Schauder Estimates

Introduction to PDE

1 Schauder Estimates: the Laplacian

The fundamental solution of the Laplacian is given by the Newtonian potential

$$N(x) = \begin{cases} \frac{1}{2\pi}\log|x|, & \text{if } n = 2, \\ \frac{1}{(2-n)\omega_n}|x|^{2-n}, & \text{if } n \geq 3. \end{cases} \tag{1}$$

Here ω_n is the area of the unit sphere in $\mathbb{R}^n$. Note that N is radial and it is singular at $x = 0$.

Proposition 1 *Let $f \in C_0^2(\mathbb{R}^n)$, and let*

$$u(x) = \int_{\mathbb{R}^n} N(x-y)f(y)dy. \tag{2}$$

Then $u \in C^2(\mathbb{R}^n)$ and

$$\Delta u = f$$

Lemma 1 *Let $N \in L^1_{loc}(\mathbb{R}^n)$ and let $\phi \in C_0^1(\mathbb{R}^n)$. Then $u = N * \phi$ is in $C^1(\mathbb{R}^n)$ and*

$$\nabla u(x) = \int N(x-y)\nabla\phi(y)dy$$

Idea of proof of the lemma. First of all, the notation: L^p_{loc} is the space of functions that are locally in L^p and that means that their restrictions to compacts are in L^p. The space C_0^1 is the space of functions with continuous derivatives of first order, and having compact support. Because ϕ has compact support

$$u(x) = \int N(x-y)\phi(y)dy = \int N(y)\phi(x-y)dy$$

is well defined. Fix $x \in \mathbb{R}^n$ and take $h \in \mathbb{R}^n$ with $|h| \leq 1$. Note that

$$\begin{array}{l} \frac{1}{|h|}\left(u(x+h) - u(x) - h \cdot \int N(x-y)\nabla\phi(y)dy\right) = \\ \int N(y)\frac{1}{|h|}\left(\phi(x+h-y) - \phi(x-y) - h \cdot \nabla\phi(x-y)\right)dy \end{array}$$

The functions $y \mapsto \frac{1}{|h|}(\phi(x+h-y) - \phi(x-y) - h \cdot \nabla\phi(x-y))$ for fixed x and $|h| \le 1$ are supported all in the same compact, are uniformly bounded, and converge to zero as $h \to 0$. Because of Lebesgue dominated, it follows that u is differentiable at x and that the derivative is given by the desired expression. Because

$$\nabla u(x) = \int N(y)\nabla\phi(x-y)dy$$

and $\nabla\phi$ is continuous, it follows that ∇u is continuous. This finishes the proof of the lemma.

Idea of proof of the Proposition 1. By the Lemma, u is C^2 and

$$\Delta u(x) = \int N(x-y)\Delta f(y)dy$$

Fix x. The function $y \mapsto N(x-y)\Delta f(y)$ is in $L^1(\mathbb{R}^n)$ and compactly supported in $|x-y| < R$ for a large enough R that we'll keep fixed. Therefore

$$\Delta u(x) = \lim_{\epsilon \to 0} \int_{\{y;\epsilon<|x-y|<R\}} N(x-y)\Delta f(y)dy$$

We will use Green's identities for the domains $\Omega_\epsilon^R = \{y; \epsilon < |x-y| < R\}$. This is legitimate because the function $y \mapsto N(x-y)$ is C^2 in a neighborhood of $\overline{\Omega_\epsilon^R}$. Note that, because of our choice of R, $f(y)$ vanishes identically near the outer boundary $|x-y| = R$. Note also that $\Delta_y N(x-y) = 0$ for $y \in \Omega_\epsilon^R$. From the Green formula we have

$$\begin{aligned}&\int_{\{y;\epsilon<|x-y|<R\}} N(x-y)\Delta\ f(y)dy = \\ &\int_{|x-y|=\epsilon} N(x-y)\partial_\nu f(y)dS - \int_{|x-y|=\epsilon} f(y)\partial_\nu N(x-y)dS\end{aligned}$$

