Harmonic functions, Second Order Elliptic Equations I

Introduction to PDE

1 Green's Identities, Fundamental Solution

Let Ω be a bounded open set in $\mathbb{R}^n$, $n \geq 2$, with smooth boundary $\partial\Omega$. The fact that the boundary is smooth means that at each point $x \in \partial\Omega$ the external unit normal vector $\nu(x)$ is a smooth function of x. (If the boundary is C^k then this function is C^{k-1}. Locally, $\partial\Omega$ is an embedded hypersurface - a manifold of codimension 1). Green's identities are

$$\int_{\Omega}(v\Delta u + \nabla u \cdot \nabla v)dx = \int_{\partial\Omega}(v\partial_\nu u)dS \tag{1}$$

and

$$\int_{\Omega}(v\Delta u - u\Delta v)dx = \int_{\partial\Omega}(v\partial_\nu u - u\partial_\nu v)dS. \tag{2}$$

In the two identities above, u and v are real valued functions twice continiously differentiable in Ω, $(u, v \in C^2(\Omega))$, with first derivatives that have continuous extensions to $\overline{\Omega}$. The Laplacian Δ is

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial^2 x_j} = \sum_{j=1}^{n} \partial_j^2 u,$$

the gradient is

$$\nabla u = (\partial_1 u, \ldots, \partial_n u),$$

partial derivatives are

$$\partial_i = \frac{\partial}{\partial x_i},$$

the normal derivative is

$$\partial_\nu u = \nu \cdot \nabla u$$

where $x \cdot y = \sum_{j=1}^n x_j y_j$. Finally, dS is surface measure.

Exercise 1 *Prove that (2) follows from (1) and that (1) follows from the familiar Green's identity*

$$\int_\Omega (\nabla \cdot \Phi)\, dx = \int_{\partial\Omega} (\Phi \cdot \nu) dS \tag{3}$$

where Φ is a $C^1(\Omega)$ vector field (n functions) that has a continuous extension to $\overline{\Omega}$.

Exercise 2 *Prove that the Laplacian commutes with rotations: $\Delta(f(Rx)) = (\Delta f)(Rx)$, where R is a $O(n)$ matrix, i.e. $RR^t = R^t R = \mathbb{I}$ with R^t the transpose. Consequently, rotation invariant functions (i.e., radial functions, functions that depend only on $r = |x|$) are mapped to rotation invariant functions. Show that, for radial functions,*

$$\Delta f = f_{rr} + \frac{n-1}{r} f_r$$

where $r = |x|$, $f_r = \frac{x}{r} \cdot \nabla f$.

The equation $\Delta f = 0$ in the whole plane has affine solutions. If we seek solutions that are radial, that is

$$f_{rr} + \frac{n-1}{r} f_r = 0, \quad \text{for } r > 0,$$

then, multiplying by r^{n-1}, we see that $r^{n-1} f_r$ should be constant. There are only two possibilities: this constant is zero, and then f itself must be a constant, or this constant is not zero, and then f is a multiple of r^{2-n} (plus a constant) if $n > 2$ or a multiple of $\log r$ (plus a constant).

The fundamental solution is

$$N(x) = \begin{cases} \frac{1}{2\pi} \log |x|, & \text{if } n = 2, \\ \frac{1}{(2-n)\omega_n} |x|^{2-n}, & \text{if } n \geq 3. \end{cases} \tag{4}$$

Here ω_n is the area of the unit sphere in $\mathbb{R}^n$. Note that N is radial, and it is singular at $x = 0$.

Proposition 1 *Let $f \in C_0^2(\mathbb{R}^n)$, and let*

$$u(x) = \int_{\mathbb{R}^n} N(x-y)f(y)dy. \tag{5}$$

Then $u \in C^2(\mathbb{R}^n)$ and

$$\Delta u = f$$

Lemma 1 *Let $N \in L^1_{loc}(\mathbb{R}^n)$ and let $\phi \in C_0^1(\mathbb{R}^n)$. Then $u = N * \phi$ is in $C^1(\mathbb{R}^n)$ and*

$$\nabla u(x) = \int N(x-y)\nabla\phi(y)dy$$

Idea of proof of the lemma. First of all, the notation: L^p_{loc} is the space of functions that are locally in L^p and that means that their restrictions to compacts are in L^p. The space C_0^1 is the space of functions with continuous derivatives of first order, and having compact support. Because ϕ has compact support

$$u(x) = \int N(x-y)\phi(y)dy = \int N(y)\phi(x-y)dy$$

is well defined. Fix $x \in \mathbb{R}^n$ and take $h \in \mathbb{R}^n$ with $|h| \leq 1$. Note that

$$\begin{array}{c} \frac{1}{|h|}\left(u(x+h) - u(x) - h \cdot \int N(x-y)\nabla\phi(y)dy\right) = \\ \int N(y)\frac{1}{|h|}\left(\phi(x+h-y) - \phi(x-y) - h\cdot\nabla\phi(x-y)\right)dy \end{array}$$

The functions $y \mapsto \frac{1}{|h|}\left(\phi(x+h-y) - \phi(x-y) - h\cdot\nabla\phi(x-y)\right)$ for fixed x and $|h| \leq 1$ are supported all in the same compact, are uniformly bounded, and converge to zero as $h \to 0$. Because of Lebesgue dominated, it follows that u is differentiable at x and that the derivative is given by the desired expression. Because

$$\nabla u(x) = \int N(y)\nabla\phi(x-y)dy$$

and $\nabla\phi$ is continuous, it follows that ∇u is continuous. This finishes the proof of the lemma.

Idea of proof of the Proposition 1. By the Lemma, u is C^2 and

$$\Delta u(x) = \int N(x-y)\Delta f(y)dy$$

Fix x. The function $y \mapsto N(x-y)\Delta f(y)$ is in $L^1(\mathbb{R}^n)$ and compactly supported in $|x-y| < R$ for a large enough R that we'll keep fixed. Therefore

$$\Delta u(x) = \lim_{\epsilon \to 0} \int_{\{y; \epsilon < |x-y| < R\}} N(x-y)\Delta f(y) dy$$

We will use Green's identities for the domains $\Omega_\epsilon^R = \{y; \epsilon < |x-y| < R\}$. This is legitimate because the function $y \mapsto N(x-y)$ is C^2 in a neighborhood of $\overline{\Omega_\epsilon^R}$. Note that, because of our choice of R, $f(y)$ vanishes identically near the outer boundary $|x-y| = R$. Note also that $\Delta_y N(x-y) = 0$ for $y \in \Omega_\epsilon^R$. From (2) we have

$$\int_{\{y; \epsilon < |x-y| < R\}} N(x-y)\Delta\ f(y) dy = \\ \int_{|x-y|=\epsilon} N(x-y)\partial_\nu f(y) dS - \int_{|x-y|=\epsilon} f(y)\partial_\nu N(x-y) dS$$

