Take Home Final, due Monday May 9 at 9 a.m. in my mail box

Functional Analysis

1. (i) $U : \mathbb{C}^n \to \mathbb{C}$ is the Fourier transform of a distribution $u \in \mathcal{E}'(\mathbb{R}^n)$ with $\text{supp} u \subset B(0, R) = \{ x \in \mathbb{R}^n \mid |x| \leq R \}$ if and only if there exist $N$ and $C$ such that

$$|U(\zeta)| \leq C(1 + |\zeta|)^N e^{R|\text{Im} \zeta|}$$

holds for all $\zeta \in \mathbb{C}^n$.

(ii) $U$ is the Fourier transform of a function $u \in \mathcal{D}(\mathbb{R}^n)$ supported in $B(0, R)$ if and only if, for every $N$ there exists $C_N$ such that

$$|U(\zeta)| \leq C_N (1 + |\zeta|)^{-N} e^{R|\text{Im} \zeta|}$$

holds for all $\zeta \in \mathbb{C}^n$.

**Hint.** For the direct (“only if”) implication of part (i) consider the function $e(x, \zeta) = \psi(|\zeta|(|x| - R)) e^{-i(x \cdot \zeta)}$ where $\psi$ is a smooth function of a positive variable $r$, supported in $0 \leq r \leq 1$ and identically one on $0 \leq r \leq \frac{1}{2}$. Show that for $x$ in the support of $e$ we have $|x| \leq R + \frac{1}{|\zeta|}$, $|\partial_\zeta e| \leq C(1 + |\zeta|)^{N_1} e^{R|\text{Im} \zeta|}$. For the converse (“if”) implication of part (ii) use a contour integral $\xi + i\eta$ for fixed $\eta$ in the inverse Fourier transform formula to obtain

$$|u(x)| \leq C e^{-x \cdot \eta} e^{R|\text{Im} \eta|}$$

with a constant that is independent of $\eta$. Let $\eta = \lambda x$ and send $\lambda \to \infty$. For the converse implication (“if”) of part (i), mollify and use the converse part of (ii).

2. We start by recalling some definitions. A finite Borel regular measure in $\mathbb{R}^n$ is a positive measure such that the Borel sets are measurable, $\mu(\mathbb{R}^n) < \infty$, and for any measurable set $A$ $\mu(A) = \inf \{ \mu(U) \mid A \subset U, U \text{ open} \}$. The Fourier transform of a finite Borel measure is

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^n} e^{-i(x \cdot \xi)} d\mu(x)$$
We say that a finite regular Borel measure $\mu$ in $\mathbb{R}^n$ has an atom at $x$ if $\mu(\{x\}) > 0$. A measure without atoms is said to be continuous.

(i) Prove that the set of atoms of a finite regular Borel measure $\mu$ is at most countable and that the measure can be written as

$$\mu = \sum_j a_j \delta_{x_j} + \mu_C$$

where $x_j$ is the sequence of distinct atoms, $\mu_C$ is Borel regular, $a_j \geq 0$ and \( \sum_j a_j = A < \infty \).

(ii) Let $f \in L^\infty(\mathbb{R})$. Assume that

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T |f(t)|^2 dt = 0.$$ 

Then, for any $g \in L^\infty(\mathbb{R})$,

$$\lim_{T \to \infty} \left( \frac{1}{2T} \int_{-T}^T |(f + g)(t)|^2 dt - \frac{1}{2T} \int_{-T}^T |g(t)|^2 dt \right) = 0.$$ 

(iii) If $\mu = \sum_j a_j \delta_{x_j}$ is a countable sum of point masses with $x_j \in \mathbb{R}$, $a_j > 0$, $A = \sum a_j < \infty$ then

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T |\hat{\mu}(\xi)|^2 d\xi = \sum_j a_j^2.$$ 

(iv) If $\mu$ is a continuous finite regular Borel measure then

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T |\hat{\mu}(\xi)|^2 d\xi = 0.$$ 

2
(v) If $\mu = \sum_j a_j \delta_{x_j} + \mu_C$ is a finite regular Borel measure with atoms at $x_j$, $a_j \geq 0$, and with $\mu_C$ continuous, then

