The wave equation

Intoduction to PDE

1 The Wave Equation in one dimension

The equation is

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0. \tag{1}$$

Setting $\xi_1 = x + ct$, $\xi_2 = x - ct$ and looking at the function $v(\xi_1, \xi_2) = u\left(\frac{\xi_1+\xi_2}{2}, \frac{\xi_1-\xi_2}{2c}\right)$, we see that if u satisfies (1) then v satisfies

$$\partial_{\xi_1}\partial_{\xi_2} v = 0.$$

The "general" solution of this equation is

$$v = f(\xi_1) + g(\xi_2)$$

with f, g arbitrary functions. In other words, the "general" solution of (1) is

$$u(x,t) = f(x+ct) + g(x-ct) \tag{2}$$

The two functions $w_1 = f(x+ct)$ and $w_2 = g(x-ct)$ solve

$$\partial_t w_1 - c\partial_x w_1 = 0$$

and

$$\partial_t w_2 + c\partial_x w_2 = 0$$

describing propagation to the left (in positive time) and propagation to the right of a wave of arbitrary constant shape. The solution to the initial value problem

$$\begin{cases} u(x,0) = u_0(x) \\ \partial_t u(x,0) = u_1(x) \end{cases} \tag{3}$$

is obtained from the equations $u(x,t) = f(x+ct) + g(x-ct)$ and $\partial_t u(x,t) = cf'(x+ct) - cg'(x-ct)$ by setting $t = 0$. We obtain

$$\begin{cases} u_0 = f + g \\ \frac{1}{c} u_1 = f' - g' \end{cases}$$

and so, differentiating the first equation, solving and then integrating, we obtain

$$\begin{cases} f = \frac{1}{2}(u_0 + \frac{1}{c} U_1) + \text{ constant} \\ g = \frac{1}{2}(u_0 - \frac{1}{c} U_1) + \text{ constant} \end{cases}$$

where U_1 is some primitive of u_1. Returning to the formula for u we obtain

$$u(x,t) = \frac{1}{2}\left(u_0(x+ct) + u_0(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(\xi) d\xi \tag{4}$$

If u_0, u_1 have compact support K then at any $T > 0$ the solution $u(x,t)$ will have compact support K_T formed with the union of the points on the rays $x \pm ct =$ constant starting at $t = 0$ in K and arriving at T in K_T. The signal travels with finite speed c. A point lying at distance larger than cT from K cannot be reached by the signal in less than T time.

2 The wave equation in $\mathbb{R}^n$

The equation is

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0 \tag{5}$$

with $c > 0$. We prescribe

$$\begin{cases} u(x,0) = f(x) \\ \frac{\partial u}{\partial t}(x,0) = g(x) \end{cases} \tag{6}$$

We use the method of spherical means due to Poisson. For a function $h = h(x_1, \dots, x_n)$ we define $M_h(x,r)$ by

$$M_h(x,r) := \frac{1}{\omega_n r^{n-1}} \int_{|y-x|=r} h(y) dS(y) \tag{7}$$

Clearly

$$M_h(x,r) = \frac{1}{\omega_n} \int_{|\xi|=1} h(x + r\xi) dS(\xi)$$

We can use this formula to extend the definition of $M_h(x,r)$ for all $r \in \mathbb{R}$. Note that for fixed x this is an even function of r,

$$M_h(x,-r) = M_h(x,r)$$

and also

$$M_h(x,0) = h(x)$$

if h is continuous. Let us compute $\partial_r M_h$:

$$\partial_r M_h = \fint_{\mathbb{S}^{n-1}} \xi \cdot \nabla h dS = r^{-1} \fint_{\mathbb{S}^{n-1}} \xi \cdot \nabla_\xi (h(x+r\xi)) dS$$

The vector ξ is the external normal to the unit ball. Thus, using the Gauss formula

$$\begin{array}{c} \partial_r M_h(x,r) = \frac{1}{\omega_n r} \int_{|\xi|<1} \Delta_\xi h(x+r\xi) d\xi = \frac{r}{\omega_n} \int_{|\xi|<1} \Delta_x h(x+r\xi) dx \\ = \frac{r}{\omega_n} \Delta_x \int_{|\xi|<1} h(x+r\xi) d\xi = \Delta_x \frac{r^{1-n}}{\omega_n} \int_{|x-y|<1} h(y) dy \\ = \Delta_x \frac{r^{1-n}}{\omega_n} \int_0^r d\rho \int_{|y-x|=\rho} h(y) dS(y) = \Delta_x \left[r^{1-n} \int_0^r \rho^{n-1} M_h(x,\rho) d\rho \right] \\ = r^{1-n} \int_0^r \rho^{n-1} \Delta_x M_h(x,\rho) d\rho \end{array}$$

Multiplying by r^{n-1} and differentiating we obtain

$$\partial_r \left(r^{n-1} \partial_r M_h(x,r) \right) = r^{n-1} \Delta_x M_h(x,r)$$

which is the Darboux equation:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} \right) M_h(x,r) = \Delta_x M_h(x,r) \tag{8}$$

