1. Let $F \subset L^2(\mathbb{R}^n)$ be the set $F = \{ f \in L^2; (1 + |\xi|)^\epsilon \hat{f} \in L^1(\mathbb{R}^n) \}$ where $\hat{f}$ is the Fourier transform and $\epsilon > 0$. Consider the norm $\|f\| = \|f\|_{L^2} + \|(1 + |\xi|)^\epsilon \hat{f}\|_{L^1}$ for $f \in F$.

(a) Prove that $F$ is a Banach space.
(b) Prove that $F$ is continuously embedded in $C^\alpha(\mathbb{R}^n)$ for $0 < \alpha < \epsilon$. (Recall that $C^\alpha$ is the space of Hölder continuous functions of order $\alpha$).
(c) Let $\phi \in \mathcal{D}_K(\mathbb{R}^n)$ have compact support included in $K \subset \subset \mathbb{R}^n$. Prove that the map $f \mapsto T(f) = (\phi f)|_K$,

$$T : F \to C(K)$$

has range in

$$C(K) = \{ g; g : K \to \mathbb{C}, g \text{ continuous} \}.$$ 

$C(K)$ is a Banach space with norm $\|g\|_{C(K)} = \sup_{x \in K} |g(x)|$. Prove that the linear map $T$ is a compact map between the Banach spaces $F$ and $C(K)$.

2. (Poisson summation formula). Let $f \in L^1(\mathbb{R}^n)$. Assume that $|\hat{f}(k)| \leq C(1 + |k|)^{-n-1}$ holds for all $k \in \mathbb{R}^n$ and also $|f(x)| \leq C(1 + |x|)^{-n-1}$ holds for all $x \in \mathbb{R}^n$. Here $\hat{f}(k) = \int e^{-ix \cdot k} f(x)dx$ is the Fourier transform. Without loss of generality we take $f(x) = (2\pi)^{-n} \int e^{ix \cdot k} \hat{f}(k) dk$ for all $x$.

(a) Consider the function

$$f_p(x) = \sum_{j \in \mathbb{Z}^n} f(x + 2\pi j)$$

for $x \in Q = [-\pi, \pi]^n$. Prove that this series converges in norm in $L^1(Q)$. Prove also that the Fourier coefficients of $f_p(x)$,

$$\hat{f}_p(m) = \int_Q e^{-ix \cdot m} f_p(x) dx$$

1
are given by the restriction of \( \hat{f} \) to \( \mathbb{Z}^n \):
\[
\hat{f}_p(m) = \hat{f}(m).
\]

(b) Prove that if two functions in \( L^1(Q) \) have the same Fourier coefficients, then they coincide.

c) Prove that
\[
\sum_{j \in \mathbb{Z}^n} f(x + 2\pi j) = (2\pi)^{-n} \sum_{m \in \mathbb{Z}^n} \hat{f}(m)e^{im \cdot x}
\]
holds everywhere, and the sums are absolutely and uniformly convergent. The classical Poisson summation formula is obtained by taking \( x = 0 \).

3. We say that a point \( x \in B \subset X \) of a subset \( B \) of a locally convex space \( X \) is extremal in \( B \) if from \( x = tb_1 + (1 - t)b_2 \) with \( b_1, b_2 \in B \) and \( t \in (0, 1) \) it follows that \( b_1 = b_2 = x \). Let \( Q \subset \mathbb{R}^n \) be the unit cube, \( Q = [0, 1]^n \). Describe the extremal points of the unit ball \( B = \{x; \|x\| \leq 1\} \) in the following Banach spaces:
   
   (a) \( L^p(Q) \), \( 1 < p < \infty \),
   (b) \( L^\infty(Q) \),
   (c) \( L^1(Q) \),
   (d) \( C(Q) \) (continuous functions with uniform convergence).
   (e) Show that \( L^1(Q) \) is not the dual of any normed linear space.
   (f) Show that \( C(Q) \) is not the dual of any linear normed space.
   (g) Show that any extremal point in the unit ball of the dual \( (C(Q))^* \) of \( C(Q) \) is of the form \( \alpha \delta(x - x_0) \), with \( \alpha \) a (complex or real) number with \( |\alpha| = 1 \), and \( \delta(x - x_0) \) the Dirac delta function concentrated at \( x_0 \in Q \).

4. Consider the operator
\[
(Lu)(x, y) = -(\Delta u)(x, y) + \sin(2\pi x)\cos(4\pi y) \int_Q \sin(6\pi t) \sin(14\pi s) u(t, s) dtds
\]
defined for \( u \in H^2(Q) \cap H^1_0(Q) \) where \( Q = (0, 1) \times (0, 1) \) and \( \Delta \) is the Laplacian.
(a) Is this a bounded operator? Is it compact? Is it invertible (that means, is 0 in the resolvent set)? Is it selfadjoint?

(b) Let \( f \in L^2(\Omega) \). Can you always solve the equation \( Lu = f \) with \( u \in H^2(\Omega) \cap H^1_0(\Omega) \)? Sometimes? Never?

(Please provide proofs for your answers.)

5. Consider the operator defined in \( \mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \) by the equation \( A(f) = u \) where \( u \in \mathcal{S}(\mathbb{R}^n) \) is the solution of

\[
\Delta^2 u + u = f
\]

and \( \Delta^2 \) is the square of the Laplacian, and \( \mathcal{S}(\mathbb{R}^n) \) is the Schwartz class of rapidly decaying functions.

(a) Show that \( A \) admits a unique extension to a bounded, selfadjoint operator in \( L^2(\mathbb{R}^n) \).

(b) Compute the spectrum \( \sigma(A) \) of \( A \), the point spectrum and and the continuous spectrum.

(c) Use the Fourier transform to compute the spectral measure of \( A \).

(Here I want a formula for the honest measure \( \mu_f \) on \( \mathbb{R} \), \( d\mu_f(\lambda) = d(E_\lambda f, f) \), for any \( f \) fixed in \( L^2(\mathbb{R}^n) \).)

6. Let

\[
P_u = -\sum_{i,j=1}^{n} \partial_i (a_{ij}(x) \partial_j u)
\]

be a uniformly elliptic operator in the bounded open set \( \Omega \subset \mathbb{R}^n \) with smooth boundary. Assume that \( a_{ij}(x) \) are bounded and their first derivatives exist, are continuous and bounded. Let

\[
(Ku)(x) = \int_{\Omega} k(x, y) u(y) dy
\]

where \( k(x, y) \) is real valued, symmetric, \( k(x, y) = k(y, x) \) and \( k \in L^2 \)

\[
\int_{\Omega \times \Omega} k^2(x, y) dx dy < \infty.
\]
Consider the operator
\[ Lu = Pu + Ku \]
defined on \( D = H^1_0(\Omega) \cap H^2(\Omega) \).

(a) Prove that \( L : D \subset L^2(\Omega) \to L^2(\Omega) \) is selfadjoint.

(b) Assume that \( L \) is one-to-one. Prove that \( L \) is onto and that there exists a constant \( C \) so that

\[ \| u \|_{H^2(\Omega)} \leq C \| f \|_{L^2(\Omega)} \]

holds for any \( f \in L^2(\Omega), \ u \) such that \( Lu = f \).

**Important:** You may assume without proof the Poincaré inequality

\[ \int_{\Omega} |\nabla u|^2 \, dx \geq c \int_{\Omega} |u|^2 \, dx \]

valid for all functions in \( H^1_0(\Omega) \).