# Inviscid limit for SQG in bounded domains

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ABSTRACT. We prove that the limit of any weakly convergent sequence of Leray-Hopf solutions of dissipative SQG equations is a weak solution of the inviscid SQG equation in bounded domains.

## 1. Introduction

The behavior of high Reynolds number fluids is a broad, important and mostly open problem of nonlinear physics and of PDE. Here we consider a model problem, the surface quasi-geostrophic equation, and the limit of its viscous regularizations of certain types. We prove that the inviscid limit is rigid, and no anomalies arise in the limit.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Denote

$$\Lambda = \sqrt{-\Delta}$$

where $-\Delta$ is the Laplacian operator with Dirichlet boundary conditions. The dissipative surface quasi-geostrophic (SQG) equation in $\Omega$ is the equation

$$\partial_t \theta^\nu + u^\nu \cdot \nabla \theta^\nu + \nu \Lambda^s \theta^\nu = 0, \quad \nu > 0,\ s \in (0, 2], \tag{1.1}$$

where $\theta^\nu = \theta^\nu(x,t)$, $u^\nu = u^\nu(x,t)$ with $(x,t) \in \Omega \times [0,\infty)$ and with the velocity $u^\nu$ given by

$$u^\nu = R_D^\perp \theta^\nu := \nabla^\perp \Lambda^{-1} \theta^\nu, \quad \nabla^\perp = (-\partial_2, \partial_1). \tag{1.2}$$

We refer to the parameter $\nu$ as "viscosity". Fractional powers of the Laplacian $-\Delta$ are based on eigenfunction expansions. The inviscid SQG equation has zero viscosity

$$\partial_t \theta + u \cdot \nabla \theta = 0, \quad u = R_D^\perp \theta. \tag{1.3}$$

The dissipative SQG (1.1) has global weak solutions for any $L^2$ initial data:

THEOREM 1.1. *For any initial data $\theta_0 \in L^2(\Omega)$ there exists a global weak solution $\theta$*

$$\theta \in C_w(0,\infty; L^2(\Omega)) \cap L^2(0,\infty; D(\Lambda^{\frac{s}{2}}))$$

*to the dissipative SQG equation* (1.1). *More precisely, $\theta$ satisfies the weak formulation*

$$\int_0^\infty \int_\Omega \theta \varphi(x) dx \partial_t \phi(t) dt + \int_0^\infty \int_\Omega u\theta \cdot \nabla \varphi(x) dx \phi(t) dt - \nu \int_0^\infty \int_\Omega \Lambda^{\frac{s}{2}} \theta \Lambda^{\frac{s}{2}} \varphi(x) dx \phi(t) dt = 0 \tag{1.4}$$

*for any $\phi \in C_c^\infty((0,\infty))$ and $\varphi \in D(\Lambda^2)$. Moreover, $\theta$ obeys the energy inequality*

$$\frac{1}{2}\|\theta(\cdot,t)\|^2_{L^2(\Omega)} + \nu \int_0^t \int_\Omega |\Lambda^{\frac{s}{2}} \theta|^2 dx dr \le \frac{1}{2}\|\theta_0\|^2_{L^2(\Omega)} \tag{1.5}$$

*and the balance*

$$\frac{1}{2}\|\theta(\cdot,t)\|^2_{D(\Lambda^{-\frac{1}{2}})} + \nu \int_0^t \int_\Omega |\Lambda^{\frac{s-1}{2}} \theta|^2 dx dr = \frac{1}{2}\|\theta_0\|^2_{D(\Lambda^{-\frac{1}{2}})} \tag{1.6}$$

*for a.e. $t > 0$. In addition, $\theta \in C([0,\infty); D(\Lambda^{-\varepsilon}))$ for any $\varepsilon > 0$ and the initial data $\theta_0$ is attained in $D(\Lambda^{-\varepsilon})$.*

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We refer to any weak solutions of (1.1) satisfying the properties (1.4), (1.5), (1.6) as a "Leray-Hopf weak solution".

REMARK 1.2. Theorem 1.1 for critical dissipative SQG $s = 1$ was obtained in [**5**].

REMARK 1.3. Note that $C_c^\infty(\Omega)$ is not dense in $D(\Lambda^2)$ since the $D(\Lambda^2)$ norm is equivalent to the $H^2(\Omega)$ norm and $C_c^\infty(\Omega)$ is dense in $H_0^2(\Omega)$ which is strictly contained in $D(\Lambda^2)$.

The existence of $L^2$ global weak solutions for inviscid SQG (1.3) was proved in [**7**]. More precisely, (see Theorem 1.1, [**7**]) for any initial data $\theta_0 \in L^2(\Omega)$ there exists a global weak solution $\theta \in C_w(0, \infty; L^2(\Omega))$ satisfying

$$\int_0^\infty \int_\Omega \theta \partial_t \varphi dx dt + \int_0^\infty \int_\Omega u\theta \cdot \nabla \varphi dx dt = 0 \quad \forall \varphi \in C_c^\infty(\Omega \times (0, \infty)), \tag{1.7}$$

and such that the Hamiltonian

$$H(t) := \|\theta(t)\|^2_{D(\Lambda^{-\frac{1}{2}})} \tag{1.8}$$

is constant in time. Moreover, the initial data is attained in $D(\Lambda^{-\varepsilon})$ for any $\varepsilon > 0$.

Our main result in this note establishes the convergence of weak solutions of the dissipative SQG to weak solutions of the inviscid SQG in the inviscid limit $\nu \to 0$.

