Maximal functions, non-tangential limit

Analysis III

1 Maximal functions

Definition 1 *Let $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$. The (Hardy-Littlewood) maximal function $m_f(x)$ is defined as*

$$m_f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy$$

Here $B(x,r) \subset \mathbb{R}^n$ is the Euclidean ball centered at x with radius r, and $|B|$ denotes Lebesgue measure. We introduce a variant of the maximal function for convenience. We define

$$n_f(x) = \sup_{Q_x} \frac{1}{|Q_x|} \int_{Q_x} |f(y)|dy$$

and call it the cubic maximal function. Here $Q_x = x + Q$ are non-degenerate open cubes centered at x and with sides parallel to the axis, $Q = (-r,r)^n$ where $r > 0$ is a real number. We use the notation $x + Q = \{y | \, \exists q \in Q, \, y = x + q\}$, It is clear that there exist two constants depending on dimension of space only so that

$$cn_f(x) \le m_f(x) \le Cn_f(x)$$

holds for all f. It is also clear that if $f \in L^\infty(\mathbb{R}^n)$ then $\|m(f)\|_{L^\infty} \le \|f\|_{L^\infty}$ holds.

Lemma 1 *Let $Q_1 \subset Q_2 \subset \cdots \subset Q_k$ be a finite family of nonempty open cubes with sides parallel to the axis, centered at the origin. Assume that a bounded set $S \subset \mathbb{R}^n$ is given, and we have associated to each $x \in S$ a number $i(x) \in \{1, \ldots k\}$ and with it the cube $Q_x = x + Q_{i(x)}$, translate of the corresponding $Q_{i(x)}$. Then there exist a finite collection of points $x_1, \ldots, x_l$ such that*

$$S \subset \cup_{i=1}^{l} Q_{x_i}$$

and for any $y \in \mathbb{R}^n$ there exist at most 2^n cubes Q_{x_i} to which y may belong.

Proof. We pick x_1 so that $i(x_1)$ is largest available. Then we choose $x_2 \in S \setminus Q_{x_1}$ so that $i(x_2)$ is largest available from $i(x)$ with $x \in S \setminus Q_{x_1}$, then $x_2 \in S \setminus (Q_{x_1} \cup Q_{x_2})$ with maximal available $i(x)$, and we continue inductively. We obtain a sequence of open cubes centered at $x_i \in S$ such that $x_i \notin Q_{x_j}$ for any $i \neq j$. This implies that the distance between the centers is bounded below by the smallest of the half-lengths of sides of Q_i, i.e. by half the side length of Q_1, which is positive. Because $x_i \in S$, and S is bounded, the sequence must be finite and the union of cubes covers S. Now assume that $y \in \mathbb{R}^n$ is arbitrary. We consider hyperplanes passing through y with sides parallel to the axis. These form 2^n octants. If x_i and x_j belong to the same octant, and $y \in Q_{x_i} \cap Q_{x_j}$ then, if, say, Q_{x_j} has the bigger side of the two, then $x_i \in Q_{x_j}$, contradicting the construction. So at most one Q_{x_i} per octant can contain y.

Theorem 1 *There exists a constant c depending only on dimension such that, whenever $f \in L^1(\mathbb{R}^n)$ and*

$$F_s = \{x \in \mathbb{R}^n \mid m_f(x) > s > 0\}$$

then

$$|F_s| \leq \frac{c}{s} \|f\|_{L^1(dx)}.$$

Proof. We are going to use n_f, prove the inequality for it, and deduce it therefore for m_f. Let

$$S \subset \{x \mid n_f(x) > s\}$$

be a compact set. For any $x \in S$ there exists (by definition) a non-empty open cube Q_x centered at x such that

$$\frac{1}{|Q_x|} \int_{Q_x} |f(y)| dy > s$$

Because of the continuity of the integral, there exists a neighborhood U of x, so that for every $p \in U$ we have

$$\frac{1}{|Q_p|} \int_{Q_p} |f(y)| dy > s$$

where $Q_p = p - x + Q_x$. We cover S with finitely many such neighborhoods, and select the cubes $Q_1, \dots Q_k$, associated to them (we may order them by size). Thus, for any $p \in S$ there exists an index $i(p)$ such that

$$\frac{1}{|Q_p|} \int_{Q_p} |f(y)| dy > s$$

where $Q_p = p + Q_{i(p)}$. By the Lemma, we have a finite collection of points $x_1, \dots x_l$ so that

$$S \subset \cup_{i=1}^{l} Q_{x_i}$$

and every point $y \in \mathbb{R}^n$ belongs to at most 2^n sets Q_{x_i}. This means that

