High order linear hyperbolic equations with constant coefficients.

Intoduction to PDE

This is mostly from Fritz John, PDE, but with a small adjustment to be able to treat equations like

$$\partial_t^2 u + \Delta^2 u = 0.$$

We denote

$$\partial = (\partial_1, \ldots, \partial_n)$$

and

$$\tau = \frac{\partial}{\partial t}.$$

We will consider operators

$$P(\partial, \tau)u$$

where P is a polynomial of degree N in its $n+1$ variables, and of degree $m \le N$ in τ. We want to solve the problem

$$P(\partial, \tau)u = w, \quad \text{in } \ t > 0 \tag{1}$$

with

$$\tau^k u = f_k, \ k = 0, \ldots, m-1, \quad \text{at } \ t = 0. \tag{2}$$

We assume that the hyperplane $t = 0$ is noncharacteristic, and we normalize $P(0, 1) = 1$. We assume that the polynomial has the form

$$P(\partial, \tau) = \tau^m + P_1(\partial)\tau^{m-1} + \ldots P_m(\partial) \tag{3}$$

where $P_j(\partial)$ are polynomials of degree at most $N - m + j$. This is not the most general case we could treat by the same method, but it is sufficient for our purposes. In Fritz John $N = m$. A standard form of the initial value problem is when

$$f_0 = f_1 = \cdots = f_{m-2} = 0, \text{ and } f_{m-1} = g. \tag{4}$$

The solution of

$$\begin{cases} P(\partial, \tau)u = w, & \text{in } t > 0 \\ \tau^k u_{|t=0} = 0, & \text{for } k = 0, \dots, m-1 \end{cases} \tag{5}$$

is given by the Duhamel formula

$$u(x,t) = \int_0^t U(x,t,s)ds \tag{6}$$

where $U(x,t,s)$ solves the standard problem

$$\begin{cases} P(\partial, \tau)U(x,t,s) = 0, \quad \text{in } t > s, \\ \tau^k U_{|t=s} = 0, \quad \text{for } k = 0, \dots, m-2, \\ \tau^{m-1} U_{|t=s} = w(\cdot, s) \end{cases} \tag{7}$$

If we denote the solution of the standard problem

$$\begin{cases} P(\partial, \tau)u = 0, \quad \text{in } t > 0, \\ \tau^k u_{|t=0} = 0, \quad \text{for } k = 0, \dots, m-2, \\ \tau^{m-1} u_{|t=0} = g \end{cases} \tag{8}$$

by u_g, then the solution of the general problem

$$\begin{cases} P(\partial, \tau)u = w, \quad \text{in } t > 0, \\ \tau^k u_{|t=0} = f_k, \quad \text{for } k = 0, \dots, m-1 \end{cases} \tag{9}$$

is given by the sum of (6) above and

$$\begin{aligned} v = u_{f_{m-1}} + (\tau + P_1(\partial))\, u_{f_{m-2}} + (\tau^2 + P_1(\partial)\tau + P_2(\partial))\, u_{f_{m-3}} \\ + (\tau^{m-1} + P_1(\partial)\tau^{m-2} + \cdots + P_{m-1}(\partial))\, u_{f_0} \end{aligned} \tag{10}$$

In order to check this formula note that all the operators commute, so clearly $P(\partial, \tau)v = 0$. Now, for $k = 0, \dots, m-1$, $j = 0, \dots, m-1$

$$\tau^j \left(\tau^k + P_1(\partial)\tau^{k-1} + \dots P_k(\partial)\right) u_{f_{m-k-1}}$$

equals clearly 0 at $t = -0$ if $j + k \leq m - 2$, i.e. $j \leq m - k - 2$. It also equals zero if $j + k \geq m$ i.e. if $j \geq m - k$, because we can use the equation $P(\partial, \tau)u_{f_{m-k-1}} = 0$ and

$$\begin{aligned}&\tau^j \left(\tau^k + P_1(\partial)\tau^{k-1} + \dots P_k(\partial)\right) u_{f_{m-k-1}} \\ &= P(\partial, \tau)(\tau^{j+k-m} u_{f_{m-k-1}}) - \textstyle\sum_{l=k+1}^{m} P_l(\partial)\tau^{m-l}(\tau^{j+k-m} u_{f_{m-k-1}})\end{aligned}$$

Now the second sum vanishes because $0 \leq j + k - l \leq j - 1 \leq m - 2$ when $l \geq k + 1$. Finally, when $j + k = m - 1$ we get precisely f_{m-k-1}.