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T |\hat{\mu}(\xi)|^2 = \sum_j a_j^2$$

Hints. For (iii): use the fact that

$$\sum_{j \neq k} a_j a_k \frac{\sin((x_j - x_k)T)}{(x_j - x_k)T}$$

is uniformly (in $T$) small if $j$ or $k$ are large enough, and then take $T$ large enough to show that the rest of this sum is small as well. For (iv) prove that if $\mu$ is a continuous finite regular Borel measure, then for any $\epsilon > 0$ there exists $\delta > 0$ such that $\mu(B(x, \delta)) \leq \epsilon$ holds for all $x$. Use this to dominate the contribution near the diagonal $|x - y| \leq \delta$ in

$$\int \int \frac{\sin((x - y)T)}{(x - y)T} d\mu(x) d\mu(y)$$

For (v) use (ii), (iii), (iv).

3. (a) Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $1 \leq p < \infty$ and $m \geq 0$. Show that the dual $W_{0}^{m,p}(\Omega)'$ of $W_{0}^{m,p}(\Omega)$ (the closure of $D(\Omega)$ in $W^{m,p}(\Omega)$) is a space of distributions, i.e. for any $L \in W_{0}^{m,p}(\Omega)'$ there exists a unique $u \in D'(\Omega)$ so that $u(\phi) = L(\phi)$ for all $\phi \in D(\Omega)$. The dual is denoted $W^{-m,q}(\Omega)$, with $\frac{1}{q} - 1 + \frac{1}{p} = 1$. Prove that $f \in W^{-1,2}(\mathbb{R}^n)$ if and only if $(1 + |\xi|^2)^{-\frac{1}{2}} \hat{f}(\xi) \in L^2(\mathbb{R}^n)$ where $\hat{f}$ is the Fourier transform. (You need to justify that the Fourier transform is defined.)

(b) Let $\mu$ be a Radon measure on $K \subset \subset \Omega \subset \mathbb{R}^n$. ($K$ is compact, $\Omega$ is open). Radon measure means that $f \mapsto \int_K f d\mu$ is a continuous positive linear functional on $C(K)$, the Banach space of continuous functions. Show that $\mu$ defines a distribution in $D'(\Omega)$ naturally, by $\mu(\phi) = \int_K \phi d\mu$. (Here $\phi|_K$ is the restriction of $\phi \in D(\Omega)$ to $K$.)

Prove that $K = \{0\}$, $\mu = \delta$, $\Omega = \mathbb{R}^3$ is an example of $\mu$ that does not belong to $W^{-1,2}(\Omega)$.

(c) Let $K = S^2$ be the unit sphere and $\mu = dS$ be the surface measure. Show that $\mu \in W^{-1,2}(\mathbb{R}^3)$. 3
4. (a) Let $f_j$ be a sequence of functions in $L^1(\mathbb{R}^n)$ that is uniformly bounded, i.e. $\exists \, C > 0, \|f_j\|_{L^1} \leq C$, and is uniformly absolutely continuous i.e. $\forall \epsilon > 0, \exists \delta > 0, |A| \leq \delta \Rightarrow \int_A |f_j| \, dx \leq \epsilon$. Prove that there exists a subsequence $f_{j_k}$ and a function $f \in L^1(\mathbb{R}^n)$ such that $\lim_{k \to \infty} \int \phi(x) f_{j_k}(x) \, dx = \int \phi(x) f(x) \, dx$ holds for any continuous, compactly supported function $\phi$.

(b) Assume that $f \in L^1(\mathbb{R}^n)$. Consider the finite differences 

$$(\delta_{he_1} f)(x) = f(x_1 - h, x_2, \ldots, x_n) - f(x)$$

and assume that there exists a constant $C > 0$ such that

$$\|\delta_{he_1} f\|_{L^1} \leq C|h|$$

for all $|h| \leq 1$. Assume also that the family $h^{-1}\delta_{he_1} f$ is uniformly absolutely continuous: $\forall \epsilon > 0, \exists \delta > 0, |A| \leq \delta \Rightarrow \int_A |\delta_{he_1} f| \, dx \leq \epsilon|h|$. Show that $\partial_1 f \in L^1(\mathbb{R}^n)$.

