Degree in Infinite Dimensions

1 Schauder fixed point

Warning: Brouwer's Thm is false in infinite dimensions. Example: $\ell_2(\mathbb{N})$, with unit closed ball B. Then

$$f : B \to \partial B, \qquad f(x) = (\|x\|^2 - 1, x_1, x_2, \dots)$$

is continuous, and if it had a fixed point, the fixed point equations would be $x_1 = 0$, $x_2 = x_1$, $\dots$, $x_{n+1} = x_n$, so the fixed point would be 0, but it had to have norm equal to 1.

Definition 1. *A continuous function $F : S \subset X \to X$, where X is a Banach space, is compact if it maps bounded closed sets to relatively compact sets (sets whose closure is compact)*

Theorem 1. *Let $f : S \to X$ where S is closed and bounded in the Banach space X. Then f is compact iff it is a uniform limit of continuous finite range maps.*

Proof. If f is compact then $K = \overline{f(S)}$ is compact. Given $\epsilon > 0$ there exist $x_1 \dots x_{j(\epsilon)} \in K$ such that the balls B_i of centers x_i and radii ϵ cover K. Let ψ_i be a partition of unity for K subordinated to the cover, i.e $\psi_i \geq 0$ is supported in B_i and $\sum_i \psi_i = 1$ on K. Let

$$f_\epsilon(x) = \sum_{i=1}^{j(\epsilon)} \psi_i(f(x))x_i$$

Then $f_\epsilon(x)$ belongs to the convex hull of x_i and

$$\|f(x) - f_\epsilon(x)\| \leq \sum_{i=1}^{j(\epsilon)} \psi_i(f(x))\|f(x) - x_i\| \leq \epsilon$$

The argument in the other direction is an exercise.

Theorem 2. *(Schauder fixed point). Let S be a closed, convex, bounded subset of a Banach space X, and let $f : S \to S$ be a compact map. Then f has a fixed point.*

Proof. Consider $f_\epsilon(x)$ defined above, and let X_ϵ be the finite dimensional linear spaced spanned by x_i, $i = 1, \dots j(\epsilon)$. Since S is convex and $f_\epsilon(S)$ is contained in the convex hull of $f(S)$ we have $f_\epsilon : S \to S \cap X_\epsilon$. Therefore f_ϵ maps the closed bounded set $S \cap X_\epsilon$ to itself. This is a subset of X_ϵ so we may apply the finite dimensional Brouwer fixed point theorem, and find $x_\epsilon \in X_\epsilon \cap S$ such that $x_\epsilon = f_\epsilon(x_\epsilon)$. Now $f_\epsilon(x_\epsilon)$ has a convergent subsequence by the relative compactness of $f(S)$. Passing to the limit and using $x_\epsilon - f(x_\epsilon) = f_\epsilon(x_\epsilon) - f(x_\epsilon)$, we finish the proof.

2 Leray-Schauder Degree

If X is a Banach space and $\phi = I - K$ where $K : \overline{\Omega} \to X$ is a compact transformation, then we the image under $\phi(S)$ of a closed bounded set is closed. Indeed, if $y_n = \phi(x_n)$ with $x_n \in S$ converges to $y \in X$ then, because S is bounded and K is compact we may extract a subsequence, relabeled x_n, such that $Kx_n \to z$, and then $x_n = \phi(x_n) + Kx_n$ converges to $x = y + Kz$. By continuity, $y = x - Kz$.

If $y_0 \notin \phi(\partial\Omega)$, then it is at positive distance δ from $\partial\Omega$. We take an ϵ-approximation K_ϵ of K with range in X_ϵ, a finite dimensional subspace of X such that $y_0 \in X_\epsilon$. If $\epsilon \le \frac{\delta}{2}$ then $y_0 \notin \phi_\epsilon(\partial\Omega)$ where $\phi_\epsilon = I - K_\epsilon$. We consider

$$\phi_{\epsilon|\, X_\epsilon \cap \overline{\Omega}} : X_\epsilon \cap \overline{\Omega} \to X_\epsilon$$

Definition 2.

$$deg\,(\phi, \Omega, y_0) = deg\left(\phi_{\epsilon|\, X_\epsilon \cap \overline{\Omega}}, \Omega \cap X_\epsilon, y_0\right)$$

This is well defined by the last proposition in the chapter on finite dimensional degree. That means that we may change the finite dimensional space X_ϵ, and we may also change the finite range approximation K_ϵ. This follows by first placing both approximation ranges in a common (larger) finite dimensional space, and the using homotopy.

