Maximum principles, Harnack inequality for classical solutions

Introduction to PDE

This is mostly following Evans, Chapter 6.

1 Main Idea

We consider an elliptic operator in non-divergence form

$$p(x,D)u = -\sum_{ij} a_{ij}\partial_{ij}u + \sum_j b_j\partial_j u + cu$$

where the matrix (a_{ij}) is symmetric, and uniformly elliptic $(a_{ij}) \geq \gamma I$, for some $\gamma > 0$. In terms of regularity of the coefficients, let us assume the are continuous functions.

As opposed to the interior and boundary regularity estimates, where we worked with integral quantities, here we work with pointwise estimates on the solution. The point is the following: assume $u \in C^2(\Omega)$ attains a maximum at $x_0 \in \Omega$. Then we can immediately conclude that the gradient of u vanishes at x_0, and the Hessian of u is non-positive definite at x_0, i.e.

$$\nabla u(x_0) = 0 \text{ and } \nabla^2 u(x_0) \leq 0.$$

In particular, if say $c \equiv 0$, this implies in view of the ellipticity of the a_{ij} that

$$p(x,D)u(x_0) \geq 0$$

at a point where the maximum is attained (in the interior of Ω). So, if the (strict) reverse inequality holds on all of Ω, we shouldn't have maxima in the interior. Also, if the (not necessarily strict) reverse inequality holds, we may still have an interior maximum, as long as u is locally constant. We make these ideas precise below.

Note: For the elliptic operator $p(x,D)u = -\Delta u$, all the following results follow for free from the mean value theorem, as we saw earlier in the class.

2 Weak maximum principle

The weak maximum principle tells us that extrema of solutions to elliptic equations are dominated by their extrema on the boundary.

Theorem 1 (Weak maximum principle)**.** *Assume* $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$*, and* $c \equiv 0$ *in* Ω*.*

(i) If $p(x, D)u \leq 0$ *in* Ω *(subsolution), then* $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$*.*

(ii) If $p(x, D)u \geq 0$ *in* Ω *(supersolution), then* $\min_{\bar{\Omega}} u = \min_{\partial\Omega} u$*.*

In particular, for a smooth solution u *of* $p(x, D)u = 0$ *in* Ω*, we have* $\max_{\bar{\Omega}} |u| = \max_{\partial\Omega} |u|$*.*

Proof. We only need to prove item (i), since if u is a supersolution, then $-u$ is a subsolution, and $\min u = -\max(-u)$.

First, we show that an interior maximum cannot exist for a strict subsolution. Assume by contradiction that $p(x, D)u < 0$ in Ω, and there exits $x_0 \in \Omega$ such that $u(x_0) = \max_{\bar{\Omega}} u$. Since $c = 0$ (by assumption), and $\nabla u(x_0) = 0$ (since we evaluate at a maximum), we have

$$0 > p(x, D)u(x_0) = -\sum_{ij} a_{ij}\partial_{ij}u(x_0). \tag{1}$$

On the other hand, the Hessian (symmetric matrix) is non-positive definite when evaluated at a maximum, while the matrix (a_{ij}) (which is also symmetric) is by assumption elliptic, and hence positive definite, which implies

$$-\sum_{ij} a_{ij}\partial_{ij}u(x_0) \geq 0$$

a contradiction with (1). Thus, for a strict subsolution the maximum must occur on $\partial\Omega$.

Second, we let $\lambda > \|b\|_{L^\infty}/\gamma$, and define for any $x \in \Omega$

$$u_\epsilon(x) = u(x) + \epsilon e^{\lambda x_1}$$

where $\epsilon > 0$ is arbitrary. The above defined function satisfies

$$\begin{aligned} p(x, D)u_\epsilon(x) &= p(x, D)u(x) - \epsilon\lambda^2 a_{11} e^{\lambda x_1} + \epsilon\lambda b_1 e^{\lambda x_1} \\ &\leq -\epsilon\lambda e^{\lambda x_1} \left(\lambda\gamma - \|b\|_{L^\infty}\right) < 0 \end{aligned}$$

for any $\epsilon > 0$ in view of our choice of λ. In the above inequality we've used that due to uniform ellipticity $a_{11} \geq \gamma$. By the first step, u_ϵ attains its global maximum on the boundary. Passing $\epsilon \to 0$ concludes the proof. □

Theorem 1 can be augmented to include a zero order term c which is positive. Let us introduce the notation $u^+ = \max\{u, 0\}$ and $u^- = -\min\{u, 0\} = \max\{-u, 0\}$.

