choose $K = K_m$ so that the contribution from $\Omega\setminus K$ is arbitrarily small, uniformly in n and t.

On K, the embedding $X \hookrightarrow L^1(K)$ gives a uniform bound for $\|f_n(t)\|_{L^1(K)}$ and $\|f(t)\|_{L^1(K)}$. The off-diagonal convergence of J_n then gives

$$\sup_{0\le t\le T}\left|Q^{\text{far}}_{J_n,\eta}(f_n(t)\mathbf{1}_K) - Q^{\text{far}}_{J,\eta}(f_n(t)\mathbf{1}_K)\right| \longrightarrow 0.$$

The far kernel is continuous on $[0,T]\times K\times K$. The standard compact-kernel argument, the convergence in $C([0,T];X_{\mathrm{w}})$, and the uniform $L^1(K)$ bound therefore give

$$\sup_{0\le t\le T}\left|Q^{\text{far}}_{J_n,\eta}(f_n(t)) - Q^{\text{far}}_{J,\eta}(f(t))\right| \longrightarrow 0, \tag{61}$$

It remains to control the diagonal uniformly. For any member h of the family $\{f_n(t), f(t)\}$, write

$$|h| = h_{\le M} + h_{>M}, \qquad h_{\le M} = |h|\wedge M, \qquad h_{>M} = (|h|-M)_+.$$

The embedding $X \hookrightarrow L^{4/3,\infty}$ gives

$$\|h_{\le M}\|_{L^2}^2 \le CM^{2/3}A^{4/3}.$$

Young's inequality and $\||z|^{-1}\mathbf{1}_{\{|z|\le 2\eta\}}\|_{L^1(\mathbb{R}^2)} \le C\eta$ therefore imply

$$\iint_{|x-y|\le 2\eta}\frac{h_{\le M}(x)h_{\le M}(y)}{|x-y|}\,dx\,dy \le CM^{2/3}A^{4/3}\eta. \tag{62}$$

The boundedness $\Lambda^{-\frac12} : X \to L^2$ gives

$$\iint \frac{h_{>M}(x)h_{>M}(y)}{|x-y|}\,dx\,dy \le C\omega(M)^2, \tag{63}$$

$$\iint \frac{h_{\le M}(x)h_{>M}(y)}{|x-y|}\,dx\,dy \le CA\omega(M). \tag{64}$$

Combining the three estimates and using (59), we obtain for both the approximating and limiting near-field forms

$$\sup_n \sup_{0\le t\le T} |Q^{\mathrm{near}}_{J_n,\eta}(f_n(t))| + \sup_{0\le t\le T} |Q^{\mathrm{near}}_{J,\eta}(f(t))| \le C\big(M^{2/3}A^{4/3}\eta + A\omega(M) + \omega(M)^2\big). \tag{65}$$

We first choose M so that the last two terms are small and then choose η so that the first term is small. Together with (61), this proves (60). ☐

We apply the lemma to the transport term. Let $\varphi \in C_c^\infty([0,T]\times\Omega)$ and set

$$J_n(t,x,y) = A_{\varphi(t),\epsilon_{1,n}}(x,y), \qquad J(t,x,y) = A_{\varphi(t),0}(x,y).$$

The uniform bound is (51); off-diagonal convergence follows from Lemma 4.1 on $\mathbb{R}^2$ and from Corollary 4.1 on a bounded domain. The high amplitude cutoff hypothesis follows from (24) and (55), while the localization hypothesis is (56) or (57). Hence

$$\sup_{0\le t\le T} \big|B_{\varphi(t),\epsilon_{1,n}}(\theta_n(t),\theta_n(t)) - B_{\varphi(t),0}(\theta(t),\theta(t))\big| \longrightarrow 0. \tag{66}$$

Passing to the limit in the approximate weak formulation proves (11). Thus θ is a global weak solution with initial datum θ_0.

5.3. Compactness of the negative Sobolev Hamiltonian. Weak convergence alone would give only lower semicontinuity of the Hamiltonian. The preceding quadratic-form lemma excludes an energy defect. Let $\mathscr{G}_\epsilon$ denote $G_\epsilon(x-y)$ on $\mathbb{R}^2$ and $G_{1,\epsilon}(x,y)$ on a bounded domain. In either case, $\mathscr{G}_\epsilon$ is symmetric and

$$|\mathscr{G}_\epsilon(x,y)| \le \frac{C_\Omega}{|x-y|}$$

uniformly for $\epsilon \ge 0$. Moreover, $\mathscr{G}_{\epsilon_{1,n}} \to \mathscr{G}_0$ uniformly off the diagonal on compact subsets, by Lemma 4.1 or Corollary 4.1.