Note that the left hand side is the radial part of the Laplacian. Moreover, since

$$\partial_r M_h = \frac{r}{\omega_n} \int_{|\xi|<1} \Delta_x h(x+r\xi) d\xi$$

we have that

$$\partial_r M_h(x,0) = 0 \tag{9}$$

If $u(x,t)$ solves (5) then

$$(\Delta_x u)(x+r\xi,t) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} u(x+r\xi,t)$$

and so

$$\left(\Delta_x M_u\right)(x,t,r) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} M_u(x,t,r)$$

On the other hand, from the Darboux equation

$$\left(\Delta_x M_u\right)(x,t,r) = \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right) M_u$$

Thus, the spherical means of a solution of the wave equation solve the spherically symmetric wave equation

$$\left[\frac{\partial^2}{\partial t^2} - c^2\left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right)\right] M_u = 0 \tag{10}$$

We will continue the calculation now for $n = 3$ where the ansatz $a(r) = \frac{b(r)}{r}$ transforms the Laplacian

$$\left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right) a = \frac{1}{r}\partial_r^2 b$$

In general, the trick $a = r^{\frac{1-n}{2}} b$ gets rid of the singular term $\frac{n-1}{r}\partial_r b$ but introduces the singular term $\frac{(n-1)(n-3)}{r^2} b$. Thus, writing

$$M_u(x,t,r) = \frac{1}{r} N_u(x,t,r)$$

we obtain that the function $N_r(x,t,u)$ satisfies the one dimensional wave equation

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial r^2}\right) N_u(x,t,r) = 0 \tag{11}$$

Now

$$N_u(x,0,r) = rM_u(x,0,r) = rM_f(x,r) = N_f(x,r)$$

and

$$\partial_t N_u(x,0,r) = r\partial_t M_u(x,0,r) = rM_g(x,r) = N_g(x,r)$$

It follows that

$$N_u(x,t,r) = \frac{1}{2}\left(N_f(x,r+ct) + N_f(x,r-ct)\right) + \frac{1}{2c}\int_{r-ct}^{r+ct} N_g(x,\xi)d\xi$$

Dividing by r we obtain

$$M_u(x,t,r) = \frac{1}{2r}\left((r+ct)M_f(x,r+ct) + (r-ct)M_f(x,r-ct)\right) + \frac{1}{2cr}\int_{r-ct}^{r+ct} \xi M_g(x,\xi)d\xi$$

Now M_f and M_g are even, and therefore it follows that

$$M_u(x,t,r) = \frac{(r+ct)M_f(x,r+ct)-(ct-r)M_f(x,ct-r)}{2r} + \frac{1}{2cr}\int_{ct-r}^{r+ct} \xi M_g(x,\xi)d\xi \tag{12}$$

Recalling that $M_u(x,t,0) = u(x,t)$, we let $r \to 0$

$$\begin{aligned} M_u(x,t,0) &= \frac{d}{dr}((r+ct)M_f(x,r+ct))_{|r=0} + \frac{(r+ct)M_g(x,r+ct)}{c}_{|r=0} \\ &= \partial_t\left(tM_f(x,ct)\right) + tM_g(x,ct) \\ &= \frac{1}{4\pi c^2 t}\int_{|y-x|=ct} g(y)dS(y) + \partial_t\left(\frac{1}{4\pi c^2 t}\int_{|y-x|=ct} f(y)dS(y)\right) \end{aligned}$$

Carrying out the differentiation we have

$$\begin{aligned} u(x,t) &= \frac{1}{4\pi c^2 t}\int_{|y-x|=ct} g(y)dS(y) + \partial_t\left(\frac{1}{4\pi c^2 t}\int_{|y-x|=ct} f(y)dS(y)\right) \\ &= \frac{1}{4\pi c^2 t}\int_{|y-x|=ct} g(y)dS(y) + \frac{1}{4\pi c^2 t^2}\int_{|y-x|=ct}\left(f(y) + (y-x)\cdot\nabla f(y)\right)dS(y) \end{aligned} \tag{13}$$

The formula (13) gives the unique solution of the problem (5), (6). The domain of inflence of a point is the set in space-time where a disturbance which was initially situated at the point can occur. We see that if the disturbance was initially ($t=0$) located at $x \in \mathbb{R}^3$ then it will propagate on a spherical front $S(x,ct) = \{y \in \mathbb{R}^3 |\ |x-y| = ct\}$. The union of these spheres is the future-oriented light cone

$$\Gamma_x^+ = \{(y,t)|\ |x-y| = ct,\ t > 0\}$$

The domain of influence is the light cone. This is the (strong) Huyghens principle. If you sit at a fixed location in $\mathbb{R}^3$, and wait, the evolution is an inflating sphere of radius ct, and after a while the signal passes. This is why we can communicate (sound, light) in 3d.