THEOREM 1.4. *Let $\{\nu_n\}$ be a sequence of viscosities converging to* $0$ *and let $\{\theta_0^{\nu_n}\}$ be a bounded sequence in $L^2(\Omega)$. Any weak limit $\theta$ in $L^2(0, T; L^2(\Omega))$, $T > 0$, of any subsequence of $\{\theta^{\nu_n}\}$ of Leray-Hopf weak solutions of the dissipative SQG equation* (1.1) *with viscosity $\nu_n$ and initial data $\theta_0^{\nu_n}$ is a weak solution of the inviscid SQG equation* (1.3) *on* $[0, T]$. *Moreover, $\theta \in C(0, T; D(\Lambda^{-\varepsilon}))$ for any $\varepsilon > 0$, and when $s \in (0, 1]$ the Hamiltonian of $\theta$ is constant on* $[0, T]$.

REMARK 1.5. The same result holds true on the torus $\mathbb{T}^2$. The case of the whole space $\mathbb{R}^2$ was treated in [**1**].

REMARK 1.6. With more singular constitutive laws $u = \nabla^\perp \Lambda^{-\alpha}\theta$, $\alpha \in [0, 1)$, $L^2$ global weak solutions of the inviscid equations were obtained in [**3, 14**]. Theorem 1.4 could be extended to this case. It is also possible to consider $L^p$ initial data in light of the work [**11**].

As a corollary of the proof of Theorem 1.4 we have the following weak rigidity of inviscid SQG in bounded domains:

COROLLARY 1.7. *Any weak limit in $L^2(0, T; L^2(\Omega))$, $T > 0$, of any sequence of weak solutions of the inviscid SQG equation* (1.3) *is a weak solution of* (1.3). *Here, weak solutions of* (1.3) *are interpreted in the sense of* (1.7).

REMARK 1.8. On tori, this result was proved in [**13**]. If the weak limit occurs in $L^\infty(0, T; L^2(\Omega))$ and the sequence of weak solutions conserves the Hamiltonian then so is the limiting weak solution.

The paper is organized as follows. Section 2 is devoted to basic facts about the spectral fractional Laplacian and results on commutator estimate. The proofs of Theorems 1.1 and 1.4 are given respectively in sections 3 and 4. Finally, an auxiliary lemma is given in Appendix A.

## 2. Fractional Laplacian and commutators

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with smooth boundary. The Laplacian $-\Delta$ is defined on $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$. Let $\{w_j\}_{j=1}^\infty$ be an orthonormal basis of $L^2(\Omega)$ comprised of $L^2-$normalized eigenfunctions $w_j$ of $-\Delta$, i.e.

$$-\Delta w_j = \lambda_j w_j, \quad \int_\Omega w_j^2 dx = 1,$$

with $0 < \lambda_1 < \lambda_2 \le ... \le \lambda_j \to \infty$.
The fractional Laplacian is defined using eigenfunction expansions,

$$\Lambda^s f \equiv (-\Delta)^{\frac{s}{2}} f := \sum_{j=1}^{\infty} \lambda_j^{\frac{s}{2}} f_j w_j \quad \text{with } f = \sum_{j=1}^{\infty} f_j w_j, \quad f_j = \int_\Omega f w_j dx$$

for $s \ge 0$ and $f \in D(\Lambda^s) := \{f \in L^2(\Omega) : \big(\lambda_j^{\frac{s}{2}} f_j\big) \in \ell^2(\mathbb{N})\}$. The norm of $f$ in $D(\Lambda^s)$ is defined by

$$\|f\|_{D(\Lambda^s)} := \|(\lambda_j^{\frac{s}{2}} f_j)\|_{\ell^2(\mathbb{N})}.$$

It is also well-known that $D(\Lambda)$ and $H_0^1(\Omega)$ are isometric. In the language of interpolation theory,

$$D(\Lambda^\alpha) = [L^2(\Omega), D(-\Delta)]_{\frac{\alpha}{2}} \quad \forall \alpha \in [0,2].$$

As mentioned above,

$$H_0^1(\Omega) = D(\Lambda) = [L^2(\Omega), D(-\Delta)]_{\frac{1}{2}},$$

hence

$$D(\Lambda^\alpha) = [L^2(\Omega), H_0^1(\Omega)]_\alpha \quad \forall \alpha \in [0,1].$$

Consequently, we can identify $D(\Lambda^\alpha)$ with usual Sobolev spaces (see Chapter 1, [**16**]):

$$D(\Lambda^\alpha) = \begin{cases} H_0^\alpha(\Omega) & \text{if } \alpha \in (\frac{1}{2}, 1], \\ H_{00}^{\frac{1}{2}}(\Omega) := \{u \in H_0^{\frac{1}{2}}(\Omega) : u/\sqrt{d(x)} \in L^2(\Omega)\} & \text{if } \alpha = \frac{1}{2}, \\ H^\alpha(\Omega) & \text{if } \alpha \in [0, \frac{1}{2}). \end{cases} \tag{2.1}$$

Here and below $d(x)$ denote the distance from $x$ to the boundary $\partial\Omega$.

Next, for $s > 0$ we define

$$\Lambda^{-s} f = \sum_{j=1}^{\infty} \lambda_j^{-\frac{s}{2}} f_j w_j$$

if $f = \sum_{j=1}^\infty f_j w_j \in D(\Lambda^{-s})$ where

$$D(\Lambda^{-s}) := \left\{ \sum_{j=1}^{\infty} f_j w_j \in \mathscr{D}'(\Omega) : f_j \in \mathbb{R},\ \sum_{j=1}^{\infty} \lambda_j^{-\frac{s}{2}} f_j w_j \in L^2(\Omega) \right\}.$$

The norm of $f$ is then defined by

$$\|f\|_{D(\Lambda^{-s})} := \|\Lambda^{-s} f\|_{L^2(\Omega)} = \big( \sum_{j=1}^{\infty} \lambda_j^{-s} f_j^2 \big)^{\frac{1}{2}}.$$

It is easy to check that $D(\Lambda^{-s})$ is the dual of $D(\Lambda^s)$ with respect to the pivot space $L^2(\Omega)$.

LEMMA 2.1 (Lemma 2.1, [**14**]). *The embedding*

$$D(\Lambda^s) \subset H^s(\Omega) \tag{2.2}$$

*is continuous for all* $s \ge 0$.

