


Lecture 3

A semilinear example

Superlinear
 no cancellation
 side condition

$$\left\{ \begin{array}{l} \Delta \phi + \operatorname{div}(s \nabla \phi) + f = 0 \\ -\Delta \phi = g \end{array} \right.$$

with $f \geq 0$, smooth, given.

in $\Omega \subset \mathbb{R}^3$ $\partial\Omega$ smooth
bdy

$$g|_{\partial\Omega} = 0 \quad \Phi|_{\partial\Omega} = 0$$

$$0 = \Delta g + \sum_{j=1}^d a_j (s \frac{\partial \phi}{\partial x_j}) + f$$

A priori H_1 \rightarrow

multiply by s

$$\|\nabla f\|_{L^2(\Omega)}^2 + \int_{\Omega} \partial_j \phi \underbrace{\int \partial_j f}_{\rightarrow \frac{\partial_j f^2}{2}} = \int_{\Omega} f \phi$$

$$\|\nabla f\|_{L^2}^2 + \frac{1}{2} \int f^3 = \int f \phi$$

↗ cubic, bad for loss

although $\|f\|_{L^3(\Omega)} \leq C \|f\|_{H^1(\Omega)}$
 by Sobolev

What if $f \geq 0$? Then

$$\|\nabla f\|_{L^2}^2 \leq \int_{\Omega} f^2 \quad R_1$$

$$\Rightarrow (\text{Poincaré}) \quad \|f\|_{H^1(\Omega)} \leq C_2 \|f\|_{L^2}$$

$$\Rightarrow \begin{matrix} \text{Elliptic Reg} \\ \text{Sobolev} \end{matrix} \Rightarrow \|f\|_{H^2(\Omega)} \leq R_2$$

$R_{1,2}$ depend only on f & Ω .

Why $f \geq 0$? (Phys... but

$$f_t = \Delta f + \operatorname{div}(f \nabla \phi) + f$$

$$\text{is } f_t = \Delta f + \nabla f \cdot \nabla \phi - f^2 + f$$

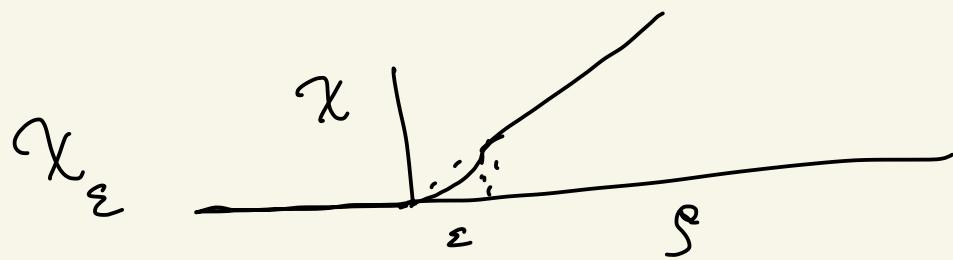
↑ ↑ transport ↑ diffusion ↑ ODE

$$f \geq 0 \Rightarrow f(x,t) \geq 0.$$

How to do $\int g \geq 0$?

$$\Delta g + \operatorname{div}(\chi_\varepsilon(g) \nabla \phi) + f = 0$$

$$-\Delta \phi = g$$



smooth, $= 0$ for $s \leq \varepsilon$
 $= g$ for $s > \varepsilon$

$$F_\varepsilon = \int_0^s \chi_\varepsilon(r) dr$$

$$F_\varepsilon(s) = 0 \text{ for } s \leq 0, F_\varepsilon \geq 0, F'_\varepsilon = \chi_\varepsilon$$

$$\| \nabla g \|_{L^2}^2 + \int_{\Omega} g F_\varepsilon(s) dx = \int_{\Omega} f g \, dx \geq 0$$

$$\implies \| g \|_{H^2} \leq R_2.$$

indep of ε

Deg th.

$$\phi_\varepsilon = f - t(-\Delta_D^{-1}) \left(\chi_\varepsilon(f) \nabla (-\Delta_D^{-1} f) \right) - (-\Delta_D^{-1}) f$$

compact in $B_R \subset H_0^1(\Omega)$

$$t \in [0, 1]$$

$$\text{No soln } \phi_\varepsilon(f) = 0 \quad \forall f \in \mathcal{B}_{R+1}$$

$$t=0 \implies f = (-\Delta_D^{-1}) f \implies f \geq 0$$

at $t > 0$, max principle

$$\Delta f + t \chi'(f) \nabla \chi \nabla f - \chi_\varepsilon(f) f + f = 0$$

\uparrow
 $\Delta \phi$

at interior negative min

$$\Delta f + 0 + 0 + f = 0 \quad \text{absurd}$$

$\boxed{\varepsilon \rightarrow 0}$

pointwise conv.