Note that there is a loss of regularity because the solution sees derivatives of f. In order to make this clear, let us note that if the initial data are spherically symmetric, then the solution is spherically symmetric as well (the equation commutes with rotations). Note that if h is spherically symmetric then $h(y) = M_h(0, |y|)$. Denoting $|y| = \rho$, looking at (12) and reading at $x = 0$, $r = \rho$ we obtain

$$u(\rho, t) = \frac{(\rho + ct)f(\rho + ct) + (\rho - ct)f(\rho - ct)}{2\rho} + \frac{1}{2c\rho}\int_{\rho - ct}^{\rho + ct} \xi g(\xi) d\xi$$

If $g = 0$ and f is continuous but is not differentiable, like for instance $f(x) = \sqrt{1 - |x|^2}$ for $|x| \le 1$, $f(x) = 0$ for $|x| > 1$, then the solution $u(\rho, t)$ becomes infinite at $\rho = 0$ at time $t = c^{-1}$. We have a linear focusing effect in L^∞. In L^2 we have conservation. Let

$$E(t) = \int_{\mathbb{R}^3} (|\nabla_x u|^2 + (\partial_t u)^2) dx$$

Then

$$\frac{d}{dt} E(t) = 0$$

We took $c = 1$. The energy balance is true in any dimension. Indeed,

$$\frac{d}{2dt} E(t) = \int_{\mathbb{R}^3} u \Box u dx$$

where

$$\Box u = (\partial_t^2 - \Delta) u$$

is the d'Alembertian.

3 Hadamard's method of descent

This method finds solutions of a PDE by considering them as particular solutions of equations in more variables. We seek solutions of

$$\partial_t^2 u - c^2(\partial_{x_1}^2 + \partial_{x_2}^2) u = 0 \tag{14}$$

as solutions of the 3D wave equation in $x_3 = 0$. We assume thus that $f(x_1, x_2, x_3) = f(x_1, x_2, 0)$ and $g(x_1, gx_2, x_3) = g(x_1, x_2, 0)$ with a slight abuse of notation. We use (13) at $x_3 = 0$ noting that

$$\{y | \, |y - x| = ct\} = \{y | \, (y_1 - x_1)^2 + (y_2 - x_2)^2 + y_3^2 = c^2 t^2\}$$

and that

$$dS(y) = (\cos \alpha)^{-1} dy_1 dy_2$$

where

$$\cos\alpha = \frac{|y_3|}{ct}.$$

The integral $\int_{|y-x|=ct} g(y_1, y_2, y_3)dS(y)$ is clearly equal to $2\int_{|y-x|=ct,\, y_3>0} gdS$ because g does not depend on y_3. Introducing

$$\rho = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$$

the integral becomes

$$\int_{|y-x|=ct} g(y)dS = 2ct \int_{D_x(ct)} g(y_1, y_2)\frac{1}{\sqrt{c^2t^2 - \rho^2}}dy_1dy_2$$

where

$$D_x(ct) = \{(y_1, y_2)|\ \rho < ct\}$$

The formula (13) becomes

$$\begin{array}{c} u(x_1, x_2, t) = \frac{1}{2\pi c}\iint_{\rho<ct} \frac{g(y_1,y_2)}{\sqrt{c^2t^2-\rho^2}}dy_1dy_2 \\ +\partial_t\left(\frac{1}{2\pi c}\iint_{\rho<ct} \frac{f(y_1,y_2)}{\sqrt{c^2t^2-\rho^2}}dy_1dy_2\right) \end{array} \tag{15}$$

We see that in $n = 2$ the strong Huyghens principle does not hold. The domain of influence of a point is the solid disk $D_x(ct)$. Once received, the signal never dies: communication is impossible in 2d.

4 The inhomogeneous Cauchy problem

We wish to solve

$$\Box u = F \tag{16}$$

with initial conditions given by (6). Because of linearity of the equation, we can solve separately the initial value and the forcing:

$$u(x, t) = u_1(x, t) + u_2(x, t)$$

with

$$\Box u_1 = 0, \quad u(x, 0) = f(x),\ \partial_t u(x, 0) = g(x)$$

and

$$\Box u_2 = F, \quad u(x, 0) = 0,\ \partial_t u(x, 0) = 0.$$

Thus, WLOG $f = g = 0$. Let us denote by $W_F(x,t,s)$ the solution of the homogeneous problem,

$$\begin{cases} \Box w = 0 \\ w(x,s) = 0, \quad \partial_t w(x,s) = F(x,s) \end{cases} \tag{17}$$

In $n = 3$ we have

$$W_F(x,t,s) = \frac{1}{4\pi c^2(t-s)} \int_{|y-x|=c(t-s)} F(y,s)dS(y) \tag{18}$$

for $t > s$. Now we claim that the solution of (16) is given by

$$u(x,t) = \int_0^t W_F(x,t,s)ds \tag{19}$$

This is Duhamel's priciple. It is the analogue of Laplace's "variation of constants" formula from ODEs. Let us check that it works.

$$\partial_t u(x,t) = W_F(x,t,t-0) + \int_0^t \partial_t W_F(x,t,s)ds = \int_0^t \partial_t W_F(x,t,s)ds$$

because $W_F(x,t,t-0) = 0$. Then

$$\partial_t^2 u = \partial_t W_F(x,t,t-0) + \int_0^t \frac{\partial^2 W_F}{\partial t^2}(x,t,s)ds$$

Now

$$\Delta_x u = \int_0^t (\Delta_x W_F)(x,t,s)ds$$

and because

$$\partial_t W_F(x,t,t-0) = F(x,t)$$

we verified that u solves the inhomogeneous equation.