We note that if $y_0 \notin \phi(\overline{\Omega})$ then $\deg(\phi, \Omega, y_0) = 0$. All results in the chapter on finite dimensional degree are valid. In particular $\deg(\phi, \Omega, y_0)$

depends only on the homotopy class of $\phi : \partial\Omega \to X \setminus \{y_0\}$, where the homotopy is of the form $\phi_t = I - K_t$, with K_t continuous in $t \in [0,1]$ and compact for each t. In particular, the image of an open set under a one-to-one map $\phi = I - K$ is open.

3 First elementary applications

First, an application of Schauder's fixed point theorem. Let $K(s,t)$ be a continuous function and let

$$Ku(s) = \int_0^1 K(s,t)f(t,u(t))dt$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and bounded. Taking $X = C([0,1])$ we have that K is a compact map on any ball $\|u\| \leq R$. By the Schauder fixed point, there exists u constinuous, such that

$$u(s) = Ku(s).$$

Indeed we want to find R such that K maps the ball of radius R into itself. Now, let $M = \sup|f|$ and $L = \sup|K|$. The range of K obeys $\|Ku\| \leq ML$, so that if we take $R \geq ML$ we are done.

We recall from functional analysis that if K is a *linear* compact operator then $I - K$ is Fredholm of index zero. That is, range is closed, of finite codimension, kernel is finite dimensional, and

$$\dim \ker(I-K) = \operatorname{codim} \operatorname{Range}(I-K).$$

We recall here also $P(x,D)$ linear elliptic operators in Sobolev spaces and Hölder spaces, and embeddng theorems.

Now an application involving elliptic operators. Let $P = P(x,\partial)$ be an elliptic operator of order m

$$P(x,D)u = \sum_{|\alpha| \leq m} a_\alpha(x)\partial^\alpha u$$

with principal symbol

$$p_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha$$

that does not vanish for $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$. We consider boundary conditions on $\partial\Omega$ that are good: $Bu = 0$ on $\partial\Omega$ imply that the $P : X \to Y$ is a Fredholm operator (kernel finite dimensional, closed range with finite dimensional codimension. In many cases the index of P is zero, i.e. the dimension of the kernel equals the dimension of the coimage. Examples are the Laplacian with Neumann or Dirichlet BC.

Now we consider a sublinear function $g(x, \partial^\alpha u)$ with $|\alpha| \le m - 1$, satisfying

$$|g(x, \partial^\alpha u)| \le C(1 + \sum_{|\alpha| \le 1} |\partial^\alpha u|)^r$$

with $r < 1$, uniformly for $x \in \overline{\Omega}$ and arbitrary entries $\partial^\alpha u \in \mathbb{R}^M$ where M is the number of such things. We consider the equation

$$P(x, D)u = g(x, \partial^\alpha u)$$

with boundary conditions $Bu = 0$. We assume that the index of P is zero and P is injective. Then there exists a $C^\infty(\overline{\Omega})$ solution. (Assuming the boundary, and all coefficients are smooth all the way to the boundary).

The idea of the proof is to take $I - P^{-1}g(x, \partial^\alpha u)$ and apply degree theory. We may choose the space $X = C^{m-1}(\overline{\Omega}) \cap \{Bu = 0\}$.