Theorem 2 (Weak maximum principle with zero order term)**.** *Assume* $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$*, and* $c \geq 0$ *in* Ω*.*

(i) If $p(x, D)u \leq 0$ *in* Ω*, then* $\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u^+$*.*

(ii) If $p(x, D)u \geq 0$ *in* Ω*, then* $\min_{\bar{\Omega}} u \geq -\max_{\partial\Omega} u^-$*.*

In particular, if $p(x, D)u = 0$ *in* Ω*, we have* $\max_{\bar{\Omega}} |u| = \max_{\partial\Omega} |u|$*.*

Proof. Upon changing u to $-u$, we only need to prove (i). Let

$$U = \{x \in \Omega \colon u(x) > 0\}.$$

Then, for all $x \in U$ we have

$$\tilde{p}(x, D)u(x) := p(x, D)u(x) - c(x)u(x) \leq -c(x)u(x) \leq 0.$$

Applying Theorem 1 to the elliptic operator $\tilde{p}(x, D)$, which has no zero order term, we obtain that

$$\max_{\bar{U}} u = \max_{\partial U} u \leq \max_{\partial\Omega} u^+.$$

In the above inequality we used that on $\partial U \setminus \partial\Omega$ we have $u = 0$, by the definition of U and the continuity of u. Lastly, we trivially have

$$\max_{\bar{\Omega}\setminus U} u \leq 0 \leq \max_{\partial\Omega} u^+$$

which concludes the proof. □

3 Strong maximum principle

The strong maximum principle tells us that for a solution of an elliptic equation, extrema can be attained in the interior if and only if the function is a constant. The key ingredient for the proof of the strong maximum principle is the following lemma, due to E. Hopf.

Lemma 1 (Hopf's Lemma)**.** *Assume $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$, and $c \equiv 0$ in Ω. If*

$$p(x, D)u \leq 0 \text{ in } \Omega$$

and there exits $x_0 \in \partial\Omega$ such that

$$u(x_0) > u(x) \text{ for all } x \in \Omega \tag{2}$$

and Ω has the interior ball property at x_0 (i.e., there exits a ball $B \subset \Omega$ such that $x_0 \in \partial B$, smooth domains have this property), then we have

$$\frac{\partial u}{\partial \nu}(x_0) > 0 \tag{3}$$

where ν is the outer normal vector to B at x_0. If $c \geq 0$, (3) holds under the extra assumption that $u(x_0) \geq 0$.

Proof. Recall that $\frac{\partial u}{\partial \nu}(x_0) = \nabla u(x_0) \cdot \nu$, so that if x_0 is such that (2) holds, we directly get $\frac{\partial u}{\partial \nu}(x_0) \geq 0$. The point is here that we have a strict inequality in (3).

Assume $c \geq 0$, and denote the ball contained in Ω such that $x_0 \in \partial B \cap \partial\Omega$, by $B_r(x_1)$. We introduce the function

$$v(x) = e^{-\lambda|x-x_1|^2} - e^{-\lambda r^2}$$

where $\lambda > 0$ is to be chosen later. Note that $v(x_0) = 0$, by the definition of x_1 and r, and in fact $v = 0$ on $\partial B_r(x_1)$.

Applying the elliptic operator, we see that

$$\begin{aligned}
p(x, D)v(x) &= -\sum_{ij} a_{ij}\partial_{ij}v(x) + \sum_j b_j\partial_j v(x) + cv(x) \\
&= e^{-\lambda|x-x_1|^2} \sum_{ij} a_{ij}\left(-4\lambda^2(x-x_1)_i(x-x_1)_j + 2\lambda\delta_{ij}\right) \\
&\quad + e^{-\lambda|x-x_1|^2}\left(c - \sum_j b_j 2\lambda(x-x_1)_j\right) - ce^{-\lambda r^2} \\
&\leq e^{-\lambda|x-x_1|^2}\left(-4\lambda^2\gamma^2|x-x_1|^2 + 2\lambda|\operatorname{Tr} a| + 2\lambda|b||x-x_1| + |c|\right) \\
&\leq e^{-\lambda|x-x_1|^2}\left(-2\lambda^2\gamma^2|x-x_1|^2 + 2\lambda|\operatorname{Tr} a| + |b|^2/\gamma + |c|\right)
\end{aligned}$$

