Littlewood-Paley decomposition and applications.

Intoduction to PDE

1 Littlewood-Paley decomposition

We start with a smooth, nonincreasing, radial nonnegative function $\phi(r)$ satisfying

$$\begin{cases} \phi(r) = 1, & \text{for } 0 \le r \le a, \\ \phi(r) = 0, & \text{for } b \le r, \\ 0 < a < b. \end{cases}$$

We define

$$\psi_0(r) = \phi\left(\frac{r}{2}\right) - \phi(r),$$

$$(\Delta_{-1}u)(x) = (\phi(D)u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \phi(|\xi|)\widehat{u}(\xi) d\xi, \tag{1}$$

$$(\Delta_0 u)(x) = (\psi_0(D)u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \psi_0(|\xi|)\widehat{u}(\xi) d\xi, \tag{2}$$

$$\psi_j(r) = \psi_0(2^{-j}r)$$

and

$$(\Delta_j u)(x) = (\psi_j(D)u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \psi_j(|\xi|)\widehat{u}(\xi) d\xi, \tag{3}$$

where

$$\widehat{u}(\xi) = \mathcal{F}u(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x) dx.$$

We choose $a = \frac{1}{2}$, $b = \frac{5}{8}$. We set also

$$S_k(u) = \sum_{j=-1}^{k} \Delta_j(u) \tag{4}$$

Proposition 1. *If $u \in \mathcal{S}'(\mathbb{R}^n)$, then*

$$u = \sum_{j=-1}^{\infty} \Delta_j u,$$

$$\mathrm{supp}\mathcal{F}(\Delta_j u) \subset 2^j \left[\frac{1}{2}, \frac{5}{4}\right],$$

for $j \geq 0$, and in particular

$$\Delta_j \Delta_k \neq 0 \Rightarrow |j-k| \leq 1, \quad \text{for } j,k \geq 0.$$

Moreover,

$$(\Delta_j + \Delta_{j+1} + \Delta_{j+2})\Delta_{j+1} = \Delta_{j+1},$$

for $j \geq 0$,

$$\Delta_j \left(S_{k-2}(u)\Delta_k(v)\right) \neq 0 \Rightarrow k \in [j-2, j+2]$$

for $j \geq 2$, $k \geq 2$.

Proposition 2. *(Bernstein inequalities)*

$$\|\Delta_j u\|_{L^q(\mathbb{R}^n)} \leq C 2^{j(\frac{n}{p}-\frac{n}{q})} \|\Delta_j u\|_{L^p(\mathbb{R}^n)}, \quad q \geq p \geq 1,$$

$$\|S_j u\|_{L^q(\mathbb{R}^n)} \leq C 2^{j(\frac{n}{p}-\frac{n}{q})} \|S_j u\|_{L^p(\mathbb{R}^n)}, \quad q \geq p \geq 1,$$

and

$$2^{jm}\|\Delta_j u\|_{L^p(\mathbb{R}^n)} \leq C \sum_{|\alpha|=m} \|\partial^\alpha \Delta_j u\|_{L^p(\mathbb{R}^n)} \leq C 2^{jm} \|\Delta_j u\|_{L^p(\mathbb{R}^n)}$$

We introduce the inhomogeneous Besov space with norm

$$\|u\|_{B^s_{p,q}(\mathbb{R}^n)} = \left\| \left\{ 2^{sj} \|\Delta_j u\|_{L^p(\mathbb{R}^n)} \right\}_j \right\|_{\ell^q(\mathbb{N})}$$

Proposition 3. *(Littlewood-Paley) Let $1 < p < \infty$. Then $(\mathbb{I} - \Delta)^{\frac{s}{2}} u \in L^p(\mathbb{R}^n)$ if and only if $\Delta_j u \in L^p(\mathbb{R}^n)$ for all $j \geq -1$ and*

$$\|u\|_{W^{s,p}(\mathbb{R}^n)} \sim \left\| \sqrt{\sum_{j \geq -1} 2^{2js} |\Delta_j(u)|^2} \right\|_{L^p(\mathbb{R}^n)}$$

Proposition 4. *Embeddings:*

$$B^s_{p,r}(\mathbb{R}^n) \subset B^{s-\left(\frac{n}{p}-\frac{n}{q}\right)}_{q,r}(\mathbb{R}^n), \quad q \geq p \geq 1,$$

$$B^0_{p,2}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n) \subset B^0_{p,p}(\mathbb{R}^n) \quad p \geq 2,$$
$$B^0_{p,p}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n) \subset B^0_{p,2}(\mathbb{R}^n) \quad p \leq 2.$$

Products: Consider two functions, $u = \sum_{k\geq -1} \Delta_k u$ and $v = \sum_{l\geq -1} \Delta_l(v)$. Then we have the Bony decomposition

$$\Delta_j(uv) = I_j(u,v) + I_j(v,u) + R_j(u,v) \tag{5}$$

with

$$I_j(u,v) = \sum_{k\in[j-2,j+2]} \Delta_j(S_{k-2}(u)\Delta_k(v)) \tag{6}$$

and

$$R_j(u,v) = \sum_{|k-l|\leq 1} \Delta_j(\Delta_k u \Delta_l v) \tag{7}$$

2 Applications

Proposition 5. *Let* $u, v \in H^s(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n)$. *Then*

$$\|uv\|_{H^s(\mathbb{R}^n)} \leq C[\|u\|_{L^\infty(\mathbb{R}^n)}\|v\|_{H^s(\mathbb{R}^n)} + \|v\|_{L^\infty(\mathbb{R}^n)}\|u\|_{H^s(\mathbb{R}^n)}]$$

Here the trick is to write

$$\Delta_j(uv) = \Delta_j(u(\mathbb{I} - S_{j-2})v) + \Delta_j(uS_{j-2}v)$$

The first term is handled thus

$$\begin{aligned}
&\textstyle\sum_j \|\Lambda^s \Delta_j(u(\mathbb{I} - S_{j-2})v)\|^2_{L^2} \\
&\leq \textstyle\sum_j 2^{2js}\|u((\mathbb{I} - S_{j-2})v)\|^2_{L^2} \\
&\leq C\|u\|^2_{L^\infty} \textstyle\sum_j 2^{2js}\|(\mathbb{I} - S_{j-2})v\|^2_{L^2} \\
&\leq C\|u\|^2_{L^\infty} \textstyle\sum_j 2^{2js} \sum_{k\geq j-2} \|\Delta_k v\|^2_{L^2} \\
&\leq C\|u\|^2_{L^\infty} \textstyle\sum_k \|\Lambda^s\Delta_k v\|^2_{L^2} \sum_{j\leq k+2} 2^{2s(j-k)} \\
&\leq C\|u\|^2_{L^\infty}\|v\|^2_{H^s}
\end{aligned}$$

The other term satisfies

$$\Delta_j(uS_{j-2}(v)) = \sum_{|k-j|\leq 2} (\Delta_k u S_{j-2}(v))$$

and is handled using the uniform bound on $\|S_{j-2}v\|_{L^\infty}$:

$$\begin{array}{l} \sum_j \|\Lambda^s \Delta_j(uS_{j-2}(v)\|_{L^2}^2 \\ \leq C \sum_{|l|\leq 2} \sum_j \|S_{j-2}(v)\|_{L^\infty}^2 2^{2js} \|\Delta_{j+l} u\|_{L^2}^2 \\ \leq C\|v\|_{L^\infty}^2 \|u\|_{H^s}^2 \end{array}$$