The external unit normal at the boundary is $\nu = -(x-y)/|x-y|$. The first integral vanishes in the limit because $|\nabla f|$ is bounded, $N(x-y)$ diverges like ϵ^{2-n} (or $\log \epsilon$) and the area of boundary vanishes like ϵ^{n-1}:

$$\left| \int_{|x-y|=\epsilon} N(x-y)\partial_\nu f(y) dS \right| \le C\epsilon \|\nabla f\|_\infty$$

in $n > 2$, and the same thing replacing ϵ by $\epsilon \log \epsilon^{-1}$ in $n = 2$. The second integral is more amusing, and it is here that it will become clear why the constants are chosen as they are in (4). We start by noting carefully that

$$-\partial_\nu N(x-y) = \frac{1}{\omega_n}|x-y|^{1-n}.$$

Therefore, in view of the fact that $|x-y| = \epsilon$ on the boundary we have

$$-\int_{|x-y|=\epsilon} f(y)\partial_\nu N(x-y) dS = \frac{1}{\epsilon^{n-1}\omega_n} \int_{|x-y|=\epsilon} f(y) dS$$

Passing to polar coordinates centered at x we see that

$$-\int_{|x-y|=\epsilon} f(y)\partial_\nu N(x-y) dS = \frac{1}{\omega_n} \int_{|z|=1} f(x+\epsilon z) dS$$

and we do have

$$\lim_{\epsilon \to 0} \frac{1}{\omega_n} \int_{|\ z|=1} f(x+\epsilon z) dS = f(x)$$

because f is continuous.

Remark 1 *The fundamental solution solves*

$$\Delta N = \delta.$$

This is rigorously true in the sense of distributions, but formally it can be appreciated without knowledge of distributions by using without justification the fact that δ is the identity for convolution $\delta * f = f$, the rule $\Delta(N * f) = \Delta N * f$ and the Proposition above.

2 Harmonic functions

Definition 1 *We say that a function u is harmonic in the open set $\Omega \subset \mathbb{R}^n$ if $u \in C^2(\Omega)$ and if*

$$\Delta u = 0$$

holds in Ω.

We denote by $B(x, r)$ the ball centered at x of radius r, by $A_f(x, r)$ the surface average

$$A_f(x, r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B(x,r)} f(y) dS$$

and by $V_f(x, r)$ the volume average

$$V_f(x, r) = \frac{n}{\omega_n r^n} \int_{B(x,r)} f(y) dy$$

Proposition 2 *Let Ω be an open set, let u be harmonic in Ω and let $\overline{B(x, r)} \subset \Omega$. Then*

$$u(x) = A_u(x, r) = V_u(x, r)$$

holds.

Idea of proof. Let $\rho \le r$ and apply (2) with domain $B(x, \rho)$ and functions $v = 1$ and u. We obtain

$$\int_{\partial B(x,\rho)} \partial_\nu u \, dS = 0$$

On the other hand, because

$$A_u(x, \rho) = \frac{1}{\omega_n} \int_{|z|=1} u(x + \rho z) dS$$

it follows that

$$\frac{d}{d\rho}A_u(x,\rho) = \frac{1}{\omega_n}\int_{|z|=1} z\cdot\nabla u(x+\rho z)\,dS = \frac{1}{\rho^{n-1}\omega_n}\int_{\partial B(x,\rho)} \partial_\nu u\,dS = 0$$

So $A_u(x,\rho)$ does not depend on ρ for $0<\rho\le r$. But, because the function u is continuous, $\lim_{\rho\to 0} A_u(x,\rho) = u(x)$, so that proves

$$u(x) = A_u(x,r).$$

The relation

$$V_f(x,r) = nr^{-n}\int_0^r \rho^{n-1}A_f(x,\rho)d\rho,$$

valid for any integrable function, implies the second equality, in view of the fact that $A_f(x,\rho) = u(x)$ does not depend on ρ.

Remark 2 *The converse of the mean value theorem holds. If u is C^2 and $u(x) = A_u(x,r)$ for each x and r sufficiently small, then u is harmonic. Indeed, in view of the above, the integral $\int_{B(x,r)}\Delta u(y)dy$ must vanish for each x and r small enough, and that implies that there cannot exist a point x where $\Delta u(x)$ does not vanish.*

Theorem 1 *(Weak maximum principle.) Let Ω be open, bounded. Let $u\in C^2(\Omega)\cap C^0(\overline{\Omega})$ satisfy*

$$\Delta u(x)\ge 0,\quad \forall x\in\Omega.$$

Then

$$\max_{x\in\overline{\Omega}} u(x) = \max_{x\in\partial\Omega} u(x).$$

Idea of proof. If $\Delta u > 0$ in Ω then clearly u cannot have an interior maximum, so its maximum must be achieved on the boundary. If $\Delta u\ge 0$, then a useful trick is to add $\epsilon|x|^2$. The function $u(x)+\epsilon|x|^2$ achieves its maximum on the boundary, for any positive ϵ. Therefore,

$$\max_{x\in\overline{\Omega}} u(x) + \epsilon\min_{x\in\overline{\Omega}}|x|^2 \le \max_{x\in\partial\Omega} u(x) + \epsilon\max_{x\in\partial\Omega}|x|^2$$

and the result follws by taking the limit $\epsilon\to 0$.

Remark 3 *If u is harmonic, then by applying the previous result to both u and $-u$ we deduce that*

$$\max_{x\in\overline{\Omega}}|u(x)| = \max_{x\in\partial\Omega}|u(x)|$$

Definition 2 *A continuous function $u \in C^0(\Omega)$ is subharmonic in Ω if $\forall x \in \Omega$, $\exists r > 0$ so that*

$$u(x) \leq A_u(x,\rho)$$

holds $\quad \forall \rho,\ 0 < \rho \leq r$.

Exercise 3 *Let $u \in C^2(\Omega)$, and assume that $\Delta u(x) \geq 0$ holds for any $x \in \Omega$. Prove that u is subharmonic.*

Theorem 2 *(Strong maximum principle) Let Ω be open, bounded and connected. Let $u \in C^0(\Omega)$ be subharmonic. Then, either u is constant, or*

$$u(x) < \sup_{x \in \Omega} u(x)$$

holds.

Idea of proof. Let $M = \sup_{x\in\Omega} u(x)$. Consider the sets $S_1 = \{x \in \Omega;\ u(x) < M\}$ and $S_2 = \{x \in \Omega;\ u(x) = M\}$. The two sets are disjoint, and S_1 is open. We show that S_2 is open as well. Indeed, take $x \in S_2$. Then

$$0 \leq A_u(x,\rho) - M = \frac{1}{\rho^{n-1}\omega_n} \int_{\partial B(x,\rho)} (u(y) - M) dS \leq 0.$$

Because the integrand is non-positive we deduce that $u(y) = M$ for all y such that $|y - x| = \rho$, with any $0 < \rho < r$. This means that S_2 is open.

