Homework 7 = Higher order elliptic regularity = due April 19

Functional Analysis

We say that the operator $P = P(x, \partial) = \sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha$ is (uniformly) elliptic in $\Omega \subset \mathbb{R}^n$ if there exists $\gamma > 0$ such that

$$\left| \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha \right| \geq \gamma |\xi|^m$$

holds for all $\xi \in \mathbb{R}^n$ and all $x \in \Omega$. Here $m \geq 2$ is an integer. You are going to prove interior regularity for solutions of $Pu = f$. We use the notation $\|u\|_s$ for the norm in $H^s$. We use Sobolev spaces $H^s_0(\Omega)$. We recall that these can be defined for all $s$ as the completion of $\mathcal{D}(\Omega)$ in the $\| \cdot \|_s$ norm. The norm is defined via Fourier transform.

1. Consider the constant coefficient case $P = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha$. Assume the operator is elliptic. Prove that there exists a constant $C_s$ (depending on $s$ and $\gamma$) such that

$$\|u\|_s \leq C_s (\|u\|_0 + \|Pu\|_{s-m})$$

holds for any $s \geq m$ and any $u \in H^s(\mathbb{R}^n)$.

**Hint:** Use Fourier transform and ellipticity.

2. Assume that the coefficients $a_\alpha$ are $C^\infty$ and $P$ is elliptic in $\Omega$. Let $x_0 \in \Omega$ and let $s \geq m$. There exists $\delta > 0$ and a constant $C_s$ such that if $u \in H^s_0(B(x_0, \delta))$ then

$$\|u\|_s \leq C_s (\|u\|_{s-1} + \|Pu\|_{s-m})$$

**Hint:** Use the previous result and the fact $\|(P(x, \partial) - P(x_0, \partial))u\|_{s-m}$ can be bound by $\epsilon \|u\|_s + C \|u\|_{s-1}$.

3. Let $s \geq m$. There exists a constant $C_s$ such that

$$\|u\|_s \leq C_s (\|u\|_{s-1} + \|Pu\|_{s-m})$$
holds for all \( u \in H^s_0(\Omega) \).

**Hint:** Partition of unity \( u = \sum \phi_j u \) to achieve small supports, and commutators

\[
[P, \phi_j]u = P(\phi_j u) - \phi_j Pu \in H_{s-m}
\]

controlled by \( \|u\|_{s-1} \).

4. Let \( s \geq m, t \leq s - 1 \). There exists a constant \( C_t \) such that

\[
\|u\|_s \leq C_t (\|u\|_t + \|Pu\|_{s-m})
\]

holds for all \( u \in H^s_0(\Omega) \).

**Hint:**

\[
\|u\|_{s-1} \leq \epsilon \|u\|_s + C\epsilon \|u\|_t
\]

5. Let \( \Omega \) be open, \( P \) elliptic of order \( m \) with \( C^\infty \) coefficients and \( s \geq m \). If \( u \in H^s_{loc}(\Omega) \) and \( Pu \in H^{s-m+1}_{loc}(\Omega) \) then \( u \in H^{s+1}_{loc}(\Omega) \).

**Hint:** Let \( \psi \in D(\Omega) \). Then \( \psi u \in H^s_0(\Omega) \) and \( \psi Pu \in H^{s-m+1}_0(\Omega) \). Because also the commutator \( P(\psi u) - \psi P(u) \in H^{s-m+1}_0(\Omega) \) it follows that \( P(\psi u) \in H^{s-m+1}_0(\Omega) \). Take finite difference quotients and show, using commutators that

\[
\frac{1}{h} \delta_h(\psi u) \|_s \leq C \left( \|P(\psi u)\|_{s-m+1} + \|\psi u\|_s \right)
\]

6. Let \( P \) be an uniformly elliptic operator of order \( m \) with \( C^\infty \) coefficients in the open set \( \Omega \subset \mathbb{R}^n \). Let \( f \in H^t_{loc}(\Omega) \), with \( t \geq 0 \). If \( u \in L^2_{loc}(\Omega) \) solves \( Pu = f \) in \( D'(\Omega) \) then \( u \in H^{t+m}_{loc}(\Omega) \).