We apply Lemma 5.1 with

$$J_n = \mathscr{G}_{\epsilon_{1,n}}, \qquad J = \mathscr{G}_0, \qquad f_n = \theta_n, \qquad f = \theta.$$

All high amplitude cutoff and localization assumptions were established in the preceding subsection. We obtain, for every $T > 0$,

$$\lim_{n\to\infty} \sup_{0\le t\le T} \left| \int_\Omega \theta_n(t)\Lambda^{-1}_{\epsilon_{1,n}}\theta_n(t)\,dx - \int_\Omega \theta(t)\Lambda^{-1}\theta(t)\,dx \right| = 0. \tag{67}$$

5.4. Conservation of the negative Sobolev Hamiltonian. The regularized energy identity (31) gives

$$\int_\Omega \theta_n(t)\Lambda^{-1}_{\epsilon_{1,n}}\theta_n(t)\,dx = \int_\Omega e^{\epsilon_{2,n}\Delta}\theta_0\Lambda^{-1}_{\epsilon_{1,n}}e^{\epsilon_{2,n}\Delta}\theta_0\,dx.$$

By (32), the right-hand side converges to $\int_\Omega \theta_0\Lambda^{-1}\theta_0\,dx$. On the other hand, (67) identifies the limit of the left-hand side, uniformly for $t\in[0,T]$. Therefore

$$\int_\Omega \theta(t)\Lambda^{-1}\theta(t)\,dx = \int_\Omega \theta_0\Lambda^{-1}\theta_0\,dx \qquad (0\le t\le T).$$

Since $T>0$ is arbitrary, this identity holds for every $t \ge 0$. Equivalently,

$$\|\theta(t)\|_{\dot H^{-1/2}(\Omega)} = \|\theta_0\|_{\dot H^{-1/2}(\Omega)}, \qquad t \ge 0.$$

This completes the proof of Theorem 1.1.

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Appendix A. Optimality of the secondary Lorentz exponent

We show that the secondary Lorentz index in Proposition 2.1 cannot be increased while the principal exponent remains 4/3. The examples are radial, nonnegative, and supported near the origin. We first consider $q = \infty$ and then add logarithmic factors for $2 < q < \infty$.

Proof of the sharpness statement in Proposition 2.1. For $q = \infty$, take

$$f(x) = \mathbf{1}_{\{|x|<1/100\}}|x|^{-3/2}.$$

Then $f \in L^{4/3,\infty}(\mathbb{R}^2)$, whereas the annulus estimate used below gives $I_{1/2}f(x) \gtrsim |x|^{-1}$ near the origin, and hence $I_{1/2}f \notin L^2$. It remains to treat $2 < q < \infty$.

We fix $2 < q < \infty$ and choose $a > 1/q$. We write $L(r) = \log(1/r)$ and choose $r_0 \in (0, e^{-e})$ sufficiently small that the radial function

$$f(x) = \mathbf{1}_{\{|x|<r_0\}}|x|^{-3/2}L(|x|)^{-1/q}\big(\log L(|x|)\big)^{-a}$$

is radially decreasing on its support. For a radial decreasing function in two dimensions, the rearrangement formula for Lorentz norms gives

$$\|f\|^q_{L^{4/3,q}} \simeq \int_0^{r_0} \left[r^{3/2}f(r)\right]^q \frac{dr}{r} = \int_{L(r_0)}^{\infty} \frac{dL}{L(\log L)^{aq}} < \infty,$$

because $aq > 1$. Thus $f \in L^{4/3,q}(\mathbb{R}^2)$.

We next show that $I_{1/2}f \notin L^2$. Let $|x| = r < r_0/4$ and restrict the potential integral to $2r < |y| < 4r$. On this annulus, $|x - y| \simeq r$, and the explicit radial profile gives

$$f(y) \simeq r^{-3/2}L(r)^{-1/q}(\log L(r))^{-a}.$$

Since the annulus has area comparable to r^2 and $f \geq 0$, it follows that

$$I_{1/2}f(x) = c_2 \int_{\mathbb{R}^2} \frac{f(y)}{|x-y|^{3/2}}\,dy \gtrsim r^{-1}L(r)^{-1/q}(\log L(r))^{-a}.$$

Consequently,

$$\|I_{1/2}f\|^2_{L^2} \gtrsim \int_0^{r_0/4} \frac{dr}{rL(r)^{2/q}(\log L(r))^{2a}} = \int_{L(r_0/4)}^{\infty} \frac{dL}{L^{2/q}(\log L)^{2a}} = \infty,$$

because $2/q < 1$. Hence $f \in L^{4/3,q}(\mathbb{R}^2)$ but $|D|^{-1/2}f \notin L^2(\mathbb{R}^2)$, and the asserted estimate cannot hold for any $q > 2$. □

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