Remark 4 *The result implies that, if $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies $\Delta u \geq 0$ in the bounded open connected domain Ω, then either u is constant, or*

$$u(x) < \max_{x \in \partial\Omega} u(x)$$

holds.

3 Green's functions, Poisson kernel

Exercise 4 *Let Ω be open, bounded, with smooth boundary. Let $u \in C^2(\Omega)$ with first derivatives that are continuous up to the boundary $\nabla u \in C(\overline{\Omega})$. Let $x \in \Omega$ Then*

$$u(x) = \int_{\partial\Omega} (u\partial_\nu N - N\partial_\nu u)\, dS + \int_\Omega N(x-y)\Delta u(y) dy \tag{6}$$

is true. The function N in the boundary integral is computed at $x - y$, with $y \in \partial\Omega$; the normal derivative refers to the external normal at y.

Hint: Take a small ball $B(x,r) \subset \Omega$ and write (2) in $\Omega \setminus B(x,r)$, with functions u and $N(x-y)$. Then use the calculation from the proof of the Proposition 1.

Let us consider now the inhomogeneous Dirichlet problem

$$\begin{cases} \Delta u = f, \\ u_{|\partial\Omega} = g \end{cases}$$

where f is some given function in Ω and g is a continuous function on $\partial\Omega$. Suppose that we can find, for each $x \in \Omega$, a harmonic function $n^{(x)}(y)$ such that

$$\begin{cases} \Delta_y n^{(x)}(y) = 0 \\ n^{(x)}(y) = N(x-y), \quad \text{for}\, y \in \partial\Omega. \end{cases}$$

Then, applying (2) we have

$$0 = -\int_\Omega n^{(x)}(y)\Delta u(y)dy + \int_{\partial\Omega}(N\partial_\nu u - u\partial_\nu n^{(x)})dS.$$

Adding to (6) we deduce that

$$G(x,y) = N(x-y) - n^{(x)}(y)$$

provides the solution to the Dirichlet problem,

$$u(x) = \int_\Omega G(x,y)f(y)dy + \int_{\partial\Omega} g(y)\partial_\nu G(x,y)dS \tag{7}$$

Note that the Green's function satisfies

$$\begin{cases} \Delta_y G(x,y) = \delta(x-y) \\ G(x,y) = 0 \quad \text{for}\, y \in \partial\Omega \end{cases}$$

with $\delta(x-y)$ the δ function concentrated at x. When $f = 0$ we obtain the representation of harmonic functions $u(x)$ for $x \in \Omega$ that satisfy $u(y) = g(y)$ for $y \in \partial\Omega$:

$$u(x) = \int_{\partial\Omega} P(x,y)g(y)dS \tag{8}$$

with

$$P(x,y) = \partial_\nu G(x,y) \tag{9}$$

the Poisson kernel. The representations (7) and (8) are useful, but the Green's function and the Poisson kernel for a general domain are hard to obtain explicitly. Two important examples that can be computed are the half plane and the ball. The main idea, in both cases, is to reflect the singularity away. The reflection is rather straightforward for the half-plane, less so for the ball.

Let $\Omega = \{x \in \mathbb{R}^n,\ x_n > 0\}$. Set,

$$G(x, y) = N(x - y) - N(x^* - y)$$

where $x^* = (x_1, \dots, x_{n-1}, -x_n)$. This is a Green's function for Ω. It is convenient to write a point in Ω as $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$ and $x_n > 0$. Then $x^* = (x', -x_n)$, the Poisson kernel is a function of $x' - y'$ and x_n, where $y = (y', 0)$ represents a point on the boundary,

$$P(x' - y', x_n) = \frac{2x_n}{\omega_n}\left(|x' - y'|^2 + x_n^2\right)^{-\frac{n}{2}}$$

Proposition 3 *Let g be a continuous bounded function of $n - 1$ variables, and let $n > 1$. The function*

$$u(x) = \int_{\mathbb{R}^{n-1}} P(x' - y', x_n) g(y') dy'$$

is harmonic in $x_n > 0$ and the limit $\lim_{x_n \to 0} u(x', x_n) = g(x')$ holds.

Let now $\Omega = B(0, R)$. Let $x \in \Omega$ and set $x^* = \frac{R^2}{|x|^2}x$. Note that if $y \in \partial\Omega$ then

$$\frac{|x^* - y|}{|x - y|} = \frac{R}{|x|}$$

does not depend on y. For $n > 2$ take the fundamental solutions $N(x - y)$ and $N(x^* - y)$ and write

$$G(x, y) = N(x - y) - \left(\frac{|x|}{R}\right)^{2-n} N(x^* - y)$$

Clearly, for $y \in \partial\Omega$, $G(x, y)$ vanishes. The second term is not singular in $y \in \Omega$, so it is harmonic in y. The Poisson kernel is

$$P(x, y) = \frac{1}{R\omega_n}\frac{R^2 - |x|^2}{|x - y|^n} \tag{10}$$

This works even in $n = 2$.

Proposition 4 *Let g be a continuous function on $\partial B(0,R)$ in $\mathbb{R}^n$, $n \geq 2$. The function*

$$u(x) = \begin{cases} \int_{\partial B(0,R)} P(x,y)g(y)dS, & \textit{for } |x| < R \\ u(x) = g(x), & \textit{for } |x| = R \end{cases}$$

given by the Poisson kernel (10) is harmonic in $|x| < R$, and continuous in $|x| \leq R$.

The proof uses the following properties of the Poisson kernel:

$$\begin{gathered} P(x,y) \in C^\infty,\ P(x,y) > 0 \quad \text{for } |x| < R, |y| = R, \\ \Delta_x P(x,y) = 0, \quad \text{for } |x| < R, |y| = R, \\ \int_{\partial B(0,R)} P(x,y)dS = 1 \quad \forall\, x,\ |x| < R, \\ \lim_{x \to z,\, |x| < R} P(x,y) = 0, \quad \text{uniformly for } |y| = |z| = R, |z - y| \geq \delta > 0. \end{gathered}$$

4 Method of Perron

Theorem 3 *Let Ω be an open bounded set, f a continuous function on $\partial\Omega$. We will assume that $\partial\Omega$ has the barrier property (see below). Then there exists a harmonic function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $u = f$ on $\partial\Omega$.*

The method of Perron is based on subharmonic functions and the observation that all such functions that have boundary values not larger than f, have to lie below u. If a solution exists, then it is subharmonic, so clearly $u(x) = \sup_S s(x)$ where the supremum is taken among all subharmonic functions s that are dominated by f on the boundary. The rest of this section shows how to turn this idea into math.

We will denote by $\sigma(\Omega)$ the set of subharmonic functions in Ω, that is, the set of continuous functions in Ω such that $u(x) \leq A_u(x,r)$ for all r sufficiently small:

$$\sigma(\Omega) = \{u \in C(\Omega);\ \exists \rho > 0, \forall\, r < \rho,\ u(x) \leq A_u(x,r)\}$$

(We use the notation $A_f(x,r)$ from the previous section on the Laplacian).