The steps of the proof are instructive. First we establish a priori estimates. For example, we can look at $W^{m,p}(\Omega)$, $p > n$, and assuming a solution, obtain uniform bounds

$$\|u\|_{m,p} \le C_{m,p}$$

with constant independent of anything. This comes from $r < 1$ and ellipticity. We could have had a fully nonlinear equation here (right hand side depending on all m derivatives). Then we show that this means that solutions have to belong to a fixed ball of X. This uses Sobolev embedding and $p > n$ and the fact that the right hand side sees $m - 1$ derivatives only. Then we take a stricly larger ball $B \subset X$. There are no solution on the boundary of this ball. Also, by embeddings, $K(u) = P^{-1}g(x, \partial^\alpha u)$ is compact (because its range is bounded in the Hölder space $C^{m-1,\gamma}(\Omega)$, with $\gamma = 1 - \frac{n}{p}$. By homotopy to I vis $I - tK$, the degree $\deg(I - K, B, 0) = 1$, and therefore there is a solution. Smoothness follows by bootstrapping.

This was sublinear, but set the stage. Here is a semilinear example that is not trivial: the existence of steady solutions of Navier-Stokes equations with arbitrary forcing in both 2 and 3 dimensions.

The equation

$$Au + B(u,u) = f$$

where A is the Stokes operator and $B(u,v) = \mathbb{P}(u \cdot \nabla v)$ has solutions $u \in V$ for any $f \in L^2(\Omega)^d$ with $\mathbb{P}f = f$.

Here Ω is an open bounded set with smooth boundary, $d = 2, 3$ and $\mathbb{P}$ is the projector on divergence-free functions in L^2. We recall notations: V is the closure of the space of divergence-free $C_0^\infty(\Omega)$ vectors in the topology of $H^1(\Omega)^d$, $d = 2, 3$. The Stokes operator is $A = -\mathbb{P}\Delta$ with domain $\mathcal{D}(A) = V \cap H^2(\Omega)^d$. The function

$$K(u) = A^{-1}B(u,u) : V \to V$$

is compact. This follows because $A^{-\frac{3}{4}}B(u,u)$ is continuous

$$\|A^{-\frac{3}{4}}B(u,v)\|_V \le C\|u\|_V\|v\|_V$$

(see [2]). For any $t \in [0,1]$, the equation

$$u + tK(u) = tA^{-1}f$$

has no solutions on the boundary of the ball $B_R = \{u \mid \|u\|_V < R\}$ for $R > \|A^{-1}f\|_V$. Indeed, any solution in V obeys

$$\|u\|_V^2 = t\langle A^{-1}f, u\rangle_V.$$

Therefore, $\phi(u) = u + K(u) - A^{-1}f$ obeys $\deg(\phi, B_R, 0) = 1$ and the equation has solution in B_R.

Finally, for a quasilinear example: Damped and driven Euler equations in 2D.

Consider a bounded domain $\Omega \subset \mathbb{R}^2$. Consider a time independent force $F \in H^1(\Omega)$ and a positive constant $\gamma > 0$. Then there exist $H^1(\Omega)$ solutions of the damped Euler equations

$$\gamma u + u \cdot \nabla u + \nabla p = F, \quad \operatorname{div} u = 0$$

in Ω with $u \cdot n = 0$ on $\partial\Omega$.

The proof starts by adding artificial viscosity, thus producing a semilinear equation. We take the vorticity-stream formulation of the equation, $\omega = \Delta\psi$, $u = \nabla^\perp\psi$. The vorticity equation is

$$\gamma\omega + u \cdot \nabla\omega = f$$

with $f = \nabla^\perp \cdot F$. This we want to solve in L^2. We take first $\nu > 0$ and seek solutions of

$$-\nu\Delta\omega + \gamma\omega + u \cdot \nabla\omega = f$$

with the artificial boundary condition $\omega = 0$ at $\partial\Omega$. We should think of this as being

$$\nu\Delta^2\psi + \gamma(-\Delta\psi) + J(\psi, \Delta\psi) = f$$

where $J(f, g) = \partial_1 f \partial_2 g - \partial_2 f \partial_1 g$ is the Poisson bracket. The boundary conditions are $\psi = \Delta\psi = 0$ at $\partial\Omega$. (These are "good").

We start by showing there exist solutions at fixed ν. Then we pass to the limit as $\nu \to 0$. At fixed ν.

References

[1] L. Nirenberg, Topics in Nonlinear Functional Analysis, CIMS, 1973-1974.

[2] P. Constantin, C. Foias, Navier-Stokes Equations, U. Chicago Press, 1988.