Decay for Schrödinger and related equations.

Intoduction to PDE

This is from my paper "Decay estimates for Schrödinger equations", Com. Math. Phys **127** (1990), 101-108.

The system we study (Schrödinger-Davey-Stewartson-Zakharov) is

$$\begin{aligned} i\partial_t u + L_1 u = a|u|^2 u + vu \\ L_2 v = L_3(|u|^2) \end{aligned} \tag{1}$$

where $a \in \mathbb{R}$, and L_i are second order constant coefficient diferential operators

$$L_i = g_i^{jk} \frac{\partial^2}{\partial x^j \partial x^k}, \quad i = 1, 2, 3. \tag{2}$$

where the constant real matrices g_i^{jk} are symmetric and invertible. We assume L_2 to be elliptic, and we write then the system as a single equation

$$i\partial_t u + P(D)u = L(|u|^2)u \tag{3}$$

where $P(D) = L_1$ and we drop the index:

$$P(D) = g^{jk} \frac{\partial^2}{\partial x^j \partial x^l} \tag{4}$$

and

$$L = a\mathbb{I} + L_2^{-1} L_3 \tag{5}$$

The properties we will use for the linear operator L are: $L : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is bounded for any $1 < p < \infty$, L commutes with translations (and hence with differentiation, L is real (i.e. it commutes with complex conjugation).

1 Sobolev lemma and the linear equation

We denote by g_{jk} the inverse matrix $(g_{jk}) = (g^{jk})^{-1}$. We do *not* assume that $P(D)$ is elliptic. When $P(D) = \Delta$ then

$$iu_t + P(D)u = 0$$

is the free Schrödinger equation. We introduce the differential operators

$$Q_j((x,t),D) = Q_j = 2t\partial_j - ig_{jk}x^k \tag{6}$$

They commute with the free equation:

$$[i\partial_t + P(D), Q_j] = 0 \tag{7}$$

This can be checked by hand. We can also easily check using the Fourier transform that

$$Q_j = e^{itP(D)}(-ig_{jk}x^k)e^{-itP(D)} \tag{8}$$

Indeed,

$$\begin{gathered}\mathcal{F}(e^{itP(D)}(-ig_{jk}x^k)e^{-itP(D)}u)(\xi) = \\ e^{-itg^{ab}\xi_a\xi_b}(g_{jk}\partial_{\xi_k}(e^{itg^{cd}\xi_c\xi_d}\mathcal{F}(u)(\xi)) \\ = g_{jk}\partial_{\xi_k}\mathcal{F}(u)(\xi) + g_{jk}(itg^{cd}(\xi_c\delta_{dk} + \xi_d\delta_{ck})\mathcal{F}(u)(\xi) \\ = \mathcal{F}[(-ig_{jk}x^k + 2t\partial_j)u](\xi)\end{gathered}$$

where we used

$$\mathcal{F}(u)(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}u(x)dx$$

and δ_{jk} the Kronecker delta.

We have as well that

$$Q_j = 2te^{i\psi}\partial_j e^{-i\psi} \tag{9}$$

where

$$\psi(x,t) = \frac{g_{jk}x^j x^k}{4t} \tag{10}$$

and where the right hand side of (9) is considered as a product of operators (multiplication by $e^{-i\psi}$ followed by differentiation, followed by multiplication by $2te^{i\psi}$). The operators Q_j commute. They generate a Lie algebra denoted $\mathcal{A}$. The Lie algebra generated by the collection Q_j, ∂_j is denoted $\mathcal{B}$. For any

Lie algebra $\mathcal{A}$ of differential operators with generators $A_1, \ldots A_N$ we use the notation

$$|u(x,t)|_{\mathcal{A},m} = \sum_{j=0}^{m} \left(\sum_{|\alpha|=j} (A^\alpha u(x,t))^2 \right)^{\frac{1}{2}} \tag{11}$$

where

$$A^\alpha = A_1^{\alpha_1} \cdots A_N^{\alpha_N}, \quad \text{and } |\alpha| = \alpha_1 + \ldots \alpha_N.$$

We define generalized $W^{m,p}$ norms via

$$\|u(\cdot,t)\|_{\mathcal{A},m,p} = \left(\int_{\mathbb{R}^n} |u(x,t)|^p_{\mathcal{A},m} dx \right)^{\frac{1}{p}}. \tag{12}$$

Lemma 1. *There exists a constant $C = C(n)$ such that*

$$|u(x,t)| \le C|t|^{-\frac{n}{2}} \|u(\cdot,t)\|_{\mathcal{A},[\frac{n}{2}]+1,2} \tag{13}$$

holds for all (x,t) and all u.