SQG Consider first dissipative

$$u \cdot \nabla \theta + \Lambda \theta = f \quad \text{in } \mathbb{T}^2$$

$$\text{with } f \in L^2, \quad \int_{\mathbb{T}^2} f = 0,$$

where $u = R^+ \theta$ and

$$R = \nabla (-\Delta)^{-\frac{1}{2}} \quad \text{in } \mathbb{T}^2$$

On Fourier side: $\widehat{Rf}(k) = \frac{ik}{|k|} \widehat{f}(k)$

This is quasilinear:

$$\Lambda = (-\Delta)^{\frac{1}{2}}$$

In Fourier: $\widehat{\Lambda \theta}(k) = |k| \widehat{\theta}(k)$.

Now we claim $f \in L^2$

if θ solution in $H^2(\mathbb{T}^2)$

Note weak solution

$$u = R^+ \theta \in H^{1/2} + H^{\frac{1}{2}} L^4$$

$\Rightarrow u \theta \in L^2 \Rightarrow$ Eqn holds
in L^2

Steps of proof

1° approx by semilinear

$$-\Delta \theta + u \cdot \nabla \theta + \lambda \theta = f$$

$$u = R^+ \theta$$

2° Pass to limit $\rightarrow 0$.

- uniform bounds in $H^{1/2}$

- Not enough for strong conv in L^4
- but enough for passing to lim

Step 3 $H^{1/2}$

$$\varphi \theta + (-\Delta)^{-1} (u \cdot \nabla \theta + \lambda \theta) = (-\Delta)^{-1} f$$

uniform bound

$$\left\| \Delta^{\frac{1}{2}} \theta \right\|_{L^2}^2 \leq \left\| \Delta^{\frac{1}{2}} f \right\|_{L^2}^2$$

$$\int_C (u \cdot \nabla \theta + \lambda \theta) \cdot \theta = \int f \theta$$

Ball in $H^{\frac{1}{2}}$ of radius $R > \left\| \Delta^{\frac{1}{2}} f \right\|_{L^2}$

$$H^{\frac{1}{2}} \subset L^4$$

$$k(u) = (-\Delta)^{-\frac{1}{2}} (f - u \cdot \nabla \theta - \lambda \theta)$$

compact in $H^{\frac{1}{2}}$

$$\Delta^{\frac{1}{2}} \Delta^{-\frac{1}{2}} \sigma(u \theta) = \Delta^{\frac{1}{2}} \underbrace{R(u \theta)}_{\in L^2}$$

source
of
compactness

bdd

$$\deg(\phi_r, B_R, 0) = 1 \Rightarrow \exists \theta \in H^{\frac{1}{2}}$$

$$-\Delta \theta + \nabla \theta + \operatorname{div}(u\theta) = f \quad u = R^\perp \theta \\ \theta \in H^{1/2}$$

$$f \in L^2; \quad \text{Elliptic reg: } \not\rightarrow \theta \in H^1 \\ u\theta \in L^2 \quad \bar{\Delta}^2 \operatorname{div}(u\theta) = \underbrace{\bar{\Delta}^1 R \cdot (u\theta)}_{\in L^2} \uparrow$$

Step 2° limit We have a sequence

θ_n corresponding to $v_n \rightarrow 0$

θ_n bdd in $H^{1/2}(\pi^2)$

WLOG $\theta_n \rightarrow \theta$ in $H^{1/2}(\pi^2)$

$\theta_n \rightarrow \theta$ weak conv L^p , $p < 4$

(but $\theta_n \rightarrow \theta$ weak L^4)

$\leftarrow \int \operatorname{div}(R^\perp \theta_n \varphi) \varphi dx \rightarrow \text{test function}$

$$u_n = R^+ \theta_n$$

$$\int u_n \theta_n \cdot \nabla \phi \, dx$$



$$\int u \theta \cdot \nabla \phi$$

$$u = R^+ \theta$$

because $u_n \theta_n \rightarrow u \theta$ in L^1 .

Remark Same in $\mathcal{L} \subset \mathbb{R}^2$.

Need only to define the objects:

$$\Delta = (-\Delta_D)^{1/2}, \quad R = \nabla \tilde{\Lambda}^{1/2}$$

$-\Delta_D$: Dirichlet bc

$$-\Delta w_j = \lambda_j w_j \quad w_j \Big|_{\partial \Omega} = 0$$

$$0 < \lambda_1 \leq \dots \leq \lambda_j \leq \dots \rightarrow \infty$$

$$R : L^4(\Omega) \rightarrow L^4(\Omega) \quad \text{bdd}$$

$$R : L^2(\Omega) \rightarrow L^2(\Omega) \quad \text{bdd}$$

Now $S \otimes G$

$$\gamma \theta + u \cdot \nabla \theta = f \quad \begin{matrix} \gamma > 0, \\ f \in L^2 \end{matrix}$$

Claim $\# f \in L^2, \exists \theta \in L^2$ weak sol

$$\gamma \int \theta \varphi - \int u \theta \cdot \nabla \varphi = \int f \theta$$

Note $\int f = 0$ wlog (take out $\gamma \int \theta = \int f$)

Steps 1° viscous approx

2° limit

↑ here we use weak cont

Next