LEMMA 2.2. *For* $s, r \in \mathbb{R}$ *with* $s > r$*, the embedding* $D(\Lambda^s) \subset D(\Lambda^r)$ *is compact.*

PROOF. Let $\{u_n\}$ be a bounded sequence in $D(\Lambda^s)$. Then $\{\Lambda^r u_n\}$ is bounded in $D(\Lambda^{s-r})$. Choosing $\delta > 0$ smaller than $\min(s-r, \frac{1}{2})$ we have $D(\Lambda^{s-r}) \subset D(\Lambda^\delta) = H^\delta(\Omega) \subset L^2(\Omega)$ where the first embedding is continuous and the second is compact. Consequently the embedding $D(\Lambda^{s-r}) \subset L^2(\Omega)$ is compact and thus there exist a subsequence $n_j$ and a function $f \in L^2(\Omega)$ such that $\Lambda^r u_{n_j}$ converge to $f$ strongly in $L^2(\Omega)$. Then $u_{n_j}$ converge to $u := \Lambda^{-r} f$ strongly in $D(\Lambda^r)$ and the proof is complete. $\square$

A bound for the commutator between $\Lambda$ and multiplication by a smooth function was proved in [**5**] using the method of harmonic extension:

THEOREM 2.3 (Theorem 2, [**5**]). *Let $\chi \in B(\Omega)$ with $B(\Omega) = W^{2,d}(\Omega) \cap W^{1,\infty}(\Omega)$ if $d \geq 3$, and $B(\Omega) = W^{2,p}(\Omega)$ with $p > 2$ if $d = 2$. There exists a constant $C(d,p,\Omega)$ such that*

$$\|[\Lambda, \chi]\psi\|_{D(\Lambda^{\frac{1}{2}})} \leq C(d,p,\Omega)\|\chi\|_{B(\Omega)}\|\psi\|_{D(\Lambda^{\frac{1}{2}})}.$$

Pointwise estimates for the commutator between fractional Laplacian and differentiation were established in [**7**]:

THEOREM 2.4 (Theorem 2.2, [**7**]). *For any $p \in [1,\infty]$ and $s \in (0,2)$ there exists a positive constant $C(d,s,p,\Omega)$ such that for all $\psi \in C_c^\infty(\Omega)$ we have*

$$|[\Lambda^s, \nabla]\psi(x)| \leq C(d,s,p,\Omega)d(x)^{-s-1-\frac{d}{p}}\|\psi\|_{L^p(\Omega)}$$

*holds for all $x \in \Omega$.*

This pointwise bound implies the following commutator estimate in Lebesgue spaces.

THEOREM 2.5. *Let $p,\ q \in [1,\infty]$, $s \in (0,2)$ and $\varphi$ satisfy*

$$\varphi(\cdot)d(\cdot)^{-s-1-\frac{d}{p}} \in L^q(\Omega).$$

*Then the operator $\varphi[\Lambda^s, \nabla]$ can be uniquely extended from $C_c^\infty(\Omega)$ to $L^p(\Omega)$ such that there exists a positive constant $C = C(d,s,p,\Omega)$ such that*

$$\|\varphi[\Lambda^s, \nabla]\psi\|_{L^q(\Omega)} \leq C\|\varphi(\cdot)d(\cdot)^{-s-1-\frac{d}{p}}\|_{L^q(\Omega)}\|\psi\|_{L^p(\Omega)} \tag{2.3}$$

*holds for all $\psi \in L^p(\Omega)$.*

The inequality (2.3) is remarkable because the commutator between an operator of order $s \in (0,2)$ and an operator of order $1$ is an operator of order $0$.

## 3. Proof of Theorem 1.1

We use Galarkin approximations. Denote by $\mathbb{P}_m$ the projection in $L^2(\Omega)$ onto the linear span $L^2_m$ of eigenfunctions $\{w_1, ..., w_m\}$, i.e.

$$\mathbb{P}_m f = \sum_{j=1}^m f_j w_j \quad \text{for } f = \sum_{j=1}^\infty f_j w_j. \tag{3.1}$$

The $m$th Galerkin approximation of (1.1) is the following ODE system in the finite dimensional space $L^2_m$:

$$\begin{cases} \dot{\theta}_m + \mathbb{P}_m(u_m \cdot \nabla\theta_m) + \nu\Lambda^s\theta_m = 0 & t > 0, \\ \theta_m = P_m\theta_0 & t = 0, \end{cases} \tag{3.2}$$

with $\theta_m(x,t) = \sum_{j=1}^m \theta_j^{(m)}(t)w_j(x)$ and $u_m = R_D{}^\perp\theta_m$ satisfying $\operatorname{div} u_m = 0$. Note that (3.2) is equivalent to

$$\frac{d\theta_l^{(m)}}{dt} + \sum_{j,k=1}^m \gamma_{jkl}^{(m)}\theta_j^{(m)}\theta_k^{(m)} + \nu\lambda_l^{\frac{s}{2}}\theta_l^{(m)} = 0, \quad l = 1,2,...,m, \tag{3.3}$$

with

$$\gamma_{jkl}^{(m)} = \lambda_j^{-\frac{1}{2}}\int_\Omega \left(\nabla^\perp w_j \cdot \nabla w_k\right) w_l dx.$$