Lemma 2 *If Ω is connected and $u \in \sigma(\Omega) \cap C(\overline{\Omega})$ then*

$$\max_{\Omega} u \leq \max_{\partial\Omega} u$$

holds.

We proved this in the first section on the Laplacian.

Definition 3 *If $u \in \sigma(\Omega)$ and $\overline{B(x,r)} \subset \Omega$, we set $u_{x,r}$ to be the function*

$$u_{x,r}(y) = \begin{cases} u(y), & \text{if } y \notin B(x,r) \\ h(y), & \text{if } y \in B(x,r) \end{cases}$$

where $\Delta h = 0$ in $B(x,r)$ and $h(y) = u(y)$ for $y \in \partial B(x,r)$.

The existence and uniqueness of h have been established earlier.

Lemma 3 *If $u \in \sigma(\Omega)$ and $\overline{B(x,r)} \subset \Omega$ then*

$$u(y) \leq u_{x,r}(y)$$

holds for all $y \in \Omega$, and moreover

$$u_{x,r} \in \sigma(\Omega).$$

Proof. From definitions, it is clear that we need to check the inequality $u \leq u_{x,r}$ only for $y \in B(x,r)$. There it follows from the fact that $u - h \in \sigma(B(x,r)) \cap C(\overline{B(x,r)})$ and vanishes on the boundary of $B(x,r)$ so it's non-positive by the maximum principle Lemma 2. For the second part, we need to show that

$$u_{x,r}(y) \leq A_{u_{x,r}}(y,\rho)$$

for small ρ. If $y \notin \overline{B(x,r)}$ then clearly this is true because u is subharmonic and $u_{x,r} = u$ in the open set $\Omega \setminus \overline{B(x,r)}$. If $y \in B(x,r)$ then $u_{x,r}$ is subharmonic near y because it is harmonic near y. It remains to check the case $y \in \partial B(x,r)$. But there $u_{x,r}(y) = u(y) \leq A_u(y,\rho) \leq A_{u_{x,r}}(y,\rho)$ for small enough ρ, because u is subharmonic and $u \leq u_{x,r}$.

Lemma 4 *If $u \in \sigma(\Omega)$ then $u \leq A_u(x,r)$ holds for all r such that $\overline{B(x,r)} \subset \Omega$.*

Indeed, if $\overline{B(x,r)} \subset \Omega$ then $u(x) \leq u_{x,r}(x) = A_{u_{x,r}}(x,r) = A_u(x,r)$.

Lemma 5 *A function u is harmonic in Ω if and only if $u \in \sigma(\Omega)$ and $-u \in \sigma(\Omega)$.*

Indeed, if $u \in \sigma(\Omega)$ and $\overline{B(x,r)} \subset \Omega$ then $u \leq u_{x,r}$; if also $-u \in \sigma(\Omega)$ then $u = u_{x,r}$ and therefore u is harmonic in $B(x,r)$.

Lemma 6 *If $u \in C(\Omega)$ and if $u(x) = A_u(x,\rho)$ for sufficiently small ρ then u is harmonic in Ω.*

Indeed, then $u \in \sigma(\Omega)$ and $-u \in \sigma(\Omega)$.

Definition 4 *Let $f \in C(\partial\Omega)$ and consider*

$$\sigma_f(\overline{\Omega}) = \{u \in \sigma(\Omega) \cap C(\overline{\Omega});\;\; u(y) \le f(y), \forall\, y \in \partial\Omega\}.$$

We define, for $x \in \Omega$,

$$w_f(x) = \sup_{u \in \sigma_f(\overline{\Omega})} u(x)$$

Note that, if

$$m = \inf_{\partial\Omega} f, \quad M = \sup_{\partial\Omega} f$$

then the constant function m belongs to $\sigma_f(\overline{\Omega})$, so that the latter is not empty. Also, for any $u \in \sigma_f(\overline{\Omega})$, we have from the maximmum principle that $u \le M$, and therefore $w_f(x)$ is finite for any x.

Lemma 7 *If $u_1, \dots, u_k \in \sigma_f(\overline{\Omega})$ then $v = \max\{u_1, \dots, u_k\} \in \sigma_f(\overline{\Omega})$.*

Indeed, clearly $v \in C(\overline{\Omega})$ and $v \le f$ on $\partial\Omega$. If $\overline{B(x,r)} \subset \Omega$ then

$$v(x) \le \max\{A_{u_1}(x,r), \dots A_{u_k}(x,r)\} \le A_v(x,r)$$

Proposition 5 *Let $h_1, h_2, \dots$ be harmonic functions in an open bounded set B, and assume that $h_k \in C(\overline{B})$ and that there exists a constant C such that $\sup_{x \in \partial B} |h_k| \le C$ holds for all k. Then there exists a subsequence of h_k that converges uniformly on compact subsets of B to a limit function that is harmonic.*

Idea of proof. Let $K \subset\subset B$ and let $d > 2\text{dist}(K, \partial B)$ Let $x \in K$ and let $r < d$. Then, by the maximum principle, $|h_k(y)| \le C$ for $|x-y| = r$. On the other hand, using the Poisson formula for the ball $B(x,r)$ and differentiating it, we obtain $|\nabla h_k(x)| \le \frac{n}{r} C$. This bound is uniform for all $x \in K$ and all k an therefore the sequence h_k is uniformly bounded and equicontinuous on the compact K. By Arzela-Ascoli, the sequence has a uniformly convergent subsequence.

Lemma 8 *The function w_f is harmonic in Ω.*

Proof. Let $\overline{B(x,r)} \subset \Omega$, and let x_k be any sequence in $B(x,\rho)$, with $\rho < r$. Because of the definition, there exist functions $u_k^j \in \sigma_f(\overline{\Omega})$ such that

$$w_f(x_k) = \lim_{j\to\infty} u_k^j(x_k)$$

holds for any $k = 1, 2, \ldots$. Let $u^j = \max\{u_1^j, \ldots, u_j^j\}$. Because $u^j \in \sigma_f(\Omega)$ is larger than u_k^j, it follows that $w_f(x_k) \geq u^j(x_k) \geq u_k^j(x_k)$ so, as $j \to \infty$, $\lim_{j\to\infty} u^j(x_k) = w_f(x_k)$ holds for all k. Replacing u^j by $\max\{u^j, m\}$ we may assume without loss of generality $m \leq u^j \leq M$ in Ω. Now we replace u^j by ${u^j}_{x,\rho}$. These are harmonic functions, larger than u^j and still in $\sigma_f(\Omega)$ so, the limit still is achieved. We have found thus a sequence of harmonic functions h_j in $B(x,r)$ such that $m \leq h_j \leq M$ and $\lim_{j\to\infty} h_j(x_k) = w_f(x_k)$ holds for all k. By the previous proposition there exists a subsequence of h_j (denoted again h_j) that converges uniformly on $B(x,\rho)$ to a harmonic function W. The function W may depend on the sequence x_k and the subsequnece of h_j. Nevertheless, for any sequence x_k we found a harmonic function W such that $w_f(x_k) = W(x_k)$. This is enough. First, by choosing $x_k \to x$, it follows from the continuity of W that $w_f(x_k)$ converges, for any sequence $x_k \to x$. This implies that w_f is continuous. Now, by taking x_k dense, we find that the continuous function w_f agrees with a harmonic function on a dense set, so w_f is harmonic in $B(x,\rho)$.