5 Decay of solutions

We discuss solutions of the homogeneous wave equation

$$\Box u = 0 \tag{20}$$

in $\mathbb{R}^n$ with initial data (6). From the representation (13) it is clear that, when $n = 3$ and f, g have compact support, say $f \in C_0^1(B(0,R))$, $g \in C_0(B(0,R))$ for some $R > 0$, then it follows that

$$|u(x,t)| \le Ct^{-1}$$

for large t, uniformly for all $(x,t) \in \mathbb{R}^4$. Indeed, the term involving g is bounded by

$$t^{-1}\left|\int_{|x-y|=t} g(y)dS\right| \le Ct^{-1}H^2(\{y|\,|x-y|=t, |y|<R\})$$

where $H^2(S)$ is the two dimensional Hausdorff measure (area in our case). Because the area of any part of a sphere inside $B(0,R)$ is bounded by a constant depending only on R, this term is done. Similarly, the second term is bounded

$$\begin{array}{c} t^{-2}\left|\int_{|x-y|=t}(f(y)+(y-x)\cdot\nabla f(y))dS(y)\right| \\ \le Ct^{-1}\left(\frac{|x|}{t}+t^{-1}\right)H^2(\{y|\,|x-y|=t, |y|<R\}) \end{array}$$

If $\frac{|x|}{t} \le 2$ we are done, and if $\frac{|x|}{t} \ge 2$ then $|y| \ge |x|-t \ge t$, so $H^2(\{y|\,|x-y| = t, |y| < R\}) = 0$ for $t > R$. We see also that the decay is sharp: we can choose $g \ge 0$, $g > 1$ on some fixed region and $f = 0$ and then there are nontrivial contributions of the order Ct^{-1}. (Of course, we cannot fix x, that would contradict the strong Huyghens principle; we are talking about *uniform* bounds, i.e. we are allowed to move x around). We present now two approaches to prove the same result for general dimensions, namely

$$|u(x,t)| \le Ct^{-\frac{n-1}{2}}$$

for t large, uniformly in $\mathbb{R}^{n+1}$. The first appoach is based on ocillatory integrals; the second, due to Klainerman, uses the symmetries of the wave equation.

5.1 Oscillatory integrals

For the first approach, we write

$$u(x,t) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\left[\sin(t|\xi|)|\xi|^{-1}\widehat{g}(\xi)+\cos(t|\xi|)\widehat{f}(\xi)\right]d\xi$$

which we rewrite as

$$u(x,t) = u_+(x,t) + u_-(x,t)$$

with

$$u_\pm(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x\cdot\xi \pm t|\xi|)} A_\pm(\xi) d\xi$$

and with

$$A_\pm(\xi) = \frac{1}{2}\left(\widehat{f}(\xi) \pm i^{-1}|\xi|^{-1}\widehat{g}(\xi)\right)$$

Both terms $u_\pm$ decay. Let us look at u_+, the other term is treated the same way. Without loss of generality we may take coordinates aligned with the direction of x, so, without loss of generality we may assume

$$x = Re_n$$

where $e_n = (0,0,\ldots,1)$, $R = |x|$. Thus

$$u_+(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(R\xi_n + t|\xi|)} A_+(\xi) d\xi$$

Now we take a small number ϵ and take a smooth cutoff on the unit sphere $\chi\left(\frac{\xi}{|\xi|}\right)$ supported in $|\xi_n|^2 \geq (1-\epsilon^2)|\xi|^2$. We write

$$u_+(x,t) = a(x,t) + b(x,t)$$

where

$$a(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(R\xi_n + t|\xi|)} \alpha(\xi) d\xi$$

with

$$\alpha(\xi) = A_+(\xi)\chi\left(\frac{\xi}{|\xi|}\right)$$

and

$$b(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(R\xi_n + t|\xi|)} \beta(\xi) d\xi$$

with

$$\beta(\xi) = A_+(\xi)\left[1 - \chi\left(\frac{\xi}{|\xi|}\right)\right]$$

Now on the support of β we have that $\rho = \sqrt{\xi_1^2 + \dots \xi_{n-1}^2}$ satisfies $\rho \geq \epsilon|\xi|$. In the integral representing b we write

$$e^{it|\xi|} = \frac{|\xi|}{it\rho}\frac{d}{d\rho}e^{it|\xi|} = \left[\frac{|\xi|}{it\rho}\frac{d}{d\rho}\right]^N e^{it|\xi|} = L^N(e^{it|\xi|}),$$

where $L = \frac{|\xi|}{it\rho}\frac{d}{d\rho}$ and N is arbitrary. We integrate by parts, taking advantage of the fact that $e^{iR\xi_n}$ does not depend on ρ:

$$b(x,t) = (2\pi)^{-n}(-1)^N \int_{\mathbb{R}^n} e^{i(R\xi_n + t|\xi|)}(L^*)^N(\beta(\xi))d\xi$$