The local existence of $\theta_m$ on some time interval $[0, T_m]$ follows from the Cauchy-Lipschitz theorem. On the other hand, the antisymmetry property $\gamma_{jkl}^{(m)} = -\gamma_{jlk}^{(m)}$ yields

$$\frac{1}{2}\|\theta_m(\cdot,t)\|^2_{L^2(\Omega)} + \nu\int_0^t\int_\Omega |\Lambda^{\frac{s}{2}}\theta_m|^2 dx dr = \frac{1}{2}\|\mathbb{P}_m\theta_0\|^2_{L^2(\Omega)} \le \frac{1}{2}\|\theta_0\|^2_{L^2(\Omega)} \tag{3.4}$$

for all $t \in [0, T_m]$. This implies that $\theta_m$ is global and (3.4) holds for all positive times. The sequence $\theta_m$ is thus uniformly bounded in $L^\infty(0,\infty; L^2(\Omega)) \cap L^2(0,\infty; D(\Lambda^{\frac{s}{2}}))$. Upon extracting a subsequence, we have $\theta_m$ converge to some $\theta$ weakly-* in $L^\infty(0,\infty; L^2(\Omega))$ and weakly in $L^2(0,\infty; D(\Lambda^{\frac{s}{2}}))$. In particular, $\theta$ obeys the same energy inequality as in (3.4). On the other hand, if one multiplies (3.3) by $\lambda_l^{-1/2}\theta_l^{(m)}$ and uses the fact that $\gamma_{jkl}^{(m)}\lambda_l^{-1/2} = -\gamma_{lkj}^{(m)}\lambda_j^{-1/2}$, one obtains

$$\frac{1}{2}\|\theta_m(\cdot,t)\|^2_{D(\Lambda^{-\frac{1}{2}})} + \nu\int_0^t\int_\Omega |\Lambda^{\frac{s-1}{2}}\theta_m|^2 dx dr = \frac{1}{2}\|\mathbb{P}_m\theta_0\|^2_{D(\Lambda^{-\frac{1}{2}})}. \tag{3.5}$$

We derive next a uniform bound for $\partial_t\theta_m$. Let $N > 0$ be an integer to be determined. For any $\varphi \in D(\Lambda^{2N})$ we integrate by parts to get

$$\begin{aligned}\int_\Omega \partial_t\theta_m\varphi dx &= -\int_\Omega \mathbb{P}_m \operatorname{div}(u_m\theta_m)\varphi dx - \int_\Omega \nu\Lambda^s\theta_m\varphi dx\\ &= \int_\Omega (u_m\theta_m)\cdot\nabla(\mathbb{P}_m\varphi)dx - \int_\Omega \nu\theta_m\Lambda^s\phi dx.\end{aligned}$$

The first term is controlled by

$$\left|\int_\Omega (u_m\theta_m)\cdot\nabla(\mathbb{P}_m\varphi)dx\right| \le \|u_m\theta_m\|_{L^1(\Omega)}\|\nabla\mathbb{P}_m\varphi\|_{L^\infty(\Omega)} \le C\|\mathbb{P}_m\varphi\|_{H^3(\Omega)}.$$

According to Lemma A.1, for $N$ and $k$ satisfying $N > \frac{k}{2} + 1$ there exists a positive constant $C_{N,k}$ such that

$$\|\mathbb{P}_m\varphi\|_{H^k(\Omega)} \le C_{N,k}\|\varphi\|_{D(\Lambda^{2N})} \quad \forall m \ge 1,\ \forall\varphi \in D(\Lambda^{2N}). \tag{3.6}$$

With $k = 3$ and $N = 3$ we have

$$\left|\int_\Omega (u_m\theta_m)\cdot\nabla(\mathbb{P}_m\varphi)dx\right| \le C\|\varphi\|_{D(\Lambda^6)}.$$

On the other hand,

$$\left|\int_\Omega \nu\theta_m\Lambda^s\varphi dx\right| \le C\|\theta_m\|_{L^2(\Omega)}\|\varphi\|_{D(\Lambda^2)}.$$

We have proved that

$$\left|\int_\Omega \partial_t\theta_m\varphi dx\right| \le C\|\varphi\|_{D(\Lambda^6)} \quad \forall\varphi \in D(\Lambda^6).$$

Because $L^2(\Omega)\times D(\Lambda^6) \ni (f,g) \mapsto \int_\Omega fg dx$ extends uniquely to a bilinear from on $D(\Lambda^{-6})\times D(\Lambda^6)$, we deduce that $\partial_t\theta_m$ are uniformly bounded in $L^\infty(0,\infty; D(\Lambda^{-6}))$. Note that we have used only the uniform regularity $L^\infty(0;\infty; L^2(\Omega))$ of $\theta_m$. We have the embeddings $D(\Lambda^{\frac{s}{2}}) \subset D(\Lambda^{(s-1)/2}) \subset D(\Lambda^{-6})$ where the first one is compact by virtue of Lemma 2.2, and the second is continuous. Fix $T > 0$. Aubin-Lions' lemma (see [**15**]) ensures that for some function $f$ and along some subsequence $\theta_m$ converge to $f$ weakly in $L^2(0,T; D(\Lambda^{\frac{s}{2}}))$ and strongly in $L^2(0,T; D(\Lambda^{(s-1)/2}))$. In principle, both $f$ and the subsequence might depend on $T$, however, we already know that $\theta_m \to \theta$ weakly in $L^2(0,\infty; D(\Lambda^{\frac{s}{2}}))$. Therefore, $f = \theta$ and the convergences to $\theta$ hold for the whole sequence. Similarly, applying Aubin-Lions' lemma with the embeddings $L^2(\Omega) \subset D(\Lambda^{-\varepsilon}) \subset D(\Lambda^{-6})$ for sufficiently small $\varepsilon > 0$ we obtain that $\theta_m \to \theta$

strongly in $C([0,T];D(\Lambda^{-\varepsilon}))$. Integrating (3.2) against an arbitrary test function of the form $\phi(t)\varphi(x)$ with $\phi\in C_c^\infty((0,T))$, $\varphi\in D(\Lambda^6)$ yields

$$\int_0^T\int_\Omega \theta_m\varphi(x)dx\partial_t\phi(t)dt+\int_0^T\int_\Omega u_m\theta_m\cdot\nabla\mathbb{P}_m\varphi(x)dx\phi(t)dt-\nu\int_0^T\int_\Omega \Lambda^{\frac{s}{2}}\theta_m\Lambda^{\frac{s}{2}}\varphi(x)dx\phi(t)dt=0.$$