Definition 5 *(Barrier property). We assume that for any $y \in \partial\Omega$ there exists a barrier function $Q_y \in \sigma(\Omega) \cap C(\overline{\Omega})$ such that*

$$Q_y(y) = 0, \quad Q_y(x) < 0 \;\forall\, x \in \partial\Omega,\, x \neq y$$

Lemma 9 *Assume the barrier property. Then, for any $y \in \partial\Omega$,*

$$\lim \inf_{x\to y, x\in\Omega} w_f(x) \geq f(y)$$

holds.

Indeed, let $\epsilon > 0, K > 0$ be constants, and let $u(x) = f(y) - \epsilon + KQ_y(x)$. Clearly $u \in C(\overline{\Omega}) \cap \sigma(\Omega)$ and $u \leq f - \epsilon$ on $\partial\Omega$, and $u(y) = f(y) - \epsilon$. Then, because f is continuous, there exists $\delta = \delta(\epsilon)$ so that $f(x) > f(y) - \epsilon$ holds for $x \in \partial\Omega$, $|x - y| < \delta$. Therefore, $u(x) \leq f(x)$ for $|x - y| \leq \delta$. On the other hand, for $|x - y| \geq \delta$ we know that $\sup Q_y$ is strictly negative, so there exists a choice of $K = K(\epsilon)$ that large enough so that $u(x) \leq f(x)$ holds for $|x-y| \geq \delta$. This means that $u \in \sigma_f(\overline{\Omega})$ and then it follows that $u(x) \leq w_f(x)$ holds for all $x \in \Omega$. But then,

$$f(y) - \epsilon = \lim_{x \to y} u(x) \leq \lim \inf_{x \in \Omega,\ x \to y} w_f(x)$$

Finally, we have

Lemma 10 *Assume the barrier property. Then* $f(y) = \lim_{x \in \Omega,\ x \to y} w_f(x)$ *holds.*

In view of the lemma above, it is enough to prove only $\lim\sup_{x \to y} w_f(x) \leq f(y)$. We consider $-w_{-f}$. Writing $u = -U$ we have $-w_{-f} = \inf U$ for all U such that $-U \in C(\overline{\Omega}) \cap \sigma(\Omega)$, satisfying $-U \leq -f$ on $\partial\Omega$. Then, if $u \in \sigma_f(\overline{\Omega})$, it follows that $u - U \leq 0$ on $\partial\Omega$, therefore the same holds in Ω. Then it follows that $w_f \leq -w_{-f}$ in Ω. Applying the previous lemma to w_{-f} we obtain that

$$\lim \sup_{x \in \Omega,\ x \to y} w_f(x) \leq \lim \sup_{x \in \Omega,\ x \to y} -w_{-f}(x) = -\lim \inf_{x \in \Omega,\ x \to y} w_{-f} \leq f(y).$$

This completes the proof of the lemma, and of the theorem.

Remarks.

1. Let $y \in \partial\Omega$ and assume it has a local barrier, i.e, there exists a neighborhood N of y in $\mathbb{R}^n$ and a subharmonic function $Q \in \sigma(\Omega \cap N) \cap C(\overline{(\Omega \cap N)})$ satisfying $Q(x) < 0$ for $x \in \overline{\Omega \cap N} \setminus \{y\}$, $Q(y) = 0$. Then there exist a barrier at y relative to Ω. Indeed, let $B \subset\subset N$ be a ball centered at y. Consider $M = \sup\left\{Q(x) \mid x \in \overline{N \cap \Omega} \setminus B\right\}$. Consider the function $Q_y(x)$ equal to $Q_y(x) = \max(Q(x), M)$ for $x \in \overline{\Omega} \cap B$, and $Q_y(x) = M$ for $x \in \overline{\Omega} \setminus B$. Then Q_y is a barrier for Ω at y.

2. If Ω has the exterior ball property at $y \in \partial\Omega$ then the domain has a barrier at y. The exterior ball property simply means that there exists a ball $B = B(X, R)$ so that $\overline{\Omega} \cap \overline{B} = \{y\}$. Indeed, the function $Q(x) = |x - X|^{2-n} - R^{2-n}$ is a barrier for Ω at y if $n > 2$, and $Q(x) = \log\left(\frac{R}{|x-X|}\right)$ isa barrier if $n = 2$.

3. We say that the boundary $\partial\Omega$ is *regular* if there is a barrier at every point in $\partial\Omega$. The Perron method can be stated as

Theorem 4 *Let* Ω *be an open bounded set in* $\mathbb{R}^n$. *Then the problem*

$$\begin{cases} \Delta u(x) = 0, & \forall x \in \Omega \\ u(x) = f(x), & \forall x \in \partial\Omega \end{cases}$$

has a a solution for every $f \in C(\partial\Omega)$ if, and only if the boundary $\partial\Omega$ is regular.

The "only if" part of the theorem follows by noting that if $y \in \partial\Omega$ and $f(x) = -|x-y|$, if we can find a harmonic function u that solves $u(x) = f(x)$ on the boundary, then obviously u is a barrier at y.

5 Dirichlet principle, variational solutions

Let Ω be a bounded open set in $\mathbb{R}^n$. Let $f \in C(\Omega)$, $g \in C(\partial\Omega)$ a and let

$$\mathcal{A} = \{w \in C^2(\overline{\Omega}); \quad w_{|\partial\Omega} = g\}.$$

Let

$$I[w] = \int_\Omega \left(\frac{1}{2}|\nabla w|^2 + wf\right) dx. \tag{11}$$

Proposition 6 *$u \in C^2(\overline{\Omega})$ solves*

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases} \tag{12}$$

if, and only if

$$u = \min_{w \in \mathcal{A}} I[w].$$

Note carefully that this is not an existence theorem, rather, it states the equivalence of two possible existence theorems. One theorem asserts that the Poisson problem with data f and g has a solution u with the desired smoothness. The other theorem asserts that one can minimize the integral $I[w]$ and find a true minimum, in the class of admissible functions $\mathcal{A}$. The variational method considers the minimization program. The program consists in two steps. The first step is to establish the existence of a minimum. Unfortunately, the natural function spaces for $I[w]$ are not spaces of continuous functions, but rather Sobolev spaces based on L^2. This presents an opportunity to generalize: the right-hand side f will be allowed to be in L^2, because the method does not require more. There is a price to pay: the minimum thus obtained is not smooth. The second step is to show that if the function f is smooth then the solution is smooth. There are two ways of doing this, via Sobolev spaces and embedding thms, or via Hölder spaces. Although the method of Perron gave $C^2(\Omega)$ solutions for continuous g, spaces of continuous functions are not robust enough for analysis: certain natural integral operators, like convolution with N are not well behaved in spaces of continuous functions, but they are well-behaved in L^p and Hölder spaces.