$$\sum_{i=1}^{l} \chi_{Q_{x_i}}(y) \le 2^n \chi_{\cup Q_i}(y)$$

where χ_Q denotes the characteristic function of the set Q. Now

$$\begin{array}{c} |S| \le |\cup Q_{x_i}| \le \sum_i |Q_{x_i}| \\ \le \frac{1}{s} \sum_i \int_{Q_{x_i}} |f(y)| dy = \frac{1}{s} \int \left(\sum_i \chi_{Q_{x_i}}(y) \right) |f(y)| dy \\ \le 2^n \frac{1}{s} \int_{\cup Q_i} |f(y)| dy \le \frac{2^n}{s} \|f\|_{L^1(dx)}. \end{array}$$

This concludes the proof. The map $f \mapsto m_f$ is sublinear, in the sense that $m_{f+g} \le m_f + m_g$ and $m_{cf} = |c| m_f$. If $h \in L^p$ we take the two functions $g(x)$ and $f(x) = h(x) - g(x)$ where

$$g(x) = \begin{cases} h(x), & \text{if } |h(x)| \le 1 \\ 0, & \text{if } |h(x) > 1. \end{cases}$$

Clearly $g \in L^\infty$ and $f \in L^1$, and by sublinearity

$$m_h \le m_f + m_g.$$

Therefore m_h is almost everywhere finite. The maximal operator is not bounded in L^1, but is of weak type $1-1$. (That means precisely the inequality in the theorem).

Definition 2 *Let T be a sublinear operator mapping $L^p(\mathbb{R}^n)$, $1 \le p \le \infty$ into a space of measurable functions. We say that T is of weak-type (p, q), if there exists a constant C such that*

$$|\{x \mid |T(f)(x)| > s\}| \le \left(C s^{-1} \|f\|_{L^p(dx)} \right)^q$$

holds for all $f \in L^p(\mathbb{R}^n)$.

The maximal operator is bounded all L^p, $1 < p$. In order to see this, we refer to the distribution function

$$\lambda_g(s) = |\{x \mid |f(x)| > s\}|$$

and recall that

$$\|g\|_{L^p(dx)}^p = -\int_0^\infty s^p d\lambda_g(s) = p\int_0^\infty s^{p-1}\lambda_g(s)ds$$

Theorem 2 *There exist constants C_{pn} depending on dimension n and on $p > 1$ such that*

$$\|m_f\|_{L^p(dx)} \leq C_{pn}\|f\|_{L^p(dx)}$$

holds for every $f \in L^p(dx)$.

Proof. Let $1 < p \leq \infty$, let $s > 0$. We let

$$f_s(x) = \begin{cases} f(x) & \text{if } |f(x)| > s, \\ 0 & \text{if } |f(x)| \leq s, \end{cases}$$

and

$$f^s(x) = \begin{cases} f(x) & \text{if } |f(x)| \leq s, \\ 0 & \text{if } |f(x)| > s, \end{cases}$$

Then clearly $f = f_s + f^s$, $f_s \in L^1(dx)$, $f^s \in L^\infty(dx)$. We denote by $\lambda, \lambda_s, \lambda^s$ the distribution functions of m_f, m_{f_s}, m_{f^s}. By subadditivity,

$$m_f \leq m_{f_s} + m_{f^s}$$

and hence $\lambda(2s) \leq \lambda_s(s) + \lambda^s(s)$. But $m_{f^s} \leq \|f^s\|_{L^\infty(dx)} \leq s$, so $\lambda^s(s) = 0$. So, $\lambda(2s) \leq \lambda_s(s)$ holds. We obtain therefore

$$\begin{array}{c} \|m_f\|_{L^p(dx)}^p = p2^p\int_0^\infty s^{p-1}\lambda(2s)ds \leq p2^p\int_0^\infty s^{p-1}\lambda_s(s)ds \\ \leq p2^pC\int_0^\infty s^{p-2}\|f_s\|_{L^1(dx)} = Cp2^p\int_0^\infty s^{p-2}\int_{|f(x)|>s}|f(x)|dx \\ = Cp2^p\int_{\mathbb{R}^n}|f(x)|\int_0^{|f(x)|}s^{p-2}ds = C\frac{p2^p}{p-1}\int_{\mathbb{R}^n}|f(x)|^p dx \end{array}$$