By Lemma A.1,

$$\|(\mathbb{I}-\mathbb{P}_m)\varphi\|_{L^\infty(\Omega)}\le C\|(\mathbb{I}-\mathbb{P}_m)\varphi\|_{H^3(\Omega)}\to 0\quad \text{as } m\to\infty.$$

The weak convergence of $\theta_m$ in $L^2(0,T;D(\Lambda^{\frac{s}{2}}))$ allows one to pass to the limit in the two linear terms. The strong convergence of $\theta_m$ in $L^2(0,T;L^2(\Omega))$ together with the weak convergence of $u_m$ in the same space allows one to pass to the limit in the nonlinear term and conclude that $\theta$ satisfies the weak formulation (1.4) with $\varphi\in D(\Lambda^6)$. In fact, $\theta\in L^2(0,\infty;D(\Lambda^{\frac{s}{2}}))\subset L^2(0,\infty;L^p(\Omega))$ for some $p>2$, hence $u\theta\in L^2(0,\infty;L^q(\Omega))$ for some $q>1$. In addition, if $\varphi\in D(\Lambda^2)$ then $\nabla\varphi\in L^r$ for all $r<\infty$, and thus the nonlinearity $\int_\Omega u\theta\cdot\nabla\varphi dx$ makes sense. Then because $D(\Lambda^2)$ is dense in $D(\Lambda^6)$, (1.4) holds for $\varphi\in D(\Lambda^2)$.

We now pass to the limit in (3.5). The strong convergence $\theta_m\to\theta$ in $C(0,T;D(\Lambda^{-\varepsilon}))$ gives the convergence of the first term. On the other hand, the strong convergence $\theta_m\to\theta$ in $L^2(0,T;D(\Lambda^{(s-1)/2}))$ yields the convergence of the second term. The right hand side converges to $\frac{1}{2}\|\theta_0\|^2_{D(\Lambda^{-\frac{1}{2}})}$ since $\mathbb{P}_m\theta_0$ converge to $\theta_0$ in $L^2(\Omega)$. We thus obtain (1.6).

Since $\theta_m\to\theta$ in $C([0,T];D(\Lambda^{-\varepsilon}))$ we deduce that

$$\theta_0=\lim_{m\to\infty}\mathbb{P}_m\theta_0=\lim_{m\to\infty}\theta_m|_{t=0}=\theta|_{t=0}\quad \text{in } D(\Lambda^{-\varepsilon}).$$

For a.e. $t\in[0,T]$, $\theta_m(t)$ are uniformly bounded in $L^2(\Omega)$, and thus along some subsequence $m_j$, a priori depending on $t$, we have $\theta_{m_j}(t)$ converge weakly to some $f(t)$ in $L^2(\Omega)$. But we know $\theta_m(t)\to\theta(t)$ in $D(\Lambda^{-\varepsilon})$. Thus, $f(t)=\theta(t)$ and $\theta_m(t)\rightharpoonup\theta(t)$ in $L^2(\Omega)$ as a whole sequence for a.e. $t\in[0,T]$. Recall that $\frac{d}{dt}\theta_m$ are uniformly bounded in $L^\infty(0,T;D(\Lambda^{-6}))$. For all $\varphi\in D(\Lambda^6)$ and $t\in[0,T]$ we write

$$\langle\theta_m(t),\varphi\rangle_{L^2(\Omega),L^2(\Omega)}=\langle\theta_m(0),\varphi\rangle_{L^2(\Omega),L^2(\Omega)}+\int_0^t\langle\frac{d}{dt}\theta_m(r),\varphi\rangle_{D(\Lambda^{-6}),D(\Lambda^6)}dr.$$

Because $\frac{d}{dt}\theta_m$ converge to $\frac{d}{dt}\theta$ weakly-* in $L^\infty(0,T;D(\Lambda^{-6}))$, letting $m\to\infty$ yields

$$\langle\theta(t),\varphi\rangle_{L^2(\Omega),L^2(\Omega)}=\langle\theta_0,\varphi\rangle_{L^2(\Omega),L^2(\Omega)}+\int_0^t\langle\frac{d}{dt}\theta(r),\varphi\rangle_{D(\Lambda^{-6}),D(\Lambda^6)}dr$$

for a.e. $t\in[0,T]$. Taking the limit $t\to0$ gives

$$\lim_{t\to0}\langle\theta(t),\varphi\rangle_{L^2(\Omega),L^2(\Omega)}=\langle\theta_0,\varphi\rangle_{L^2(\Omega),L^2(\Omega)}$$

for all $\varphi\in D(\Lambda^6)$. Finally, since $D(\Lambda^6)$ is dense in $L^2(\Omega)$ and $\theta\in L^\infty(0,T;L^2(\Omega))$ we conclude that $\theta\in C_w(0,T;L^2(\Omega))$ for all $T>0$.

## 4. Proof of Theorem 1.4

First, using approximations and commutator estimates we justify the commutator structure of the SQG nonlinearity derived in [**7**].