The external unit normal at the boundary is $\nu = -(x-y)/|x-y|$. The first integral vanishes in the limit because $|\nabla f|$ is bounded, $N(x-y)$ diverges like ϵ^{2-n} (or $\log\epsilon$) and the area of boundary vanishes like ϵ^{n-1}:

$$\left| \int_{|x-y|=\epsilon} N(x-y)\partial_\nu f(y)dS \right| \le C\epsilon\|\nabla f\|_\infty$$

in $n > 2$, and the same thing replacing ϵ by $\epsilon\log\epsilon^{-1}$ in $n = 2$. The second integral is more amusing, and it is here that it will become clear why the constants are chosen as they are in (1). We start by noting carefully that two minuses make a plus, and

$$-\partial_\nu N(x-y) = \frac{1}{\omega_n}|x-y|^{1-n}.$$

Therefore, in view of the fact that $|x-y| = \epsilon$ on the boundary we have

$$-\int_{|x-y|=\epsilon} f(y)\partial_\nu N(x-y)dS = \frac{1}{\epsilon^{n-1}\omega_n}\int_{|x-y|=\epsilon} f(y)dS$$

Passing to polar coordinates centered at x we see that

$$-\int_{|x-y|=\epsilon} f(y)\partial_\nu N(x-y)dS = \frac{1}{\omega_n}\int_{|z|=1} f(x+\epsilon z)dS$$

and we do have

$$\lim_{\epsilon\to 0}\frac{1}{\omega_n}\int_{|\ z|=1} f(x+\epsilon z)dS = f(x)$$

because f is continuous. We have thus:

Proposition 2 *N is the fundamental solution of the Laplacian:*

$$\Delta N = \delta.$$

Indeed, by Proposition 1

$$\Delta(N * \phi) = \phi,\ \forall\ \phi \in \mathcal{D}$$

and thus $\Delta N * \phi = \phi$ which implies $\Delta N = \delta$ in $\mathcal{D}'$.

We compute now second derivatives of Newtonian potentials and derive Schauder estimates for the Poisson equation. We take a bounded open set Ω and a function $f \in C^\alpha(\Omega)$, $0 \le \alpha \le 1$. We extend the function f by zero outside Ω. Let

$$w(x) = \int_{\mathbb{R}^n} N(x-y)f(y)dy = \int_\Omega N(x-y)f(y)$$

be the Newtonian potential of f.

Lemma 2 *Let f be bounded and (locally) C^α in Ω. Then $w \in C^2(\Omega)$, $\Delta w = f$, and, for any $x \in \Omega$*

$$\begin{array}{l} \partial_i\partial_j w(x) = \\ \int_{\Omega_1} \partial_i\partial_j N(x-y)(f(y)-f(x))dy - f(x)\int_{\partial\Omega_1} \partial_i(N(x-y))\nu_j(y)dS(y) \end{array} \tag{3}$$

where Ω_1 is a bounded domain containing Ω and in which the divergence theorem holds.

Proof. We let

$$u(x) = \int_{\Omega_1} \partial_i\partial_j N(x-y)(f(y)-f(x))dy - f(x)\int_{\partial\Omega_1} \partial_i(N(x-y))\nu_j(y)dS(y)$$

Clearly, in view of the fact that the second derivatives of the Newtonian kernel obey

$$|\partial_i\partial_j N(x-y)| \le C|x-y|^{-n}$$

and the local Hölder continuity of f, the function $u(x)$ is well defined. We also know (by calculations above) that $w \in C^1$. Let $v = \partial_i w$ and let us set

$$v_\epsilon(x) = \int_\Omega (\partial_i N(x-y))\eta\left(\frac{|x-y|}{\epsilon}\right) f(y)dy$$

where $\epsilon > 0$ is small and $\eta(r)$ is a positive smooth function which vanishes for $0 \le r \le 1$ and is identically equal to $\eta(r) = 1$ for $r \ge 2$. So we are cutting off smoothly the singular region. Now v_ϵ is clearly C^1 , and differentiating we have