We assume that f, g are smooth and decay fast enough. Then $A_\pm(\xi)$ are smooth and decay fast enough. In order to have this behavior for small ξ as well, we need to assume that $\widehat{g}(\xi) = O(|\xi|)$, for instance that $g = Dg_1$ with D some directional derivative and g_1 smooth and decaying at infinity. Then we can assure that on the support of β we have

$$|(L^*)^N(\beta)(\xi)| \leq (t|\xi|)^{-N} C_{N,\epsilon} \sum_{|j|\leq N} |\partial_\xi^j A_+(\xi)|$$

with $C_{N,\epsilon}$ depending on ϵ, N and χ. We have thus

$$|b(x,t)| \leq C_{N,\epsilon} t^{-N} \int_{\mathbb{R}^n} |\xi|^{-N} \sum_{|j|\leq N} |\partial_\xi^j A_+(\xi)| d\xi$$

This integral converges if $N \leq n-1$ (because of the behavior near zero; if we excise a region near zero, then we have arbitrary decay), so

$$|b(x,t)| \leq Ct^{-n+1}$$

a better decay than $t^{-\frac{n-1}{2}}$. Let us consider now $a(x,t)$. We denote

$$\lambda = \frac{R}{t}, \quad k = |\xi|.$$

and write the phase as

$$R\xi_n + t|\xi| = tk(\lambda\cos\theta + 1)$$

where

$$\cos\theta = \frac{\xi_n}{|\xi|}$$

We introduce polar coordinates and have

$$a(x,t) = \\ (2\pi)^{-n}\int_0^\infty \int_{\mathbb{S}^{n-2}} \int_0^\pi e^{ikt(\lambda\cos\theta+1)}\alpha(k,\theta,\omega)(\sin\theta)^{n-2}k^{n-1}d\theta dk dS_{n-2}(\omega)$$

On the support of α we have $|\cos\theta| \geq \sqrt{1-\epsilon^2}$, so it is close to 1. We disitinguish two situations: if the phase is bounded away from zero, or if the phase could vanish. The first situation occurs if $\cos\theta > 0$ (and hence it is about 1) or if $\cos\theta < 0$ (and hence it is about -1) and $\lambda < 1-\epsilon$ or if $\cos\theta < 0$, and $\lambda > (1-2\epsilon^2)^{-1}$. In these situations we write

$$e^{ikt(\lambda\cos\theta+1)} = \frac{1}{it(\lambda\cos\theta+1)}\frac{d}{dk}e^{ikt(\lambda\cos\theta+1)} \\ = \left[\frac{1}{it(\lambda\cos\theta+1)}\frac{d}{dk}\right]^N e^{ikt(\lambda\cos\theta+1)} = L^N(e^{ikt(\lambda\cos\theta+1)})$$

After cutting off and integrating by parts, the resulting expression decays faster than any power of t. We are left with the behavior of

$$a_1(x,t) = \\ (2\pi)^{-n}\int_0^\infty k^{n-1}dk \int_{\mathbb{S}^{n-2}} dS_{n-2}(\omega)\int_{\pi-\delta}^{\pi} e^{ikt(\lambda\cos\theta+1)}\alpha(k,\theta,\omega)(\sin\theta)^{n-2}d\theta$$

for $\delta = \delta(\epsilon)$ small and with $1-\epsilon \leq \lambda \leq (1-2\epsilon^2)^{-1}$. Because λ is close to 1, we write

$$a_1(x,t) = \\ (2\pi)^{-n}\int_0^\infty e^{ikt(1-\lambda)}k^{n-1}dk \int_{\mathbb{S}^{n-2}} dS(\omega)\int_{\pi-\delta}^{\pi} e^{ik\lambda t(\cos\theta+1)}\alpha(k,\theta,\omega)(\sin\theta)^{n-2}d\theta$$

Changing variables to $\frac{z^2}{2} = 1+\cos\theta$, the inner integral is of the form

$$\int_{-\gamma}^{\gamma} e^{ik\lambda t\frac{z^2}{2}}\alpha_1(k,z,\omega)z^{n-2}dz$$

with $\gamma = \sqrt{2}(1+\sqrt{1+\epsilon^2})^{-1}\epsilon$ and for an appropriate α_1. Thus

$$\left|\int_{-\gamma}^{\gamma} e^{ik\lambda t\frac{z^2}{2}}\alpha_1(k,z,\omega)z^{n-2}dz\right| \leq C(k\lambda t)^{\frac{1-n}{2}}$$

holds with C depending only on n,ϵ,χ a and angular derivatives of $A_+(\xi)$.