LEMMA 4.1. *For all $\psi\in H_0^1(\Omega)$ and $\varphi\in C_c^\infty(\Omega)$ we have*

$$\int_\Omega\Lambda\psi\nabla^\perp\psi\cdot\nabla\varphi dx=\frac{1}{2}\int_\Omega[\Lambda,\nabla^\perp]\psi\cdot\nabla\varphi\psi dx-\frac{1}{2}\int_\Omega\nabla^\perp\psi\cdot[\Lambda,\nabla\varphi]\psi dx. \tag{4.1}$$

*Here, the commutator $[\Lambda,\nabla^\perp]\psi\cdot\nabla\varphi$ is understood in the sense of the extended operator defined in Theorem 2.5.*

PROOF. Let $\psi_n \in C_c^\infty(\Omega)$ converging to $\psi$ in $H_0^1(\Omega)$. Integrating by parts and using the fact that $\nabla^\perp \cdot \nabla\varphi = 0$ gives

$$\int_\Omega \Lambda\psi_n \nabla^\perp \psi_n \cdot \nabla\varphi dx = -\int_\Omega \psi_n \nabla^\perp \Lambda\psi_n \cdot \nabla\varphi dx,$$

Because $\psi_n$ is smooth and has compact support, $\nabla^\perp\psi_n \in D(\Lambda)$, and thus we can commute $\nabla^\perp$ with $\Lambda$ to obtain

$$\begin{aligned}
&\int_\Omega \Lambda\psi_n \nabla^\perp\psi_n \cdot \nabla\varphi dx \\
&= -\int_\Omega \psi_n [\nabla^\perp, \Lambda]\psi_n \cdot \nabla\varphi dx - \int_\Omega \psi_n \Lambda\nabla^\perp\psi_n \cdot \nabla\varphi dx \\
&= -\int_\Omega \psi_n [\nabla^\perp, \Lambda]\psi_n \cdot \nabla\varphi dx - \int_\Omega \nabla^\perp\psi_n \cdot \Lambda(\psi_n \nabla\varphi) dx \\
&= -\int_\Omega [\nabla^\perp, \Lambda]\psi_n \cdot \nabla\varphi\psi_n dx - \int_\Omega \nabla^\perp\psi_n \cdot [\Lambda, \nabla\varphi]\psi_n dx - \int_\Omega \nabla^\perp\psi_n \cdot \nabla\varphi\Lambda\psi_n dx.
\end{aligned}$$

Noticing that the last term on the right-hand side is exactly the negative of the left-hand side, we deduce that

$$\int_\Omega \Lambda\psi_n \nabla^\perp\psi_n \cdot \nabla\varphi dx = \frac{1}{2}\int_\Omega [\Lambda, \nabla^\perp]\psi_n \cdot \nabla\varphi\psi_n dx - \frac{1}{2}\int_\Omega \nabla^\perp\psi_n \cdot [\Lambda, \nabla\varphi]\psi_n dx.$$

The commutator estimates in Theorems 2.3 and 2.5 then allow us to pass to the limit in the preceding representation and conclude that (4.1) holds. □

Now let $\nu_n \to 0^+$ and let $\theta_0^{\nu_n}$ be a bounded sequence in $L^2(\Omega)$. For each $n$ let $\theta_n \equiv \theta^{\nu_n}$ be a Leray-Hopf weak solution of (1.1) with viscosity $\nu_n$ and initial data $\theta_0^{\nu_n}$. In view of the energy inequality (1.5), $\theta_n$ are uniformly bounded in $L^\infty(0,\infty; L^2(\Omega))$ and satisfies

$$\int_0^\infty \int_\Omega \theta_n \varphi(x) dx \partial_t\phi(t) dt + \int_0^\infty \int_\Omega u_n\theta_n \cdot \nabla\varphi(x) dx \phi(t) dt - \nu_n \int_0^\infty \int_\Omega \Lambda^{\frac{s}{2}}\theta_n \Lambda^{\frac{s}{2}}\varphi(x) dx \phi(t) dt = 0 \tag{4.2}$$

for all $\phi \in C_c^\infty((0,\infty))$ and $\varphi \in D(\Lambda^2)$. Fix $T > 0$. Assume that along a subsequence, still labeled by $n$, $\theta_n$ converge to $\theta$ weakly in $L^2(0,T; L^2(\Omega))$. We prove that $\theta$ is a weak solution of the inviscid SQG equation. We first prove a uniform bound for $\partial_t\theta_n$ provided only the uniform regularity $L^\infty(0,T; L^2(\Omega))$ of $\theta_n$. To this end, let us define for a.e. $t \in [0,T]$ the function $f_n(\cdot,t) \in H^{-3}(\Omega)$ by

$$\langle f_n(t), \varphi\rangle_{H^{-3}(\Omega), H_0^3(\Omega)} := \int_\Omega (u_n(x,t)\theta_n(x,t) \cdot \nabla\varphi(x) - \nu_n\theta_n(x,t)\Lambda^s\varphi(x)) dx$$

for all $\varphi \in H_0^3(\Omega) \subset D(\Lambda^2)$. Indeed, we have

$$\left| \int_\Omega (u_n(x,t)\theta_n(x,t) \cdot \nabla\varphi(x) - \nu_n\theta_n(x,t)\Lambda^s\varphi(x)) dx \right| \le C\big(\|\theta_n(t)\|_{L^2(\Omega)}^2 + 1\big)\|\varphi\|_{H^3(\Omega)}.$$

This shows that $f_n$ are uniformly bounded in $L^\infty(0,T; H^{-3}(\Omega))$. Then for any $\phi \in C_c^\infty((0,T))$, it follows from (4.2) that

$$\int_0^T \theta_n \partial_t\phi dt = -\int_0^T f_n\phi dt.$$

In other words, $\partial_t\theta_n = f_n$ and the desired uniform bound for $\partial_t\theta_n$ follows. Fix $\varepsilon \in (0, \frac{1}{2})$. Aubin-Lions' lemma applied with the embeddings $L^2(\Omega) \subset D(\Lambda^{-\varepsilon}) \subset H^{-3}(\Omega)$ then ensures that $\theta_n$ converge to $\theta$ strongly in $C(0,T; D(\Lambda^{-\varepsilon})) \subset C(0,T; H^{-1}(\Omega))$. Consequently $\psi_n$ converge to $\psi := \Lambda^{-1}\theta$ strongly in $C(0,T; L^2(\Omega))$.