(c) Verify that the condition $(1 + |\xi|) \hat{f}(\xi) \in L^1(\mathbb{R}^n)$ is a sufficient condition for the uniform absolute continuity of $h^{-1}\delta_h f$ in $L^1(\mathbb{R}^n)$.

5. Let $s$ be a real number, $s > \frac{n}{2}$, and consider the space $H^s(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n)$ of functions with $s$ derivatives in $L^2$, with norm $\|f\|_s$ defined by Fourier transform.

(a) Prove that $H^s(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \cap \mathcal{F}^{-1}(L^1(\mathbb{R}^n)) \subset L^\infty(\mathbb{R}^n)$ and the embeddings are continuous. Here $\mathcal{F}(f) = \hat{f}$ is the Fourier transform and

$$\mathcal{F}^{-1}(L^1(\mathbb{R}^n)) = \{ f \in S'(\mathbb{R}^n) \mid \hat{f} \in L^1(\mathbb{R}^n) \}$$

(Not to be confused it with the image of $L^1(\mathbb{R}^n)$ under the inverse Fourier transform!)

(b) Prove that there exists a constant so that

$$\|fg\|_s \leq C \left\{ \|\hat{f}\|_{L^1(\mathbb{R}^n)} \|g\|_s + \|\hat{g}\|_{L^1(\mathbb{R}^n)} \|f\|_s \right\}$$

(a) Prove that $H^s(\mathbb{R}^n)$ is a Banach algebra, i.e. there exists a constant $C$ so that $\|fg\|_s \leq C \|f\|_s \|g\|_s$ holds for all $f, g \in H^s(\mathbb{R}^n)$.

(d) Let $K \in S(\mathbb{R}^n)$ be a smooth, rapidly decaying function. Show that there exists $\epsilon > 0$ such that, if $\|f\|_s \leq \epsilon$, then the equation

$$u(x) + \int K(x - y) u^2(y) \, dy = f(x)$$
has always a solution in $H^s(\mathbb{R}^n)$.

**Hint:** Consider successive approximations,

$$u_{n+1}(x) = f(x) - \int K(x - y)u_n^2(y)dy.$$

**6.** Let the real valued functions $V_j : \mathbb{R}^n \to \mathbb{R}$ be smooth ($C^1$) and uniformly bounded,

$$\sup_{x \in \mathbb{R}^n} \sum_{j=1}^{n} |V_j(x)| \leq C < \infty.$$

Assume that $V(x) = (V_1(x), \ldots, V_n(x))$ is divergence-free,

$$0 = \nabla \cdot V = \sum_{j=1}^{n} \frac{\partial V_j}{\partial x_j}.$$

Consider the operator

$$A = -\Delta + \frac{1}{i} V \cdot \nabla$$

i.e.

$$Au(x) = -\Delta u(x) + \frac{1}{i} \sum_{j=1}^{n} V_j(x) \frac{\partial u}{\partial x_j}(x)$$

defined for $u \in C^\infty_0(\mathbb{R}^n)$. Show that $A$ is essentially selfadjoint as operator $A : C^\infty_0(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

**7.** Let $F : B \to \mathbb{R}$ be a function defined on the reflexive Banach space $B$.

We say that $F$ is weakly lower semicontinuous if from $u = w \lim_{n \to \infty} u_n$ it follows that

$$F(u) \leq \liminf_{n \to \infty} F(u_n)$$

Here $w \lim_{n \to \infty} u_n$ means the weak limit.

(a) Consider $B = L^4([0, 2\pi])$. Take $u_n = \sin(nx)$ and

$$F(u) = \int_0^{2\pi} (1 - |u(x)|^2)^2 dx.$$
Does $u_n$ converge weakly in $B$? Is $F$ continuous (strongly)? Is $F$ weakly lower semicontinuous?

(b) Is the norm in a reflexive Banach space weakly lower semicontinuous? Are continuous functions on reflexive Banach spaces weakly lower semicontinuous?

(c) Let $B = H^1_0(0, 2\pi)$. Consider the same functional $F$ as above,

$$F(u) = \int_0^{2\pi} (1 - |u(x)|^2)^2 dx.$$ 

Is $F$ weakly lower semicontinuous?