by using that $(a_{ij}) \geq \gamma I$. Therefore, in view of the above estimate, on the open annulus $A = B_r(x_1) \setminus B_{r/2}(x_1)$ we have

$$p(x, D)v(x) \leq e^{-\lambda|x-x_1|^2}\left(-\lambda^2\gamma^2 r^2/2 + 2\lambda|\operatorname{Tr} a| + |b|^2/\gamma + |c|\right) \leq 0$$

if we ensure that λ is sufficiently large (depending on r, γ, a, b, c).

We now use that u is a subsolution, so that for $\epsilon > 0$ the function

$$u_\epsilon(x) = u(x) + \epsilon v(x) - u(x_0)$$

satisfies

$$p(x, D)u_\epsilon(x) = p(x, D)u(x) + \epsilon p(x, D)v(x) - c(x)u(x_0) \leq 0 \tag{4}$$

for all $x \in A$, either if $c = 0$, or if $c \geq 0$ and $u(x_0) \geq 0$. Note in addition that by assumption (2) we can choose ϵ so small that

$$u_\epsilon(x) \leq u(x) - u(x_0) + \epsilon v(x) \tag{5}$$

for all $x \in \partial B_{r/2}(x_1)$ (which is compact, u, v are continuous). But since $v = 0$ on $\partial B_r(x_1)$, by (2) we have that (5) holds for all $x \in \partial A$.

Combining with (4), we can apply the weak maximum principle to u_ϵ on A and conclude that $u_\epsilon(x) \leq 0$ for $x \in \bar{A}$. Since $u_\epsilon(x_0) = 0$, it follows that

$$\begin{aligned} 0 \leq \frac{\partial u_\epsilon}{\partial \nu}(x_0) = \frac{\partial u}{\partial \nu}(x_0) + \epsilon \frac{v}{\partial \nu}(x_0) &= \frac{\partial u}{\partial \nu}(x_0) - 2\lambda\epsilon(x_0 - x_1) \cdot \nu e^{-\lambda r^2} \\ &= \frac{\partial u}{\partial \nu}(x_0) - 2\lambda\epsilon r^2 e^{-\lambda r^2} \end{aligned}$$

since $\nu = x_0 - x_1$. The proof is hence completed by taking any $\epsilon \ll 1$. $\square$

Theorem 3 (Strong maximum principle)**.** *Let Ω be connected, open, and bounded. Assume $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ and $c = 0$ in Ω (resp. $c \geq 0$ in Ω).*

(i) If $p(x, D)u \leq 0$ in Ω, and there exits $x_0 \in \Omega$ such that $\max_{\bar{\Omega}} u = u(x_0)$ (add condition $u(x_0) \geq 0$ for $c \geq 0$), then u is constant.

(ii) If $p(x, D)u \geq 0$ in Ω, and there exits $x_0 \in \Omega$ such that $\min_{\bar{\Omega}} u = u(x_0)$ (add condition $u(x_0) \leq 0$ for $c \geq 0$), then u is constant.

Proof. Assume $u \not\equiv u(x_0)$, since else we're done. Consider the set $C = \{x \in \Omega \colon u(x) = u(x_0)\}$. This set is closed (hence compact since Ω bounded), and non-empty. We now define

$$U = \{x \in \Omega \colon u(x) < u(x_0)\}.$$

By assumption this set is not empty. Pick a point $y \in U$ such that dist (y, C) $<$ dist $(y, \partial\Omega)$, and let B be the largest ball centered at y, that fits inside U.