5.2 Method of commuting vector fields of Klainerman

We start by the observation that if f is a radial function in $\mathbb{R}^n$, i.e. $f(x) = F(|x|)$, then

$$f(x)^2 = -2\int_{|x|}^{\infty} F(r)F'(r)dr \leq C|x|^{-(n-1)}\|f\|_{L^2(\mathbb{R}^n)}\|\nabla f\|_{L^2(\mathbb{R}^n)}.$$

Radial functions in $H^1(\mathbb{R}^n)$ decay like $|x|^{-\frac{n-1}{2}}$. Of course, we can't expect this to be true for all functions in $H^1(\mathbb{R}^n)$. Radial functions belong to the kernel of the angular momentum operators

$$\Omega_{ij} = x_i\partial_j - x_j\partial_i.$$

It turns out that a general "decay" estimate, of the kind obeyed by the radial functions exists, provided an account is made of the size of the momentum operators. Let $\mathcal{A}$ be a Lie algebra of first order operators with smooth coefficients (vector fields), finitely generated and let $X_1, \dots X_N$ be generators. We use the notation

$$|u(x)|_{\mathcal{A},k} = \sum_{l=0}^{k}\left(\sum_{i_1,\dots i_l=1}^{N}|X_{i_i}X_{i_2}\dots X_{i_l}u(x)|^2\right)^{\frac{1}{2}} \tag{21}$$

and denote

$$\|u\|_{\mathcal{A},k} = \left(\int_{\mathbb{R}^n}|u(x)|^2_{\mathcal{A},k}dx\right)^{\frac{1}{2}} \tag{22}$$

Let $\mathcal{O}_0$ be the Lie algebra generated by the angular momentum operators Ω_{ij}, $1 \leq i,j \leq n$.

Lemma 1. *Let $m \geq [\frac{n-1}{2}] + 1$. There exists a constant $C = C(m,n)$ such that for every smooth function f on $\mathbb{S}^{n-1}$ and all $\eta \in \mathbb{S}^{n-1}$ we have*

$$|f(\eta)| \leq C\left(\int_{\mathbb{S}^{n-1}}|f(\xi)|^2_{\mathcal{O}_0,m}dS(\xi)\right)^{\frac{1}{2}}. \tag{23}$$

This follows from the Sobolev embedding.

Lemma 2. *There exists a constant $C = C(n)$ such that, for every smooth function f, and all $x \neq 0 \in \mathbb{R}^n$*

$$|f(x)| \leq C|x|^{-\frac{n-1}{2}}\|f\|^{\frac{1}{2}}_{\mathcal{O}_0,1+[\frac{n-1}{2}]}\|\nabla f\|^{\frac{1}{2}}_{\mathcal{O}_0,1+[\frac{n-1}{2}]} \tag{24}$$

Let $x \in \mathbb{R}^n$ be fixed. We write $x = r\omega$ with $\omega \in \mathbb{S}^{n-1}$. We fix ω and write

$$|f(r\omega)|^2 \leq 2r^{1-n}\left(\int_0^\infty |f(\rho\omega)|^2\rho^{n-1}d\rho\right)^{\frac{1}{2}}\left(\int_0^\infty |\partial_\rho f(\rho\omega)|^2\rho^{n-1}d\rho\right)^{\frac{1}{2}}$$

From Lemma 1 it follows that

$$|f(\rho\omega)|^2 \leq C\int_{\mathbb{S}^{n-1}} |f(\rho\xi)|^2_{\mathcal{O}_{0,m}}dS(\xi)$$

with $m = 1 + [\frac{n-1}{2}]$. Then note also that

$$\int_0^\infty \rho^{n-1}d\rho\int_{\mathbb{S}^{n-1}} |f(\rho\xi)|^2_{\mathcal{O}_{0,m}}dS(\xi) = \int_{\mathbb{R}^n} |f(x)|^2_{\mathcal{O}_{0,m}}dx$$

We note the similar inequality for $\partial_\rho f(\rho\omega)$, and we are done.

We want to use now additional vector fields that commute with the wave operator. For simplicity of notation we will use $x^0 = t = -x_0$, $x_j = x^j$ for coordinates. Indices running from 0 to 3 will be denoted by greek letters, indices running from 1 to 3 by latin ones. We denote by $\eta_{\alpha\beta}$ the Lorentz metric, $\eta^{\alpha\beta}$ its inverse (it is the same matrix) recall that $\eta_{00} = -1$, $\eta_{ij} = \delta_{ij}$, all the rest of the entries are zero. The translation operators are

$$T_\mu = \frac{\partial}{\partial x^\mu}.$$

The angular momentum operators are

$$\Omega_{\mu\nu} = x_\mu\frac{\partial}{\partial x^\nu} - x_\nu\frac{\partial}{\partial x^\mu}.$$

