Take home final exam, Math 529

Due May 15, 2017

Please email me a .pdf file or put a hard copy in my mailbox.
Please solve one of the two problems.

Problem 1. Consider the equation

$$\partial_t u + u \cdot \nabla u + \Lambda^s u = f \tag{1}$$

with

$$\nabla \cdot u = 0 \tag{2}$$

in two dimensions, on the unit torus, $\mathbb{T}^2$ i.e.

$$u(x + 2\pi e_i, t) = u(x, t), \quad i = 1, 2, \tag{3}$$

with $e_i,\ i = 1, 2$ the canonical basis of $\mathbb{R}^2$. The forces f are time independent and smooth, $f \in W^{2,\infty}(\mathbb{T}^2)^2$,

$$\Lambda = (-\Delta)^{\frac{1}{2}} \tag{4}$$

and $s > 0$. Prove that the equation has a global attractor of finite Hausdorff dimension, and give a bound for its dimension.

Problem 2.

A. Prove that the n-dimensional Burgers equation

$$\partial_t u_i + \sum_{j=1}^{n} u_j \partial_j u_i - \Delta u_i = 0, \quad i = 1, \dots n \tag{5}$$

with $u = u(x,t)$, $x \in \mathbb{R}^n$, $t \geq 0$, and initial data $u_0 \in H^s(\mathbb{R}^d)$, $s > 1 + \frac{n}{2}$ has global solutions in $H^s(\mathbb{R}^d)$.

B. Consider the two dimensional Navier-Stokes equation on the unit torus,

$$\partial_t u + \nu A u + B(u,u) = f \tag{6}$$

with $f \in H$. Prove Lemma 14.3 of the blue book: There exists a function $c(t)$ defined for all $t > 0$, depending on ν, $|f|$, and ρ such that, for every $u_1, u_0 \in B_\rho^V$ the following estimate holds:

$$|S(t)u_1 - S(t)u_0 - S'(t,u_0)(u_1 - u_0)| \leq c(t)|u_1 - u_0|^2. \tag{7}$$

The notation is from the blue book: $|f|$ is norm in H, i.e. L^2 norm, B_ρ^V is the ball in V of radius ρ, with the norm in V being H^1 norm, $S(t)u_0$ is the solution of the Navier-Stokes equation with initial datum u_0 and $S'(t,u_0)v_0 = v(t)$ is the solution of the linearization of the Navier-Stokes equation along the path $u(t) = S(t)u_0$, i.e., v solves

$$\partial_t v + \nu A v + B(u,v) + B(v,u) = 0 \tag{8}$$

with initial datum v_0.