Now we take $\phi \in C_c^\infty((0,\infty))$ and $\varphi \in C_c^\infty(\Omega)$. By virtue of Lemma 4.1, the weak formulation (1.4) gives

$$\int_0^T \int_\Omega \theta_n \varphi(x) dx \partial_t \phi(t) dt + \frac{1}{2}\int_0^T \int_\Omega [\Lambda, \nabla^\perp]\psi_n \cdot \nabla\varphi(x)\psi_n dx \phi(t) dt$$
$$- \frac{1}{2}\int_0^T \int_\Omega \nabla^\perp \psi_n \cdot [\Lambda, \nabla\varphi(x)]\psi_n dx\phi(t)dt - \nu_n \int_0^T \int_\Omega \theta_n \Lambda^s \varphi(x) dx \phi(t) dt = 0,$$

where $\psi_n := \Lambda^{-1}\theta_n$ are uniformly bounded in $L^\infty(0,T; H_0^1(\Omega))$. The weak convergence $\theta_n \rightharpoonup \theta$ in $L^2(0,T;L^2(\Omega))$ readily yields

$$\lim_{n\to\infty} \int_0^T \int_\Omega \theta_n \varphi(x) dx \partial_t \phi(t) dt = \int_0^T \int_\Omega \theta \varphi(x) dx \partial_t \phi(t) dt$$

and

$$\lim_{n\to\infty} \nu_n \int_0^T \int_\Omega \theta_n \Lambda^s \varphi(x) dx \phi(t) dt = 0.$$

Next we pass to the limit in the two nonlinear terms. Applying the commutator estimate in Theorem 2.3 we have

$$\left| \int_0^T \int_\Omega \nabla^\perp \psi_n \cdot [\Lambda, \nabla\varphi]\psi_n dx\phi dt - \int_0^T \int_\Omega \nabla^\perp \psi \cdot [\Lambda, \nabla\varphi]\psi dx\phi dt \right|$$
$$\le \left| \int_0^T \int_\Omega \nabla^\perp(\psi_n - \psi) \cdot [\Lambda, \nabla\varphi]\psi dx\phi dt \right| + \|\phi\nabla^\perp\psi_n\|_{L^2(0,T;L^2(\Omega))} \|[\Lambda, \nabla\varphi](\psi_n - \psi)\|_{L^2(0,T;L^2(\Omega))}$$
$$\le \left| \int_0^T \int_\Omega \nabla^\perp(\psi_n - \psi) \cdot [\Lambda, \nabla\varphi]\psi dx\phi dt \right| + C\|\psi_n - \psi\|_{L^2(0,T;D(\Lambda^{\frac{1}{2}}))}.$$

The first term converges to 0 due to the weak convergence of $\psi_n$ to $\psi$ in $L^2(0,T;H_0^1(\Omega))$ and the fact that $[\Lambda, \nabla\varphi]\psi \in D(\Lambda^{\frac{1}{2}}) \subset L^2(\Omega)$ in view of Theorem 2.3. By interpolation, the second term is bounded by

$$\|\psi_n - \psi\|_{L^2(0,T;D(\Lambda^{\frac{1}{2}}))} \le \|\psi_n - \psi\|^{\frac{1}{2}}_{L^2(0,T;L^2(\Omega))} \|\psi_n - \psi\|^{\frac{1}{2}}_{L^2(0,T;D(\Lambda))} \le C\|\psi_n - \psi\|^{\frac{1}{2}}_{L^2(0,T;L^2(\Omega))}$$

which also converge to 0. Finally, we apply the commutator estimate in Theorem 2.5 to obtain

$$\left| \int_0^T \int_\Omega [\Lambda, \nabla^\perp]\psi_n \cdot \nabla\varphi\psi_n dx\phi dt - \int_0^T \int_\Omega [\Lambda, \nabla^\perp]\psi \cdot \nabla\varphi\psi dx\phi dt \right|$$
$$\le \|\nabla\varphi[\Lambda, \nabla^\perp](\psi_n - \psi)\|_{L^2(0,T;L^2(\Omega))} \|\phi\psi_n\|_{L^2(0,T;L^2(\Omega))}$$
$$+ \|[\Lambda, \nabla^\perp]\psi \cdot \nabla\varphi\|_{L^2(0,T;L^2(\Omega))} \|\phi(\psi_n - \psi)\|_{L^2(0,T;L^2(\Omega))}$$
$$\le C\|\psi_n - \psi\|_{L^2(0,T;L^2(\Omega))}$$

which converge to 0. Putting together the above considerations leads to

$$\int_0^T \int_\Omega \theta\varphi(x) dx \partial_t \phi(t) dt + \int_0^T \int_\Omega u\theta \cdot \nabla\varphi(x) dx \phi(t) dt = 0, \quad \forall \phi \in C_c^\infty((0,T)),\ \varphi \in C_c^\infty(\Omega).$$

Therefore, $\theta$ is a weak solution of the inviscid SQG equation on $[0,T]$. Finally, consider $s \in (0,1]$. We have the the balance (1.6) for each $\theta_n$. Since $s \le 1$ the uniform boundedness of $\theta_n$ in $L^\infty(0,T;L^2(\Omega))$ implies

$$\lim_{n\to\infty} \nu_n \int_0^t \int_\Omega |\Lambda^{\frac{s-1}{2}}\theta_n|^2 dx dr = 0, \quad t \in [0,T].$$

In addition, $\theta_n \to \theta$ strongly in $C(0,T;D(\Lambda^{-\varepsilon})) \subset C(0,T;D(\Lambda^{-\frac{1}{2}}))$. Letting $\nu = \nu_n \to 0$ in the balance (1.6) we conclude that the Hamiltonian of $\theta$ is constant on $[0,T]$.

## Appendix A. A bound on $\mathbb{P}_m$

Recall the definition (3.1) of $\mathbb{P}_m$. The following lemma is essentially taken from [**7**]. We include the proof for the sake of completeness.