5.1 $H_0^1(\Omega)$

Definition 6 *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with smooth boundary. The space $H^1(\Omega)$ is the completion of $C^\infty(\mathbb{R}^n)$ with norm*

$$\|u\|^2_{H^1(\Omega)} = \int_\Omega (|\nabla u|^2 + |u|^2)dx.$$

Let Ω be open, arbitrary. The space $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the norm $\|u\|_{H^1(\Omega)}$.

Similar definitions are given for the whole space and the torus. In those cases $H_0^1 = H^1$. However, in bounded domains H_0^1 is strictly smaller: it represents functions that vanish at the boundary, in a weak sense.

Exercise 5 *Show that $H_0^1((0,1)) \neq H^1((0,1))$.*

Lemma 11 *(Poincaré Inequality) There exists a constant $\pi(\Omega) > 0$ depending on the open bounded domain Ω such that*

$$\int_\Omega |u|^2 \leq \pi(\Omega) \int_\Omega |\nabla u|^2 dx$$

holds for all $u \in H_0^1(\Omega)$.

Proof. Because Ω is bounded in the direction x_1, we know that there exists an interval $[a,b]$ such that $t \in [a,b]$ holds for all t such that there exists $x_2, \dots x_n$ so that $x = (t, x_2, \dots, x_n) \in \Omega$. Because both sides of the inequality are continuous in H^1 and because of the definition of H_0^1, we may assume, WLOG that $u \in C_0^\infty(\Omega)$. Then

$$u(x_1, \dots x_n)^2 = 2\int_a^{x_1} u(t, x_2, \dots, x_n)(\partial_1 u)(t, x_2, \dots, x_n)dt$$

By Schwartz:

$$u(x_1, \dots x_n)^2 \le 2\left\{\int_a^b u^2(t, x_2, \dots, x_n)dt\right\}^{\frac{1}{2}}\left\{\int_a^b |\nabla u(t, x_2, \dots, x_n)|^2\, dt\right\}^{\frac{1}{2}}$$

We keep x_1 fixed, integrate $dx_2 \dots dx_n$ and use Schwartz again:

$$\int u^2(x_1, x_2, \dots x_n)dx_2 \dots dx_n \le 2\|u\|_{L^2}\|\nabla u\|_{L^2}.$$

We integrate dx_1 on $[a, b]$, noting that the RHS is independent of x_1:

$$\|u\|_{L^2}^2 \le 2(b-a)\|u\|_{L^2}\|\nabla u\|_{L^2}.$$

Dividing by $\|u\|_{L^2}$ we obtain the inequality with $\pi(\Omega) = 4(b-a)^2$. It is clear from this proof that we do not need to use the boundedness of Ω, only the existence of some direction in which Ω is bounded.

Exercise 6 *Show that the Poincaré inequality fails in $H^1(\Omega)$ if Ω is bounded. Show that the Poincaré inequality fails in some unbounded domains: for instance in $H_0^1(\mathbb{R})$.*

The Poincaré inequality implies that the scalar product

$$(u, v) = \int_\Omega \nabla u \cdot \nabla v dx$$

is equivalent (gives the same topology) with the scalar product in $H_0^1(\Omega)$:

$$< u, v >= \int_\Omega (uv + \nabla u \cdot \nabla v)dx$$

Theorem 5 *Let Ω be a bounded open set. Let $f \in L^2(\Omega)$. Then, there exists a unique $u \in H_0^1(\Omega)$ that solves*

$$I[u] = \min_{w \in H_0^1(\Omega)} I[w]$$

where

$$I[w] = \int \left(\frac{1}{2}|\nabla w|^2 + wf\right) dx.$$

The function u satisfies the variational formulation of the problem (12) with $g = 0$:

$$(u, v) + \int_\Omega vf dx = 0 \quad \forall\ v \in H_0^1(\Omega) \tag{13}$$

Proof. The function $I[w]$ is bounded below. Indeed, using Schwartz we have

$$I[w] \geq \frac{1}{2}\int_{\Omega} |\nabla w|^2 dx - \|f\|_{L^2}\|w\|_{L^2}$$

and using the Poincaré inequality

$$I[w] \geq \frac{1}{2C}\|w\|_{L^2}^2 - \|f\|_{L^2}\|w\|_{L^2}$$

and that is bounded below. Thus, the infimum exists:

$$m = \inf_{w \in H_0^1(\Omega)} I[w].$$

Let w_k be a minimizing sequence, $I[w_k] \to m$. The sequence is bounded in $H_0^1(\Omega)$ because (w_k, w_k) are bounded. Therefore there exists a subsequence (denoted again w_k) and an element u in $H_0^1(\Omega)$ such that w_k converges weakly to u,

$$\lim_k (w_k, v) = (u, v) \quad \forall v \in H_0^1(\Omega).$$

Then, because $v \mapsto \int_{\Omega} vf dx$ is linear and continuous in H_0^1, hence weakly continuous, it follows that

$$\int_{\Omega} uf dx = \lim_k \int w_k f dx$$

Exercise 7 *In a Hilbert space, the square of the norm is weakly lower semi-continuous.*

By the above exercise,

$$\frac{1}{2}\int_{\Omega} |\nabla u|^2 dx \leq \frac{1}{2}\liminf_k \int_{\Omega} |\nabla w_k|^2 dx$$

This shows that u achieves the minimum.

$$I(u) = m.$$

The variational formulation (13) follows by looking at the function

$$q(t) = I(u + tv)$$

for fixed, but arbitrary $v \in H_0^1(\Omega)$ and t real. $q(t)$ is a quadratic polynomial in t,

$$q(t) = I(u) + t\left\{(u,v) + \int_\Omega vf\,dx\right\} + \frac{t^2}{2}(v,v)$$

that has a minimum at $t = 0$. The variational formulation is equivalent to the fact that $q'(0) = 0$. The uniqueness of u follows immediately from variational formulation with $v = u$, the inequality

$$\|u\|^2 \le C(u,u) \le 2C\|f\|_{L^2}\|u\|_{L^2},$$

thus

$$\|u\|_{L^2} \le 2C\|f\|_{L^2},$$

and because the variational formulation is linear.