Proof. Let us consider the function

$$v(x,t) = e^{-i\psi} u(x,t)$$

and apply the local Sobolev Lemma to it

$$|v(x,t)| \le C \sum_{j=0}^{1+[\frac{n}{2}]} R^{j-\frac{n}{2}} \left(\sum_{|\alpha|=j} \int_{|x-y|\le R} |\partial_y^\alpha v(y,t)|^2 dy \right)^{\frac{1}{2}}$$

This holds for any R, and the constant C is independent of R. Now we observe that

$$|v(x,t)| = |u(x,t)|$$

and, in view of (9), by induction, it holds that

$$|\partial_y^\alpha v(y,t)| = (2|t|)^{-|\alpha|} |Q^\alpha u(y,t)|.$$

The inequality (13) follows by choosing $R = 2t$.

We remove the singularity at $t = 0$ by augmenting to $\mathcal{B}$:

Lemma 2. *There exists a constant $C = C(n)$ such that*

$$|u(x,t)| \leq C(1+|t|)^{-\frac{n}{2}} \|u(\cdot,t)\|_{\mathcal{B},[\frac{n}{2}]+1,2} \tag{14}$$

holds for all (x,t) and all u.

This is trivial because for $|t| \leq 1$ we use the usual Sobolev inequality. A direct application is:

Theorem 1. *Let $u(x,t)$ be a solution of*

$$iu_t + P(D)u = 0 \tag{15}$$

with initial datum $u(x,0) = u_0(x)$. Then

$$|u(x,t)| \leq C|t|^{-\frac{n}{2}} \|u_0\|_{\mathcal{X},1+[\frac{n}{2}],2} \tag{16}$$

where $\mathcal{X}$ is the Lie algebra generated by the operators of multiplication by x^j, $j = 1, \ldots n$. More generally,

$$|u(x,t)|_{\mathcal{A},k} \leq C|t|^{-\frac{n}{2}} \|u_0\|_{\mathcal{X},k+1+[\frac{n}{2}],2} \tag{17}$$

and

$$|u(x,t)|_{\mathcal{B},k} \leq C(1+|t|)^{-\frac{n}{2}} \|u_0\|_{\mathcal{B}_0,k+1+[\frac{n}{2}],2} \tag{18}$$

where $\mathcal{B}_0$ is the Lie algebra generated by the operators of multiplication by $1, x^j$ and by ∂_j, $j = 1, \ldots, n$.

2 The nonlinear equation

Lemma 3. *Let $0 < j < m$. There exists a constant C depending on j, m and n such that*

$$\sum_{|\beta|=j} \|Q^\beta u(\cdot,t)\|_{L^{\frac{2m}{j}}(\mathbb{R}^n)} \leq C\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{j}{m}} \left(\sum_{|\alpha|=m} \|Q^\alpha u(\cdot,t)\|_{L^2(\mathbb{R}^n)} \right)^{\frac{j}{m}} \tag{19}$$

The inequality (19) in which the operators Q_j are replaced by ∂_j is a well known Gagliardo-Nirenberg inequality. Applying it to $v = e^{-i\psi}u$ and using (9) and the scale invariance of the inequality, we immediately obtain (19).