This proves the theorem. Now note that if $\phi = \frac{1}{|B_1|}\chi_{B_1}$ is the normalized characteristic function of the unit ball (normalized so as to have integral equal to one) then the maximal function can be written

$$m_f(x) = \sup_{\epsilon>0}(\phi_\epsilon * |f|)$$

where

$$\phi_\epsilon(x) = \epsilon^{-n}\phi(\frac{x}{\epsilon})$$

and $(f * g)(x) = \int f(x-y)g(y)dy$ is convolution. Similarly, if ψ is the characteristic function of the unit cube $(-\frac{1}{2}, \frac{1}{2})^n$, then

$$n_f = \sup_{\epsilon>0} (\psi_\epsilon * |f|)$$

Many operators in analysis are bounded by the maximal operator. Let us consider real positive numbers c_k, and balls centered at the origin $B_k = B(0, r_k)$. Let

$$\phi = \sum_{k=1}^{\infty} c_k \chi_k$$

where χ_k is the characteristic function of B_k. Then

$$\begin{array}{c} (\phi_\epsilon * |f|)(x) = \sum_{k=1}^{\infty} c_k \epsilon^{-n} \int_{|y|\leq \epsilon r_k} |f(x-y)|dy \\ = \sum_{k=1}^{\infty} c_k |B_k| \left(\epsilon^n |B_k|\right)^{-1} \int_{\epsilon B_k} |f(x-y|dy \\ \leq m_f(x) \sum_{k=1}^{\infty} c_k |B_k| = m_f(x) \|\phi\|_{L^1(dx)} \end{array}$$

By approximating from below by step functions we have

Proposition 1 *Let $\phi(x) = a(|x|)$ be a radial function with $a(r) > 0$, $a(r)$ decreasing. Then*

$$\sup_{\epsilon>0} (|f| * \phi_\epsilon)(x) \leq \|\phi\|_{L^1(dx)} m_f(x)$$

holds for every $f \in L^p(dx)$.

In particular, by choosing $a(r) = \frac{c_n}{(1+r^2)^{\frac{n+1}{2}}}$, $r = |x|$ we obtain,

$$(\phi_\epsilon * f)(x) = \int_{\mathbb{R}^n} f(y) p_\epsilon(x-y)dy = P_\epsilon(f)(x)$$

the Poisson integral of f where $p_t(z) = c_n \frac{t}{(t^2+|z|^2)^{\frac{n+1}{2}}}$ is the Poisson kernel for the half-space.

Proposition 2 *Let $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$ and let $P_t(f)(x)$ be its Poisson integral, $t > 0$. Then*

$$|P_t(f)(x)| \leq m_f(x)$$

Exercise 1 *Prove that there exists a positive constant C, so that*

$$m_f(x) \leq C \sup_{t>0} P_t(f)(x)$$

holds for any $f > 0$ function in L^p.

We define now, for $x_0 \in \mathbb{R}^n$ and aperture α, the cone in $\mathbb{R}^{n+1}_+$

$$\Gamma_\alpha(x_0) = \{(x,t) \in \mathbb{R}^{n+1} \mid |x - x_0| < \alpha t\}$$

Definition 3 *We say that $u(x,t)$ has the nontangential limit l at x_0 if, for any $\alpha > 0$*

$$\lim_{(x,t)\to(x_0,0),\ (x,t)\in\Gamma_\alpha(x_0)} u(x,t) = l$$

where (x,t) tends to $(x_0, 0)$ within the cone $\Gamma_\alpha(x_0)$.

Theorem 3 *If $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$ then the Poisson integral*

$$P_t(f)(x) = \int_{\mathbb{R}^n} f(y) p_t(x-y) dy$$

has the nontangential limit $f(x_0)$ at any point x_0 belonging to the set of Lebesgue points of f.

Proof. We use the fact that, for $(x,t) \in \Gamma_\alpha(x_0)$

$$p_t(x-y) \leq d_\alpha p_t(x_0 - y)$$

with d_α a positive constant depending on α and n. Then we write

$$\begin{gathered} |P_t(f)(x) - f(x_0)| = \left|\int_{\mathbb{R}^n} p_t(x-y)\,[f(y) - f(x_0)]\,dy\right| \\ \leq \int_{\mathbb{R}^n} |f(y) - f(x_0)| p_t(x-y) dy \leq d_\alpha \int_{\mathbb{R}^n} |f(y) - f(x_0)| p_t(x_0 - y) dy \end{gathered}$$

The theorem is proven on the basis of the following exercise:

Exercise 2 *Let $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$ and let $\phi(x) = a(|x|)$ be a function with $a(r)$ positive and decreasing, and with $\int_{\mathbb{R}^n} \phi(x) dx = 1$. Let $\phi_\epsilon(x) = \epsilon^{-n}\phi(\frac{x}{\epsilon})$. Then*

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^n} |f(x_0 - y) - f(x_0)| \phi_\epsilon(y) dy = 0$$

holds for any point x_0 in the Lebesgue set of f.