We recall the real Hahn-Banach theorem: If $X$ a real vector space and $p : X \to \mathbb{R}_+$ satisfies

\begin{align*}
(i) \quad & p(x + y) \leq p(x) + p(y) \quad \forall \ x, y \in X, \\
(ii) \quad & p(tx) = tp(x) \quad \forall \ t > 0, x \in X,
\end{align*}

then any real linear map $L : F \to \mathbb{R}$, defined on a linear subspace $F$ of $X$ and satisfying

$$L(f) \leq p(f), \quad \forall \ f \in F$$

has a linear extension $\tilde{L} : X \to \mathbb{R}$ satisfying

$$\tilde{L}(x) \leq p(x) \quad \forall \ x \in X.$$
associated to $A$. Prove that $p_A$ exists, and satisfies the conditions of the real Hahn-Banach thm.

(c) Show that $p(x_0 + z) \geq 1$ for all $z \in \mathcal{V}/2$. (Use the fact that $A$ is starshaped i.e. $tA \subset A$ for any $t \in [0,1]$, and consequently that $p_A^{-1}([0,1)) \subset A$.)

(d) Let $\tilde{L}$ be a real linear extension to $X$ (which obviously is a real vector space as well) of the real linear map

$$L : \mathbb{R} x_0 \to \mathbb{R}, \quad L(tx_0) = tp_A(x_0)$$

which obeys

$$\tilde{L}(x) \leq p_A(x) \quad \forall x \in X.$$ 

Define $F : X \to \mathbb{C}$ by

$$F(x) = \tilde{L}(x) - i\tilde{L}(ix)$$

Show that $F$ is (complex) linear, that

$$Re(F(x)) \leq p_A(x) \quad \forall x \in X,$$

and

$$\sup_{x \in C} Re(F(x)) \leq 1.$$ 

(e) Prove that $F$ is continuous. Use, for instance, $K = co \left( \bigcup_{|z| \leq 1} zC \right)$, the convex hull of $\bigcup_{|z| \leq 1} zC$ and the associated Minkowski functional $p_B$ of the convex, balanced, absorbing open set

$$B = K + \frac{V}{2}.$$ 

(f) Prove that $Re(F(x_0)) > 1$. 

2
2. Let $C \subset X$, $X$ locally convex space. Assume that $C$ is convex and closed. Then it is weakly closed (i.e. it is closed in the locally convex topology on $X$ generated by $X^*$, the linear continuous functionals on $X$.)

3. Let $X$ be a Banach space and let $x_n$ be a sequence which converges weakly to $x$. Prove that there exists a sequence of convex combinations of the $x_n$, $y_n = \sum_{j=1}^{n} \alpha_{j}^{(n)} x_j$, $\sum_{j=1}^{n} \alpha_{j}^{(n)} = 1$, $\alpha_{j}^{(n)} \geq 0$ that converges in norm to $x$. 