We denote by S the dilation

$$S = x^\mu\frac{\partial}{\partial x^\mu}.$$

We use summation convention. The commutation relations are

$$[T_\mu, T_\nu] = 0, \quad [T_\mu, \Omega_{\alpha\beta}] = \eta_{\mu\alpha}T_\beta - \eta_{\mu\beta}T_\alpha,$$

$$[T_\mu, S] = T_\mu,$$

$$[\Omega_{\mu\nu}, \Omega_{\alpha\beta}] = \eta_{\mu\alpha}\Omega_{\beta\nu} - \eta_{\mu\beta}\Omega_{\alpha\nu} + \eta_{\nu\alpha}\Omega_{\mu\beta} - \eta_{\nu\beta}\Omega_{\mu\alpha},$$

$$[\Omega_{\mu\nu}, S] = 0.$$

Thus, (T_μ) generate a Lie algebra denoted $\mathcal{T}$, $(\Omega_{\mu\nu})$ generate a Lie algebra denoted Ω, $(\Omega_{\mu\nu}, S)$ generate a Lie algebra denoted $\mathcal{L}$ (for Lorentz) and $(T_\mu, \Omega_{\mu\nu}, S)$ generate a Lie algebra denoted Π (for Poincaré). Note that

$$[\Box, T_\mu] = 0, \quad [\Box, \Omega_{\mu\nu}] = 0$$

and

$$[\Box, S] = 2\Box.$$

Thus the dilation of a solution of $\Box = 0$ is again a solution, and obviously the same is true for all elements in Π. We will use conservation of energy of solutions and their Π derivatives to obtain decay. We begin by recalling the classical local Sobolev lemma in $\mathbb{R}^n$:

Lemma 3. *There exists a constant $C = C(m, n)$ so that, for $m \geq 1 + [\frac{n}{2}]$, every $R > 0$, every smooth u and all $x \in B(0, R)$ we have*

$$|u(x)| \leq C \sum_{j=0}^{m} R^{j-\frac{n}{2}} \left[\int_{B(0,R)} \sum_{|\alpha|=j} |\partial^\alpha u(x)|^2 dx \right]^{\frac{1}{2}} \tag{25}$$

This is easily seen done by reducing the problem to $R = 1$ using a dilation, and then using an extension theorem in $H^m(B(0,1))$, extending u to a function in $v \in H^m(\mathbb{R}^n)$ of comparable H^m norm, and supported in $B(0, 2)$. We take a smooth compactly supported cutoff function χ equal identically to 1 in a neighborhood of $B(0, 1)$ and write the Fourier inversion formula for χv. Using the weights $(1+|\xi|^2)^{\frac{m}{2}}$ we see that the $L^1(\mathbb{R}^n)$ norm of the Fourier transform of χv is bounded by a constant multiple of the $H^m(B(0,1))$ norm of u.

Now we note that for $(t, x) = (x^0, x)$ not on the wave cone $|x|^2 = t^2$ we can express regular ∂_i derivatives in terms of the vector fields generating $\mathcal{L}$:

$$T_\nu = \frac{1}{r^2 - t^2} \left(x^\mu \Omega_{\mu\nu} + x_\nu S \right) \tag{26}$$

with obvious notation $r^2 = |(x_1, x_2, \ldots, x_n)|^2$, $r^2 - t^2 = \eta_{\alpha\beta} x^\alpha x^\beta$. We denote

$$\sigma_- = (1 + |r - t|^2)^{\frac{1}{2}}.$$

Lemma 4. *For every $k \geq 0$ there exists $C = C(n,k)$ so that for every smooth function and every point $x = (x^0, x^1, \ldots, x^n)$ we have*

$$\left| \frac{\partial^k}{\partial x_0^{\alpha_0} \ldots \partial x_n^{\alpha_n}} u(x) \right| \leq C(\sigma_-)^{-k} |u(x)|_{\Pi,k} \tag{27}$$

holds for all $\alpha_0 + \cdots + \alpha_n = k$.

The proof is done by induction. For $k = 1$ we saw that

$$\frac{\partial u}{\partial x^\nu} = \frac{1}{r-t} \frac{1}{r+t} (x^\mu \Omega_{\mu\nu} + x_\nu S)$$

so

$$\left| \frac{\partial u}{\partial x^\nu} \right| \leq \frac{C}{|r-t|} |u(x)|_{\mathcal{L},1}.$$

In order to take care of the region $|r-t| < 1$ we augment to Π. We obtain, for any function u,

$$\left| \frac{\partial u}{\partial x^\nu} \right| \leq \frac{C}{\sigma_-} |u(x)|_{\Pi,1}. \tag{28}$$