Then, there exits $x_1 \in \partial B \cap C$. By definition, $u(x_1) > u(x)$ for any $x \in U$. We apply the Hopf Lemma in U, and obtain that

$$0 < \frac{\partial u}{\partial \nu}(x_1) = \nabla u(x_1) \cdot \nu$$

where ν is the outward unit normal to B at x_1, i.e. $\nu = x_1 - y$. This contradicts that u has a maximum at x_1, and so $\nabla u(x_1) = 0$. □

The strong maximum principle is typically used to prove uniqueness of solutions to elliptic Dirichlet boundary value problems. The difference u of two such solutions obeys $p(x, D)u = 0, u|_{\partial\Omega} = 0$, and so if u is not identically 0, by the strong maximum principle u is a non-zero constant. This contradicts the homogenous boundary condition.

4 Harnack inequality and applications

The Harnack inequality is classically used to prove solutions to elliptic equations are Hölder continuous. Let us first recall what happens for harmonic functions.

Theorem 4 (Harnack inequality for harmonic functions)**.** *Assume u is a non-negative solution of $\Delta u = 0$ in Ω. Then for any open, connected subset $U \subset\subset \Omega$, we have*

$$\sup_U u \leq C \inf_U u$$

for some positive constant C that depends only on U (and Ω).

Proof. Let $\operatorname{dist}(\partial\Omega, U) = 4r$. Let $x \neq y \in U$ be such that $|x-y| \leq r$. Then, by the mean value theorem we have

$$\begin{aligned} u(x) = \fint_{B_{2r}(x)} u(z)dz &= \frac{1}{\omega_n 2^n r^n} \int_{B_{2r}(x)} u(z)dz \\ &\geq \frac{1}{\omega_n 2^n r^n} \int_{B_r(y)} u(z)dz \\ &\geq \frac{1}{2^n} \fint_{B_r(y)} u(z)dz = \frac{u(y)}{2^n}. \end{aligned}$$

Now, for any $x, y \in U$, there exists a chain of segments of length $\leq r$, of length N (that depends only on $\operatorname{diam} U$ and r), which connects x to y. The above estimate then implies $u(x) \geq 2^{-nN} u(y)$, and the proof is complete. □

For more general elliptic equations, where we may not have a mean value theorem, the statement is:

Theorem 5 (Harnack inequality)**.** *Assume* $u \geq 0$ *is a* C^2 *solution of*

$$p(x,D)u = -\sum_{ij} a_{ij}\partial_{ij}u + b\cdot\nabla u + cu = 0 \text{ in } \Omega$$

where $(a_{ij}) \geq \gamma I$, *and* $a, b, c \in L^\infty(\Omega)$. *Consider a connected* $U \subset\subset \Omega$. *Then*

$$\sup_U u \leq C \inf_U u \tag{6}$$

for some positive constant C *that depends on* $U, \Omega, \gamma, a, b, c$.

We will give the *classical* proof of the above general statement when looking at the DeGiorgi-Nash-Moser iteration. For now give a *tricky*, but elementary, proof of the Harnack inequality, in the absence of lower order terms (so $b \equiv 0 \equiv c$), and for a_{ij} which are smooth. This is from Evans.

Proof. Without loss of generality $u > 0$ (else consider $u+\epsilon$, and send $\epsilon \to 0$). If we let

$$v = \log u$$

then the Harnack inequality follows once we show that $\|\nabla v\|_{L^\infty(U)} \leq C$. Indeed, in this case for $x_1, x_2 \in U$ we have

$$\frac{u(x_2)}{u(x_1)} = e^{v(x_2)-v(x_1)} \leq e^{|x_1-x_2|\|\nabla v\|_{L^\infty(U)}}$$

and so (6) follows from $\operatorname{diam} U < \infty$.

From the equation we have

$$-\sum_{ij} a_{ij}\partial_{ij}u = 0 \Rightarrow -\sum_{ij} a_{ij}\partial_{ij}v = \sum_{ij} \partial_{ij} a_{ij}\partial_i v \partial_j v =: w. \tag{7}$$

We have introduced w here since by $(a_{ij}) \geq \gamma I$, $|\nabla v|^2 \leq |w|\gamma^{-1}$, and so a sufficient condition for (6) to holds is that $w \in L^\infty(U)$. We shall prove the latter by looking at the elliptic equation obeyed by w, and some sign considerations which hold at the maximum of w.