Now we state a Leibniz rule:

Lemma 4. *For any multi-index α it holds that*

$$Q^{\alpha}(L(|u|^2v) = \sum_{\beta+\gamma+\delta=\alpha} \frac{\alpha!}{\beta!\gamma!\delta!} L((Q^{\beta}u)\overline{Q^{\gamma}u})Q^{\delta}v \tag{20}$$

Proof. We used the noatation $\alpha! = \alpha_1! \cdots \alpha_n!$. The proof is done by induction on $|\alpha|$ and it follows from the observations that

$$Q_j(ab) = (2t\partial_j a)\, b + aQ_j(b)$$

and that

$$2t\partial_j(a\bar{b}) = (Q_j(a))\bar{b} + a\overline{(Q_j(b))}$$

The first equality is used to write

$$Q_j(L((Q^{\beta}u)\overline{Q^{\gamma}u})Q^{\delta}v) = L(2t\partial_j((Q^{\beta}u)\overline{Q^{\gamma}u}))Q^{\delta}v + L((Q^{\beta}u)\overline{Q^{\gamma}u})Q_jQ^{\delta}v$$

and the second one to finish

$$\begin{aligned} Q_j(L((Q^{\beta}u)\overline{Q^{\gamma}u})Q^{\delta}v) = L((Q_jQ^{\beta}u)\overline{Q^{\gamma}u} + (Q^{\beta}u)\overline{Q_jQ^{\gamma}u})Q^{\delta}v \\ +L((Q^{\beta}u)\overline{Q^{\gamma}u})Q_jQ^{\delta}v \end{aligned}$$

So, Q_j distributes just like a derivative in the product. The fact that the complex conjugate is inside the operator L is used crucially. Let us start by denoting

$$I_m(w)(t) = I_m = \left(\sum_{|\alpha|=m} \int_{\mathbb{R}^n} |Q^{\alpha}w|^2 dx \right)^{\frac{1}{2}} \tag{21}$$

Let us assume that w solves the equation

$$iw_t + P(D)w = L(|u|^2)w \tag{22}$$

for some given (smooth) function u. Note that this is a linear Schrödinger equation if $P(D)$ is elliptic. Then, in view of (7) and the fact that $P(D)$ is real, we have

$$\frac{1}{2}\frac{d}{dt} I_m^2 = \operatorname{Im} \sum_{|\alpha|=m} \int_{\mathbb{R}^n} Q^{\alpha}(L(|u|^2)w)\overline{Q^{\alpha}w}dx \tag{23}$$

Now, in view of our Leibniz formula (20), the right-hand side of (23) is a sum for $|\alpha| = m$ and $\alpha = \beta + \gamma + \delta$ of terms

$$\frac{\alpha!}{\beta!\gamma!\delta!}\mathrm{Im}\int_{\mathbb{R}^n} L((Q^\beta u)\overline{Q^\gamma u})(Q^\delta w)\overline{Q^\alpha w}dx \tag{24}$$

The term in (24) corresponding to $\beta = \gamma = 0$ vanishes. This is very important, because L is not bounded in L^∞. If $\beta = \delta = 0$ or $\gamma = \delta = 0$, then we estimate (24) using

$$\begin{aligned} \left|\int_{\mathbb{R}^n} L(u\overline{Q^\alpha u})w\overline{Q^\alpha w}dx\right| &\le C\|uQ^\alpha u\|_{L^2(\mathbb{R}^n)}\|wQ^\alpha w\|_{L^2(\mathbb{R}^n)} \\ &\le C\|u\|_{L^\infty(\mathbb{R}^n)}I_m(u)\|w\|_{L^\infty(\mathbb{R}^n)}I_m(w) \end{aligned} \tag{25}$$

The rest of the terms have $0 < |\delta| < m$. In these terms we apply a Hölder inequality, raising the last term to the second power, the term involving $Q^\delta w$ to the power $\frac{2m}{|\delta|}$ and the term involving L to the power $q = 2(1-\frac{|\delta|}{m})^{-1}$. Using the boundedness of L in L^q spaces and our Gagliardo-Nirenberg inequality (19) we majorize (24) by

$$C\|w(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{|\delta|}{m}} I_m(w)^{1+\frac{|\delta|}{m}}\left(\int_{\mathbb{R}^n}|Q^\beta u|^q|Q^\gamma u|^q dx\right)^{\frac{1}{q}}$$