$$\begin{array}{l} \partial_j v_\epsilon = \int_\Omega \partial_j \left[(\partial_i N(x-y)\eta\left(\frac{|x-y|}{\epsilon}\right)\right] f(y)dy \\ = \int_{\Omega_1} \partial_j \left[(\partial_i N(x-y)\eta\left(\frac{|x-y|}{\epsilon}\right)\right] f(y)dy \\ = \int_{\Omega_1} \partial_j \left[(\partial_i N(x-y)\eta\left(\frac{|x-y|}{\epsilon}\right)\right] (f(y)-f(x)dy \\ +f(x)\int_{\Omega_1} \partial_j \left[(\partial_i N(x-y)\eta\left(\frac{|x-y|}{\epsilon}\right)\right] dy \\ = \int_{\Omega_1} \partial_j \left[(\partial_i N(x-y)\eta\left(\frac{|x-y|}{\epsilon}\right)\right] (f(y)-f(x)dy \\ -f(x)\int_{\partial\Omega_1} (\partial_i N(x-y))\nu_j(y)dS(y) \end{array}$$

Therefore

$$\begin{array}{l} |u(x) - \partial_j v_\epsilon(x)| = \\ \left|\int_{\Omega_1} \partial_j \left[\left(1-\eta\left(\frac{|x-y|}{\epsilon}\right)\right)\partial_i N(x-y)\right](f(y)-f(x))dy\right| \\ \le C[f]_\alpha \int_{|z|\le 2\epsilon} (|z|^{-n} + \frac{1}{\epsilon}|z|^{-n+1})|z|^\alpha dz \\ \le C[f]_\alpha \epsilon^\alpha \end{array}$$

So $\partial_j v_\epsilon$ converges uniformly (on compacts) and because v_ϵ converegs to $v = \partial_i w$, we obtain $w \in C^2$ and $u = \partial_j \partial_i w$.

Now we are going to state and prove the basic lemma for interior Schauder estimates (for the Laplacian).

Lemma 3 *Let $B_1 = B_R(x_0)$, $B_2 = B_{2R}(x_0)$ be concentric balls in $\mathbb{R}^n$. Suppose $f \in C^\alpha(B_2)$ and let w be the Newtonian potential of f in B_2. Then $w \in C^{2,\alpha}(\overline{B_1})$ and*

$$\begin{aligned}&\sup_{x\in B_1, |\beta|\le 2} |\partial^\beta w(x)| + R^\alpha \sup_{x\neq y} \tfrac{|\partial_i\partial_j w(x) - \partial_i\partial_j w(y)|}{|x-y|^\alpha} \\ &\le C\|f\|_{C^{0,\alpha}}\end{aligned} \tag{4}$$

Proof. Because of (3) it is easy to see that the second derivatives of w are bounded by the RHS of (4). Let us consider now two points, x and $\overline{x}$ in B_1. Let us write $\delta = |x - \overline{x}|$, $\xi = \frac{1}{2}(x + \overline{x})$ and subtract the two representations (3). We obtain

$$\begin{aligned}&\partial_i\partial_j w(x) - \partial_i\partial_j w(\overline{x}) = f(x)I_1 + (f(x) - f(\overline{x}))I_2 \\ &+I_3 + I_4 + (f(x) - f(\overline{x}))I_5 + I_6\end{aligned}$$

where

$$I_1 = \int_{\partial B_2} (\partial_i N(x-y) - \partial_i(N(\overline{x}-y)))\nu_j(y)dS(y),$$

$$I_2 = \int_{\partial B_2} \partial_i N(\overline{x}-y)\nu_j(y)dS(y),$$

$$I_3 = \int_{B_{\delta(\xi)}} \partial_i\partial_j N(x-y)(f(x) - f(y))dy,$$

$$I_4 = -\int_{B_{\delta(\xi)}} \partial_i\partial_j N(\overline{x}-y)(f(\overline{x}) - f(y))dy,$$

$$I_5 = \int_{B_2\backslash B_{\delta(\xi)}} \partial_i\partial_j N(x-y)dy,$$

and

$$I_6 = \int_{B_2\backslash B_{\delta(\xi)}} (\partial_i\partial_j N(x-y) - \partial_i\partial_j N(\overline{x}-y))(f((\overline{x}) - f(y))dy$$