For higher derivatives we use the commutation relations, which look symbolically like $[\mathcal{T}, \Pi] = \mathcal{T}$. For instance, by (28), we have

$$\left|\frac{\partial^2 u}{\partial x^\mu \partial x^\nu}\right| \leq \frac{C}{\sigma_-} \left|T_\mu u(x)\right|_{\Pi,1}$$

and a term of the form $X T_\mu u$ with $X \in \Pi \cup \{I\}$ is of the form

$$X T_\mu u = T_\mu X(u) + T$$

with $T \in \mathcal{T}$, so we apply (28) to deduce

$$|X T_\mu u| \leq \frac{C}{\sigma_-} |u(x)|_{\Pi,2}$$

Now we consider a point (t,x) in the interior of the wave cone

$$|x| \leq \frac{t}{2}.$$

Then, clearly,

$$\sigma_-(t,x) \geq \frac{t}{2}. \tag{29}$$

Let $R = \frac{t}{2}$ and apply Lemma 3:

$$|u(t,x)| \leq \sum_{j=0}^{m} R^{j-\frac{n}{2}} \left(\int_{|y|\leq R} \sum_{|\alpha|\leq j} |\partial^\alpha u(y)|^2 dy \right)^{\frac{1}{2}}$$

Using, for each y the inequality (27) of Lemma 4 and recalling (29) which is valid for (t,y), we obtain

$$\begin{aligned} |u(t,x)| &\leq C \textstyle\sum_{j=0}^{m} R^{j-\frac{n}{2}} R^{-j} \left(\int_{|y|\leq R} |u(t,y)|^2_{\Pi,j} dy \right)^{\frac{1}{2}} \\ &= Ct^{-\frac{n}{2}} \textstyle\sum_{j=0}^{m} \left(\int_{|y|\leq \frac{t}{2}} |u(t,y)|^2_{\Pi,j} dy \right)^{\frac{1}{2}} \\ &\leq Ct^{-\frac{n}{2}} \|u(t,\cdot)\|_{\Pi,m} \end{aligned}$$

We proved:

Lemma 5. *Let $m \geq 1 + \left[\frac{n}{2}\right]$. There exists a constant $C = C(m,n)$ so that, for all smooth functions u and all points (t,x) with $t > 0$, $t \geq 2|x|$, we have*

$$|u(t,x)| \leq Ct^{-\frac{n}{2}} \|u(t,\cdot)\|_{\Pi,m}. \tag{30}$$

On the other hand, for $|x| \geq \frac{t}{2}$ we can use (24):

$$|u(t,x)| \leq Ct^{-\frac{n-1}{2}} \|u(t,\cdot)\|_{\Pi,2+[\frac{n-1}{2}]} \tag{31}$$

Combining (30) and (31) we have

Proposition 1. *Let $m \geq [\frac{n}{2}] + 2$. There exists a constant $C = C(m,n)$ so that for all $t > 0$, $x \in \mathbb{R}^n$ and smooth functions u we have*

$$|u(t,x)| \leq C(1+t)^{-\frac{n-1}{2}} \|u(t,\cdot)\|_{\Pi,m} \tag{32}$$

We control the region $t < 1$, $|x| \leq \frac{1}{2}$ using usual Sobolev inequalities on each time slice. If $t < 1$ but $|x| \geq \frac{1}{2}$ or if $t \geq 1$ and $|x| \geq \frac{t}{2}$ we use (31) and if $t \geq 1$, $|x| \leq \frac{t}{2}$ we use (30).

Just as Lemma 2, Proposition 1 is not a true decay result, it is just a convenient tautology. However, for solutions of the wave equation, we can easily control the right hand side.

Theorem 1. *Let $m \geq [\frac{n}{2}] + 2$. Let $u(t,x)$ solve*

$$\Box u = 0$$

and assume that the functions

$$u_0(x) = u(0,x), \quad u_1(x) = \partial_t u(0,x)$$

are such that

$$\|\nabla_x u(0,\cdot)\|_{\Pi,m} < \infty, \quad \|\partial_t u(0,\cdot)\|_{\Pi,m} < \infty$$

Then

$$|\nabla_x u(t,x)| \leq C(1+t)^{-\frac{n-1}{2}} [\|\nabla_x u(0,\cdot)\|_{\Pi,m} + \|\partial_t u(0,\cdot)\|_{\Pi,m}]$$

Note that we need to use the equation in order to express the high Π, m norms at time $t = 0$ in terms of spatial derivatives of the initial data.

The proof follows by considering $v = \partial_j \Gamma u$ where $\Gamma \in \Pi$. Note that

$$\|v(t,\cdot)\|_{L^2(\mathbb{R}^n)} \leq C[\|v(0,\cdot)\|_{L^2(\mathbb{R}^n)} + \|(-\Delta)^{-\frac{1}{2}} \partial_t v(0,\cdot)\|_{L^2(\mathbb{R}^n)}]$$

holds for any solution of the wave equation. So we have

$$\|\nabla_x \Gamma u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \leq C[\|(\nabla_x \Gamma u)(0,\cdot)\|_{L^2(\mathbb{R}^n)} + \|(\partial_t \Gamma u)(0,\cdot)\|_{L^2(\mathbb{R}^n)}$$