In the last integral we use a Hölder inequality with powers $\frac{2m}{q|\beta|}$ and $\frac{2m}{q|\gamma|}$ (their inverses do add up to 1!) and again our Gagliardo-Nirenberg inequality (19). The result is that all these terms in the (24) can be bound by

$$C\|u\|_{L^\infty(\mathbb{R}^n)}^{1+\frac{|\delta|}{m}} I_m(u)^{1-\frac{|\delta|}{m}}\|w\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{|\delta|}{m}} I_m(w)^{1+\frac{|\delta|}{m}} \tag{26}$$

We note that (25) has the form of (26) with $|\delta| = 0$. Dividing by $I_m(w)$ we obtained

$$\frac{d}{dt}I_m(w) \le C\sum_{j=0}^{m-1}\|u\|_{L^\infty}^{1+\frac{j}{m}} I_m(u)^{1-\frac{j}{m}}\|w\|_{L^\infty}^{1-\frac{j}{m}} I_m(w)^{\frac{j}{m}} \tag{27}$$

Now the exact same calculation applies to integrals

$$J_m(w) = \left(\sum_{|\alpha|=m}\int_{\mathbb{R}^n}|\partial^\alpha w|^2 dx\right)^{\frac{1}{2}} \tag{28}$$

using the usual Leibniz formula and Gagliardo-Nirenberg inequalities. Denoting

$$K_m(w) = I_m(w) + J_m(w) \tag{29}$$

we obtain by adding the two similar inequalities and using $\max\{I, J\} \le K$

$$\frac{d}{dt} K_m(w) \le C \sum_{j=0}^{m-1} \|u\|_{L^\infty}^{1+\frac{j}{m}} K_m(u)^{1-\frac{j}{m}} \|w\|_{L^\infty}^{1-\frac{j}{m}} K_m(w)^{\frac{j}{m}} \tag{30}$$

Let us introduce now

$$E_N(w) = \sum_{m=0}^{N} K_m(w) \tag{31}$$

and take $N \ge 1 + [\frac{n}{2}]$. Note that (14) implies

$$\|w\|_{L^\infty} \le C(1 + |t|)^{-\frac{n}{2}} E_N(w) \tag{32}$$

Using the same inequality for u and majorizing each K_m by E_N, we obtain from (30)

$$\frac{d}{dt} K_m(w) \le (1 + |t|)^{-n} E_N(u) E_N(w) \tag{33}$$

for $m = 0, 1, \dots N$. (Note that K_0 is conserved.) Adding in m we obtain

Theorem 2. *Let w solve the linear equation (22). For $N \ge 1 + [\frac{n}{2}]$ there exists a constant $C = C(n, N)$ such the norm $E_N(w)$ satisfies*

$$\frac{d}{dt} E_N(w) \le C(1 + |t|)^{-n} E_N(u) E_N(w) \tag{34}$$

Theorem 3. *Let $N \ge 1 + [\frac{n}{2}]$. Then there exists $\epsilon = \epsilon(N)$ and $C = C(n, N)$ such that, if u_0 satisfies*

$$\sum_{|\alpha| \le N} \int_{\mathbb{R}^n} \left[|Q^\alpha(x, 0, D) u_0(x)|^2 + |\partial^\alpha u_0(x)|^2 \right] dx \le \epsilon$$

then the solution of (3) exists for all time and satisfies

$$\sum_{|\alpha| \le N} \int_{\mathbb{R}^n} \left[|Q^\alpha(x, t, D) u(x, t)|^2 + |\partial^\alpha u(x, t)|^2 \right] dx \le C\epsilon$$

and

$$|u(x, t)| \le C \epsilon^{\frac{1}{2}} (1 + |t|)^{-\frac{n}{2}}$$

Proof. We prove first local existence (short time existence) and uniqueness in H^N. This is done by considering the map

$$u(t) \mapsto e^{itP(D)}u_0 - i\int_0^t e^{i(t-s)P(D)}L(|u(s)|^2)u(s)ds$$

for $u \in C(0,T;H^N)$ with $u(0) = u_0$. We obtain unique solutions on time intervals depending on the norm of u_0 in H^N. Then we use (34) with $u = w$ to deduce that

$$E_N(t)(1 - C_N E(0)) \leq E_N(0).$$