For I_1 we use

$$|I_1| \leq |x - \overline{x}| \int_{\partial B_2} |\nabla \partial_i N(p-y)| dS(y)$$

for some $p \in [x, \overline{x}]$. For the term involving I_5 we integrate by parts and note that the contributions at both ends are bounded. In the term involving I_6 we use

$$|I_6| \leq \delta \int_{B_2 \backslash B_\delta(\xi)} \frac{|f(\overline{x}) - f(y)|}{|p-y|^{n+1}} dy$$

with $p \in [\overline{x} - x]$. Now because $|\overline{x} - y| \leq \frac{3}{2}|\xi - y| \leq 3|p-y|$ for $|y - \xi| \geq \delta$, we can compare to the integral

$$\int_{|y-\xi|\geq\delta} |\xi - y|^{-n-1+\alpha} dy$$

and that does diverge like $\delta^{\alpha-1}$. We deduce

Theorem 1 *Let Ω be a domain in $\mathbb{R}^n$ and let $u \in C^2(\Omega)$, $f \in C^\alpha(\Omega)$ satisfy Poisson's equation $\Delta u = f$. Then $u \in C^{2,\alpha}(\Omega)$ and, for any two concentric balls $B_1 = B_R(x_0)$ and $B_2 = B_{2R}(x_0)$ we have*

$$\|u\|_{C^{2,\alpha}(B_1)} \leq C \left[\|u\|_{C^0(B_2)} + \|f\|_{C^\alpha(B_2)} \right]$$

2 Schauder estimates: general case

We consider now a uniformly elliptic equation

$$Lu = -a^{ij}(x)\partial_i\partial_j u + b^i(x)\partial_i u + c(x)u = f(x) \tag{5}$$

where $a^{ij} = a^{ji}$ are bounded, Hölder continuous and uniformly elliptic

$$\lambda|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2. \tag{6}$$

Theorem 2 *(Constant coefficients) Let $A^{ij} = A^{ji}$ be constant and satisfy (6). Denote*

$$L_0 u = -A^{ij}\partial_i\partial_j u. \tag{7}$$

If $u \in C^2(\Omega), f \in C^\alpha(\Omega)$ satisfy $L_0 u = f$, then there exists a constant C depending only on λ, Λ, n and α so that

$$|u|^*_{2,\alpha;\Omega} \leq C \left(|u|_{0,\Omega} + |f|^{(2)}_{0,\alpha;\Omega} \right) \tag{8}$$

Here we use the non-dimensional norms from Gilbarg and Trudinger:

$$|u|^*_{k,\alpha;\Omega} = |u|^*_{k,\Omega} + [u]^*_{k,\alpha;\Omega}$$

with

$$|u|^*_{k;\Omega} = \sum_{j=0}^{k} [u]^*_{j;\Omega}$$

$$[u]^*_{j;\Omega} = \sum_{|\beta|=j} \sup_{x\in\Omega} d_x^j |\partial^j u(x)|$$

and

$$[u]^*_{k,\alpha;\Omega} = \sup_{x\neq y,\, |\beta=k} d_{x,y}^{k+\alpha} \frac{|\partial^\beta u(x) - \partial^\beta u(y)|}{|x-y|^\alpha}$$

where $d_x = dist(x, \partial\Omega)$ and $d_{x,y} = \min(d_x, d_y)$. In addition

$$|f|^{(2)}_{0;\alpha;\Omega} = \sup_{x\in\Omega} d_x^2 |f(x)| + \sup_{x\neq y} d_x^{2+\alpha} \frac{|f(x) - f(y)|}{|x-y|^\alpha}$$