LEMMA A.1. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with smooth boundary. For every $N$ and $k \in \mathbb{N}$ satisfying $N > \frac{k}{2} + \frac{d}{2}$ there exists a positive constant $C_{N,k}$ such that*

$$\|\mathbb{P}_m \varphi\|_{H^k(\Omega)} \leq C_{N,k} \|\varphi\|_{D(\Lambda^{2N})} \tag{A.1}$$

*for all $m \geq 1$ and $\phi \in D(\Lambda^{2N})$; moreover, we have*

$$\lim_{m \to \infty} \|(\mathbb{I} - \mathbb{P}_m)\varphi\|_{H^k(\Omega)} = 0. \tag{A.2}$$

PROOF. As $\varphi \in D(\Lambda^{2N})$, we have $\Delta^\ell \varphi \in H_0^1(\Omega)$ for all $\ell = 0, 1, \ldots, N-1$. This allows repeated integration by parts with $w_j$ using the relation $-\Delta w_j = \lambda_j w_j$. Using Hölder's inequality and the fact that $w_j$ is normalized in $L^2$, we obtain

$$|\varphi_j| \leq \lambda_j^{-N} \|\Delta^N \varphi\|_{L^2}, \quad \varphi_j = \int_\Omega \varphi w_j dx.$$

By elliptic regularity estimates and induction, we have for all $k \in \mathbb{N}$ that

$$\|w_j\|_{H^k(\Omega)} \leq C_k \lambda_j^{\frac{k}{2}}.$$

We know from the easy part of Weyl's asymptotic law that $\lambda_j \geq C j^{\frac{2}{d}}$. Consequently, with $N > \frac{k}{2} + \frac{d}{2}$ we deduce that

$$\begin{aligned}
\sum_{j=1}^{\infty} |\varphi_j| \|w_j\|_{H^k(\Omega)} &\leq C_k \|\Delta^N \varphi\|_{L^2} \sum_{j=1}^{\infty} \lambda_j^{-N+\frac{k}{2}} \\
&\leq C_k \|\varphi\|_{D(\Lambda^{2N})} \sum_{j=1}^{\infty} j^{(-N+\frac{k}{2})\frac{2}{d}} \\
&= C_{N,k} \|\varphi\|_{D(\Lambda^{2N})}
\end{aligned}$$

where $C_{N,k} < \infty$ depends only on $N$ and $k$. Because

$$(\mathbb{I} - \mathbb{P}_m)\varphi = \sum_{j=m+1}^{\infty} \varphi_j w_j,$$

this proves both (A.1) and (A.2). The proof is complete. ☐

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## References


[1] L. C. Berselli. Vanishing Viscosity Limit and Long-time Behavior for 2D Quasi-geostrophic Equations. *Indiana Univ. Math. J.* 51(4) (2002), 905–930.

[2] T. Buckmaster, S. Shkoller, V. Vicol. Nonuniqueness of weak solutions to the SQG equation. arXiv:1610.00676, to appear in *Communications on Pure and Applied Mathematics*.

[3] D. Chae, P. Constantin, D. Córdoba, F. Gancedo, J. Wu. Generalized surface quasi-geostrophic equations with singular velocities. *Comm. Pure Appl. Math.*, **65** (2012) No. 8, 1037-1066.

[4] P. Constantin, D. Cordoba, J. Wu. On the critical dissipative quasi-geostrophic equation. *Indiana Univ. Math. J.*, 50 (Special Issue): 97–107, 2001. Dedicated to Professors Ciprian Foias and Roger Temam (Bloomington, IN, 2000).

[5] P. Constantin, M. Ignatova. Remarks on the fractional Laplacian with Dirichlet boundary conditions and applications. *Internat. Math. Res. Notices*, (2016), 1-21.

[6] P. Constantin, M. Ignatova. Critical SQG in bounded domains. *Ann. PDE* (2016) 2:8.
[7] P. Constantin, H.Q. Nguyen. Global weak solutions for SQG in bounded domains. arXiv:1612.02489, to appear in *Comm. Pure Appl. Math.*
[8] P. Constantin, H. Q. Nguyen. Local and global strong solutions for SQG in bounded domains. arXiv:1705.05342, to appear in *Physica D*, Special Issue in Honor of Edriss Titi.
[9] P. Constantin, A.J. Majda, and E. Tabak. Formation of strong fronts in the 2-D quasigeostrophic thermal active scalar. *Nonlinearity*, 7(6) (1994), 1495–1533.
[10] P. Constantin, A. Tarfulea, V. Vicol. Absence of anomalous dissipation of energy in forced two dimensional fluid equations. *Arch. Ration. Mech. Anal.* **212** (2014), 875-903.
[11] F. Marchand. Existence and Regularity of Weak Solutions to the Quasi-Geostrophic Equations in the Spaces $L^p$ or $\dot{H}^{-1/2}$. *Comm. Math. Phys.* (2008) 277(1): 45–67.
[12] I.M. Held, R.T. Pierrehumbert, S.T. Garner, and K.L. Swanson. Surface quasi-geostrophic dynamics. *J. Fluid Mech.*, 282 (1995),1–20.
[13] P. Isset and V. Vicol. Hölder continuous solutions of active scalar equations. *Ann. PDE* 1 (2015), no. 1, 1–77.
[14] H. Q. Nguyen. Global weak solutions for generalized SQG in bounded domains. *Anal. PDE*, Vol. 11 (2018), No. 4, 1029–1047.
[15] J.L. Lions, *Quelque methodes de résolution des problemes aux limites non linéaires*. Paris: Dunod-Gauth, 1969.
[16] J. L. Lions, E. Magenes, *Non-homogeneous boundary value problems and applications. Vol. I.* Translated from the French by P. Kenneth. Die Grundlehren der mathematischen Wissenschaften, Band 181. Springer-Verlag, New York-Heidelberg, 1972.
[17] S. Resnick, *Dynamical problems in nonlinear advective partial differential equations*. ProQuest LLC, Ann Arbor, MI, 1995, Thesis (Ph.D.)–The University of Chicago.

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