Now we consider the standard problem (8). Taking the Fourier transform with respect to the x variables we arrive at the problem

$$\begin{cases} P(i\xi, \tau)\widehat{u}(\xi, t) = 0, \quad t > 0, \\ \tau^k \widehat{u}(\xi, 0) = 0, \quad k \leq m - 2, \\ \tau^{m-1}\widehat{u}(\xi, 0) = \widehat{g}(\xi). \end{cases} \tag{11}$$

This is an ODE depending on parameters ξ. Let $Z(\xi, t)$ denote the fundamental solution, i.e., the solution of (11) with initial values

$$\begin{cases} Z(\xi, 0) = \tau Z(\xi, 0) = \dots = \tau^{m-2} Z(\xi, 0) = 0, \\ \tau^{m-1} Z(\xi, 0) = 1. \end{cases} \tag{12}$$

Then the Fourier inversion formula gives

$$u(x, t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} Z(\xi, t)\widehat{g}(\xi) d\xi \tag{13}$$

All that remains to do is to justify the convergence of this integral and that we can carry differentiation under the integral sign. We will prove, under some conditions on $P(\partial, \tau)$ that

$$\left|\tau^k Z(\xi, t)\right| \leq C(1 + |\xi|)^{(N-m+1)k} \tag{14}$$

for all $\xi \in \mathbb{R}^n$ and $k = 0, 1, \dots, m$ and t in a compact set. This will allow for differentiation under the integral sign if

$$\int_{\mathbb{R}^n} (1 + |\xi|)^{(N-m+1)m} |\widehat{g}| d\xi$$

is finite, which is a smoothness assumption on the initial data.

Definition 1. *(Garding) We say that the polynomial* $P(\partial, \tau)$ *is hyperbolic if there exists a real number* γ *such that, for any fixed* $\xi \in \mathbb{R}^n$*, the roots* λ *of* $P(i\xi, i\lambda)$*,*

$$P(i\xi, i\lambda) = 0 \tag{15}$$

all lie in the half plane

$$\mathrm{Im}\,\lambda > \gamma \tag{16}$$

Obviously, the point is that γ is the same for all ξ. Now we check (14). First, we represent

$$Z(\xi, t) = \frac{1}{2\pi} \int_\Gamma \frac{e^{i\lambda t}}{P(i\xi, i\lambda)} d\lambda \tag{17}$$

where Γ is a contour counterclockwise around all roots λ of $P(i\xi, i\lambda)$. Clearly (17) represents a solution of $P(i\xi, \tau)Z(\xi, t) = 0$ because each differentiation in time brings down $i\lambda$, so

$$P(i\xi, \tau)Z(\xi, t) = \frac{1}{2\pi} \int_\Gamma e^{i\lambda} d\lambda = 0.$$

Also, at $t = 0$ we have

$$\tau^k Z(\xi, 0) = \frac{1}{2\pi} \int_\Gamma \frac{i^k \lambda^k}{P(i\xi, i\lambda)} d\lambda$$

Deforming Γ to be a circle of radius R and letting $R \to \infty$ we obtain

$$\tau^k Z(\xi, 0) = 0$$

for $k = 0, 1, \ldots, m-2$ and

$$\tau^{m-1} Z(\xi, 0) = 1.$$

So (17) indeed represents Z. Now we bound the roots. Writing

$$i^m \lambda^m = -i^{m-1}\lambda^{m-1} P_1(i\xi) - \cdots - P_m(i\xi)$$

and using

$$|P_j(\xi)| \le C(1 + |\xi|)^{N-m+j}$$

we have

$$|\lambda|^m \le C(1 + |\xi|)^{N-m} \sum_{k=0}^{m-1} |\lambda|^k (1 + |\xi|)^{m-k}$$

We divide by $(1+|\xi|)^m$:

$$\left(\frac{|\lambda|}{1+|\xi|}\right)^m \leq C(1+|\xi|)^{N-m}\sum_{k=0}^{m-1}\left(\frac{|\lambda|}{1+|\xi|}\right)^k$$

Now either $\frac{|\lambda|}{1+|\xi|} \leq 1$ or, if not, then

$$\sum_{k=0}^{m-1}\left(\frac{|\lambda|}{1+|\xi|}\right)^k \leq m\left(\frac{|\lambda|}{1+|\xi|}\right)^{m-1}$$

so, in any case

$$|\lambda_k(\xi)| \leq C(1+|\xi|)^{N-m+1} \tag{18}$$

for large enough C where we denoted $\lambda_k(\xi)$ the repeated roots of the equation $P(i\xi, i\lambda) = 0$. Let us take now Γ to be the boundary of a union of possibly overlapping disks of radius 1 around the repeated roots $\lambda_k(\xi)$ of the equation $P(i\xi, i\lambda) = 0$. Note that

$$|e^{i\lambda t}| \leq e^{(1-\gamma)t} \tag{19}$$

holds for $\lambda \in \Gamma$. Because

$$P(i\xi, i\lambda) = \Pi_{k=1}^m(\lambda - \lambda_k(\xi))$$

we have that

$$|P(i\xi, i\lambda)| \geq 1, \quad \text{for } \lambda \in \Gamma$$

Now, for each $\lambda \in \Gamma$ we have at least one $\lambda_k(\xi)$ at distance 1, so

$$|\lambda| \leq C(1+|\xi|)^{N-m+1} \tag{20}$$

holds for $\lambda \in \Gamma$ at the price of increasing C by 1. It follows that

$$\left|\tau^k Z(\xi,t)\right| \leq C(1+|\xi|)^{(N-m+1)k}e^{(1-\gamma)t}$$

for $t \in \mathbb{R}$. This proves (14). Note that if $P(\partial, \tau)$ is homogeneous of degree m then Garding's hyperbolicity condition is equivalent to the condition that

$$P(i\xi, i\lambda) = 0,\ \xi \in \mathbb{R}^n \Rightarrow \lambda \in \mathbb{R}.$$