The constant coefficients case is a direct consequence of the Schauder estimates for the Laplacian, used after an appropriate change of variables. The general interior Schauder estimates are

Theorem 3 *Let Ω be an open subset of $\mathbb{R}^n$ and let $u \in C^{2,\alpha}(\Omega)$ be a bounded solution of*

$$Lu = f$$

in Ω, where Lu is given by (5) and the coefficients a^{ij}, b^i, c are bounded in C^α. Then there exists a constant C depending on λ, Λ, α, n the bound of the coeffiicents in C^α such that

$$|u|^*_{2,\alpha;\Omega} \leq C \left(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;\Omega} \right) \tag{9}$$

The idea of the proof is to localize and use the constant-coefficients result. Near a point x_0 we write the equation

$$Lu = f$$

as

$$L_0 u = F$$

where $L_0 u = a^{ij}(x_0)\partial_i\partial_j u$ and F is what needs to be. We are in a small ball around x_0 and wish to obtain estimates there. Ignoring all lower order terms, the highest order term in F is

$$H(x) = -(a^{ij}(x) - a^{ij}(x_0))\partial_i\partial_j u(x)$$

In order to use the constant coefficients estimate, we need to take the Hölder norm of F, and that involves H as well. It is clear that when we take the difference $H(x) - H(y)$ we have a small coefficient

$$\frac{1}{2}(a^{ij}(x) + a^{ij}(y) - 2a^{ij}(x_0)))$$

multiplying the difference quotient

$$\frac{\partial_i\partial_j u(x) - \partial_i\partial_j u(y)}{|x-y|^\alpha}$$

This will give rise to a term of the form $\epsilon|u|^*_{2,\alpha;\Omega}$ with small ϵ. There is however also the a term that is of the form

$$\frac{\partial_i\partial_j u(x) + \partial_i\partial_j u(y)}{2}\left[\frac{a^{ij}(x) - a^{ij}(y)}{|x-y|^\alpha}\right]$$

This is bounded by a constant (depending on the C^α norm of a^{ij}) times the C^0 norm of second derivatives of u. Nothing is small here, but there is an interpolation inequality that says that, for any $\epsilon > 0$ there exists a constant $C = C(\epsilon)$ so that

$$|u|^*_{j,\beta;\Omega} \le C|u|_{0;\Omega} + \epsilon[u]^*_{2,\alpha;\Omega}$$

if $j = 0,1,2$, $0 \le \alpha, \beta \le 1$ and $j + \beta < 2 + \alpha$. Thus intermediate terms like the one above are bounded by a small multiple of the top term plus a large multiple of the desired right-hand side. Using this strategy we derive an estimate

$$[u]^*_{2,\alpha,B} \le \epsilon[u]^*_{2,\alpha B} + C(\epsilon)\left[|u|^*_{0,\Omega} + |f|^{(2)}_{0,\Omega}\right]$$

on a small ball. The main idea of the interpolation estimate is the following. For instance, suppose we want to prove that

$$[u]^*_1 \le \epsilon[u]^*_2 + C|u|_0$$

Take a point $x \in \Omega$ and let d_x be its distance to the boundary. We take a length $d = \delta d_x$, with $\delta > 0$ small enough. Then we take the segment $[x_1, x_2]$ of length $2d$ parallel to the i-th axis so that x is its middle. Clearly

$$|\partial_i u(\overline{x})| = \frac{|u(x_1) - u(x_2)|}{2d} \leq \frac{1}{d}|u|_0$$

for some point $\overline{x}$ in the segment. Then we write

$$\begin{aligned}&|\partial_i u(x)| = |\partial_i u(\overline{x}) + \textstyle\int_{\overline{x}}^{x} \partial_i \partial_i u(y) dy| \\ &\leq \tfrac{1}{d}|u|_0 + d \sup |\partial_i \partial_i u|\end{aligned}$$

and therefore

$$d_x |\partial_i u(x)| \leq \delta^{-1}|u|_0 + 4\delta[u]_2^*$$

Further details are in Gilbarg and Trudinger, Ch. 6.