To get an elliptic equation for w, compute (using that (a_{ij}) is symmetric)

$$\partial_k w = 2\sum_{ij} a_{ij}\partial_{ki}v\partial_j v + \sum_{ij} a_{ij}\partial_i v\partial_j v$$

$$\partial_{kl} w = 2\sum_{ij} a_{ij}\partial_{kli}v\partial_j v + 2\sum_{ij} a_{ij}\partial_{ki}v\partial_{jl}v + R_1$$

where

$$|R_1| \leq \sum_{ij} |\partial_{kl} a_{ij} \partial_i v \partial_j v| + |\partial_k a_{ij} \partial_l (\partial_i v \partial_j v)| \leq \epsilon |\nabla^2 v|^2 + C(\epsilon) |\nabla v|^2$$

for any $\epsilon > 0$. Therefore,

$$-\sum_{kl} a_{kl} \partial_{kl} w = 2 \sum_{ij} a_{ij} \partial_j v \left(\sum_{kl} a_{kl} \partial_{kli} v \right) - 2 \sum_{klij} a_{kl} a_{ij} \partial_{ik} v \partial_{jl} v + R_2 \quad (8)$$

where $|R_2| \leq \epsilon |\nabla^2 v|^2 + C(\epsilon) |\nabla v|^2$.

On the other hand, from (7), we have that

$$\partial_i w = -\sum_{kl} a_{kl} \partial_{kli} v + R_3 \quad (9)$$

where

$$|R_3| \leq C |\nabla v|^2.$$

Combining (8) and (9) yields

$$-\sum_{kl} a_{kl} \partial_{kl} w + \sum_i \tilde{b}_i \partial_i w = -2 \sum_{klij} a_{kl} a_{ij} \partial_{ik} v \partial_{jl} v + R_4 \quad (10)$$

where we have denoted

$$\tilde{b}_i = \sum_{ij} a_{ij} v_j$$

and the bound $|R_4| \leq \epsilon |\nabla^2 v|^2 + C(\epsilon) |\nabla v|^2$ holds. The point here is to use ellipticity one more time and obtain

$$\sum_{klij} a_{kl} a_{ij} \partial_{ik} v \partial_{jl} v \geq \gamma^2 |\nabla^2 v|^2,$$

which inserted into (10) proves

$$-\sum_{kl} a_{kl} \partial_{kl} w + \sum_i \tilde{b}_i \partial_i w \leq -\frac{\gamma^2}{2} |\nabla^2 v|^2 + C |\nabla v|^2 \quad (11)$$

upon letting ϵ be small compared to γ^2, for some $C > 0$.

Let χ be a smooth cutoff function adapted to (U, Ω), and define

$$z = \chi^4 w.$$

The function z is continuous (recall u is C^2) and has compact support, so it attains its maximum at some point $x_0 \in \Omega$. At this point we have $\nabla z(x_0) = 0$, and therefore

$$\chi(x_0)\partial_k w(x_0) = -4\partial_k \chi(x_0) w(x_0). \tag{12}$$

But due to the ellipticity of (a_{ij}) we also have at x_0 that

$$0 \leq -\sum_{kl} a_{kl}\partial_{kl} z + \sum_i \tilde{b}_i \partial_i z = \chi^4 \left(-\sum_{kl} a_{kl}\partial_{kl} w + \sum_i \tilde{b}_i \partial_i w \right) + R_5 \tag{13}$$

where

$$|R_5| \leq C\left(\chi^2|w| + \chi^3|\nabla w|\right) + \chi^3|\tilde{b}||w| \leq C\chi^2|w| + \chi^3|\nabla v||w|.$$

In the bound for R_5 we have used (12) and the definition of $\tilde{b}$.

We combine (7), (11), and (13) with ellipticity (through $|\nabla v|^2 \leq C|w|$) and obtain

$$\begin{aligned} \chi^4 w^2 \leq \frac{C}{2}\chi^4|\nabla^2 v|^2 &\leq C\chi^4|\nabla v|^2 + C\chi^2|w| + \chi^3|\nabla v||w| \\ &\leq C\chi^2|w| + C\chi^3|w|^{3/2} \end{aligned} \tag{14}$$

for some constant C that depends on U, Ω, γ, a. But (14) shows that $\chi^2 w$ is bounded on Ω, and since $\chi \equiv 1$ on U, it also gives a bound on $\|w\|_{L^\infty(U)}$, thereby concluding the proof. □