6 Weak solutions of second order uniformly elliptic equations

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, and suppose we are given real functions $a_{ij}(x), b_i(x), c(x)$ that belong to $L^\infty(\Omega)$. We assume that $a_{ij}(x) = a_{ji}(x)$ and the uniform ellipticity condition: There exists $\gamma > 0$ such that

$$\sum_{ij} a_{ij}(x)\xi_i\xi_j \ge \gamma|\xi|^2 \tag{14}$$

holds for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$. We consider the operator

$$(P(x,D)u)(x) = -\sum_{ij}\partial_i\left(a_{ij}\partial_j u\right) + \sum_i b_i\partial_i u + cu \tag{15}$$

We associate to $P(x,D)$ a bilinear form,

$$B : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$$

$$B(u,v) = a(u,v) + (b\cdot\nabla u, v) + (cu, v)$$

where

$$a(u,v) = \int_\Omega \sum_{ij} a_{ij}(x)(\partial_i u(x))(\partial_j v(x))dx,$$

$$(b \cdot \nabla u)(x) = \sum_i b_i(x)(\partial_i u(x))$$

and we use the notation

$$(f, g) = \int_\Omega f(x)g(x)dx.$$

Definition 7 *Let $f \in L^2(\Omega)$. We say that $u \in H_0^1(\Omega)$ is a variational solution of*

$$\begin{cases} P(x, D)u = f, & \text{in } \Omega, \\ u_{|\partial\Omega} = 0 \end{cases}$$

if

$$B(u, v) = (f, v)$$

holds for all $v \in H_0^1(\Omega)$.

Theorem 6 *Let Ω be an open bounded set, $f \in L^2(\Omega)$, let $P(x, D)$ be defined as above. Assume that $\nabla \cdot b \in L^\infty(\Omega)$ and*

$$\gamma > \pi(\Omega)\left[\frac{1}{2}\|\nabla \cdot b\|_{L^\infty} + \|c_-\|_{L^\infty}\right] \tag{16}$$

where $\pi(\Omega)$ is the constant in Poincaré's inequality, and we used the notation $c_-(x) = \max\{-c(x), 0\}$. Then there exists a unique variational solution of $P(x, D)u = f$, $u \in H_0^1(\Omega)$.

Remark. Note that if $c(x) \geq 0$ and $\nabla \cdot b = 0$ a.e. in Ω then the result is valid for arbitrary $\gamma > 0$. **Proof.** The form $B(u, v)$ is bilinear, continuous and coercive. By Lax-Milgram's theorem, there exists a unique variational solution.

Theorem 7 *(Lax-Milgram) Let H be a real Hilbert space, $B : H \times H \to \mathbb{R}$ be bilinear, continuous*

$$|B(u, v)| \leq C\|u\|\|v\|, \quad \forall\, u, v \in H$$

and coercive, i.e. there exists $\gamma > 0$ such that

$$B(u, u) \geq \gamma\|u\|^2$$

holds for all $u \in H$. Let $f : H \to \mathbb{R}$ be linear and continuous Then there exists a unique $u \in H$ such that

$$B(u, v) = (f, v)$$

holds for all $v \in H$.

Proof (of Lax-Milgram). By Riesz representation, for any $u \in H$ there exists $U \in H$ such that $B(u, v) = (U, v)$ holds for all $v \in H$. Bilinearity of B implies that $U = Au$ is a linear operator. Continuity implies that A is bounded, $\|Au\| \le C\|u\|$. Coercivity implies $\|Au\| \ge \gamma\|u\|$. This last fact implies that A is one-to-one and that its range is closed. Moreover, the range of A is H, because if v is perpendicular on therange then $B(v, v) = 0$, and hence, by coercivity $v = 0$.

Let us verify now the conditions of Lax-Milgram: Continuity is clear:

$$|B(u, v)| \le \alpha\|\nabla u\|_{L^2}\|\nabla v\|_{L^2} + \beta\|\nabla u\|_{L^2}\|v\|_{L^2} + \|c\|_{L^\infty}\|u\|_{L^2}\|v\|_{L^2}$$

For coercivity we have

$$B(v, v) \ge \gamma\|\nabla v\|_{L^2}^2 - \left[\frac{1}{2}\|\nabla \cdot b\|_{L^\infty} + \|c_-\|_{L^\infty}\right]\|v\|_{L^2}^2$$

where we used the fact that, for $v \in H_0^1$ and $b \in (L^\infty)^n$ with $\nabla \cdot b \in L^\infty$ we have

$$\int_\Omega (b \cdot \nabla v)v dx = -\frac{1}{2}\int(\nabla \cdot b)v^2 dx.$$

7 Regularity of weak solutions

Theorem 8 *(Interior regularity) Assume that $a_{ij} \in C^1(\Omega)$, $b \in (L^\infty(\Omega))^n$, $c \in L^\infty(\Omega)$. Assume that $u \in H^1(\Omega)$ is a variational solution of $P(x, D)u = f$, and $f \in L^2(\Omega)$. Then $u \in H^2_{loc}(\Omega)$ and, for every relatively compact open set $U \subset\subset \Omega$, there exists a constant C depending only on U and Ω, such that*

$$\|u\|_{H^2(U)} \le C\left[\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\right]. \tag{17}$$

The method of proof is to use finite differences and the coercivity of the principal bilinear form $a(u, v)$. Let

$$((\delta_h^i)v)(x) = h^{-1}(v(x + he_i) - v(x))$$

be the difference quotient. Let $U \subset\subset \Omega$.

Theorem 9 *(a) If $1 \le p < \infty$. There exists a constant C so that for every $u \in W^{1,p}(\Omega)$ and $|h| \le \frac{1}{2}dist(U, \partial\Omega)$ we have*

$$\|\delta_h^i u\|_{L^p(U)} \le C\|\nabla u\|_{L^p(\Omega)}.$$

(b) If $1 < p \leq \infty$ and if $u \in L^p(\Omega)$ satisfies

$$\|\delta_h^i(u)\|_{L^p(U)} \leq C$$

uniformly for all $|h|$ small enough, then $u \in W^{1,p}(U)$ and

$$\|\nabla u\|_{L^p(U)} \leq C.$$

Note that in (b) we have $1 < p$. The result fails for $p = 1$. For an example of this failure, we may consider the case $n = 1$, $\Omega = (-1, 1)$, $u(x) = 1$ for $x \geq 0$, $u(x) = 0$ for $x < 0$. Then $\delta_h u$ is bounded in L^1 but the derivative $u' = \delta \notin L^1$.

Idea of proof of the interior regularity. We take a smooth cutoff function $\chi \geq 0$, so that $\chi = 1$ on an open neighborhood of U and $\chi \in C_0^\infty(V)$ with $U \subset\subset V \subset\subset \Omega$. We know by definition

$$B(u, v) = (f, v)$$

holds for all $v \in H_0^1(\Omega)$. We pick $1 \leq k \leq n$, h small enough and consider

$$v = -\delta_{-h}^k(\chi^2 \delta_h^k u)$$

Clearly, $v \in H_0^1(\Omega)$ is an admissible test function and we have

$$a(u, v) = \sum_{ij} \int_\Omega \delta_h^k(a_{ij}\partial_i u)\partial_j(\chi^2 \delta_h^k(u))dx$$

because

$$(f, \delta_{-h}^k g) = -(\delta_h^k f, g)$$

and $\delta_h^k \partial_j = \partial_j \delta_h^k$. Then, we use a product rule

$$\delta_h^k(au) = a\delta_h^k(u) + \tau_h^k(u)\delta_h^k(a)$$

with $\tau_h^k(u)(x) = u(x + he_k)$. We deduce

$$a(u, v) \geq \gamma \int_\Omega \chi^2 \left|\delta_h^k \nabla u\right|^2 dx - E(u)$$

with

$$E(u) \leq C\|\nabla u\|_{L^2(V)} \left[\|\chi \delta_h^k \nabla u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(V)}\right].$$

For the lower order terms (involving $b \cdot \nabla u$ and f) we do not "integrate by parts", i.e., we do not move δ^k_{-h} to apply to them, but we use instead the fact that

$$\|v\|_{L^2(V)} \le C\left[\|\chi\delta^k_h \nabla u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(V)}\right].$$

We deduce, after a few calculations,

$$\|u\|^2_{H^2(U)} \le C\left[\|f\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(V)}\right]$$

By taking another cutoff function, $\psi \in C_0^\infty(\Omega)$ $\psi = 1$ on a neighborhood of V and using the definition and the test function $v = \psi^2 u$ we discover after standard manipulations that

$$\int_\Omega \psi^2 |\nabla u|^2 dx \le C \int_\Omega \left[f^2(x) + u^2(x)\right] dx.$$

This finishes the proof.

Theorem 10 *Assume that $a_{ij} \in C^1(\overline{\Omega})$, assume that $\partial\Omega \in C^2$, $b, c \in L^\infty(\Omega)$ and that $f \in L^2(\Omega)$ is given. Assume that $u \in H_0^1(\Omega)$ is a variational solution of $P(x, D)u = f$, $u_{|\partial\Omega} = 0$. Then $u \in H^2(\Omega)$ and there exists a constant C such that*

$$\|u\|_{H^2(\Omega)} \le C\left[\|u\|_{L^2(\Omega)} + \|f\|_{L^2}(\Omega)\right].$$

Idea of proof. First consider the case when Ω is half-ball included in the half space $\mathbb{R}^n_+ = \{x_n > 0\} \cap \mathbb{R}^n$, $\Omega_R = \{x \mid |x| < R, x_n > 0\}$. We assume that u is a variational solution that belongs to $H^1(\Omega_R)$ and whose trace vanishes at $\{x_n = 0\}$. We use difference quotients δ^i_h with $i < n$, in the same way as in the interior regularity case, carefully, and discover that $\partial_i \nabla u$ are all controlled (by the right hand side of the desired inequality). Then use *the equation* to deduce that $\partial_n\partial_n u$ is controlled by the same thing.

The general domain case is the attacked by a partition of unity and flattening the boundary. Changing variables changes the coefficients in the second order operator, but the uniform ellipticity is preserved. If $y = \Phi(x)$ is a diffeomorphism with inverse Φ^{-1}, and if $\widetilde{u}(y) = (u \circ \Phi^{-1})(y)$, then the principal part $A(x, D)u = \sum_{ij} a_{ij}(x)\partial^2_{ij} u$ of the operator $P(x, D)u$ becomes $\widetilde{A}(y, D)\widetilde{u} = \sum_{kl} \widetilde{a}_{kl}\partial^2_{kl}\widetilde{u}$ where $\widetilde{a}_{kl}(y) = \sum_{i,j} a_{ij}(x)(\partial_{x_i}\Phi^k(x))(\partial_{x_j}\Phi^l(x))$ at $x = \Phi^{-1}(y)$. Therefore $\sum_{kl} \widetilde{a}_{kl}(y)\eta_k\eta_l = \sum_{ij} a_{ij}(x)\xi_i\xi_j$ with $\xi = (\nabla\Phi(x))^T\eta$, and the uniform ellipticity is preserved as long as $\nabla\Phi^{-1}$ and $\nabla\Phi$ have uniformly bounded norms. The bilinear form associated to P is defined in the new variables according to the change of variables formula, i.e. $\widetilde{B}$ is defined by

$$\widetilde{B}(\widetilde{u},\widetilde{v}) = B(\widetilde{u}\circ\Phi, \widetilde{v}\circ\Phi).$$

The full operator $\widetilde{P}(y,D)\widetilde{u}$ becomes in the new variables $y = \Phi(x)$

$$\widetilde{P}(y,D)\widetilde{u})(y) = -\frac{1}{Det(\nabla\Phi^{-1})}\partial_k\left(Det(\nabla\Phi^{-1})\widetilde{a}_{kl}(\partial_l\widetilde{u})\right) + \widetilde{b}_k\partial_k\widetilde{u} + \widetilde{c}\widetilde{u}$$

where $\widetilde{a}_{kl}$ are defined above, and $\widetilde{b}_k(y) = \left[(\nabla\Phi)^T\circ(\Phi^{-1})\right](b_j\circ\Phi^{-1})$, $\widetilde{c} = c\circ\Phi^{-1}$. Let Ω be a bounded domain with C^2 boundary. For each $x_0\in\partial\Omega$ there exists $r>0$ and a C^2 function $h(x')$ defined in $\mathbb{R}^{n-1}$ such that (after a rotation)

$$\Omega\cap B(x_0,r) = \{x\in B(x_0,r)\mid x_n > h(x')\}.$$

Then a diffeomorphism $\Phi:\mathbb{R}^n\to\mathbb{R}^n$ so that $x\in\Omega\cap B(x_0,r)$ if and only if $x\in B(x_0,r)$ and $\Phi(x) = y$ satisfies $y_n>0$ and $x\in\partial\Omega\cap B(x_0,r)$ if and only if $y_n = 0$, is obtained by setting

$$\begin{gathered}\Phi^i(x) = x^i,\ \text{for } i = 1,\dots,n-1,\\ \Phi^n(x) = x_n - h(x').\end{gathered}$$

Note that

$$\begin{gathered}(\Phi^{-1})^i(y) = y^i,\ \text{for } i = 1,\dots,n-1,\\ (\Phi^{-1})^n(y) - y_n + h(y')\end{gathered}$$

and $Det\nabla\Phi^{-1} = 1$. We say that Φ flattens the boundary.

If $u\in H_0^1(\Omega)$ is a variational solution of $P(x,D)u = f$ then, restricting to $B(x_0,r)\cap\Omega$ and changing variables, $\widetilde{u}(y) = u(\Phi^{-1}(y))$ we obtain a variational solution of $\widetilde{P}(y,D)\widetilde{u} = \widetilde{f}$ in a half-ball, vanishing on the flat boundary of the half-ball. We obtain therefore H^2 estimates for it. Because the boundary is compact, we can cover it with finitely many balls $B(x_0,r)$, and summing the estimates, together with the interior regularity estimates we finish the proof of the theorem.