Boundedness, Harnack inequality and Hölder continuity for weak solutions

Intoduction to PDE

We describe results for weak solutions of elliptic equations with bounded coefficients in divergence-form. The ideas of proofs come from DeGiorgi, Nash and Moser. References abound; we mostly use Gilbarg and Trudinger. We discuss equations

$$Lu = g + \partial_i f_i \tag{1}$$

where

$$Lu = -\partial_i(a_{ij}\partial_j u + b_i u) + c_j \partial_j u + du \tag{2}$$

We assume uniform ellipticity

$$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \tag{3}$$

with $\lambda > 0$ good for all x considered. We assume that $a_{ij} = a_{ji}$ are measurable and bounded,

$$\sum |a_{ij}(x)|^2 \leq \Lambda^2. \tag{4}$$

We assume that the coefficients b, c, d are bounded

$$\lambda^{-2}(\sum |b_i(x)|^2 + |c^i(x)|^2) + \lambda^{-1}|d(x)| \leq \Gamma \tag{5}$$

and that the right hand side $g \in L^{\frac{q}{2}}$, $f \in (L^q)^n$, for some $q > n$. Denoting by

$$A_i(x, z, p) = a_{ij}(x)p_j + b_i(x)z + f_i,$$

$$B(x, z, p) = c_j(x)p_j + d(x)z - g(x),$$

we say that u is a $W^{1,2}(\Omega)$ weak subsolution in Ω if

$$\int_\Omega (\partial_i v)A_i(x, u, \nabla u) + vB(x, u, \nabla u)dx \leq 0 \tag{6}$$

holds for all $v \in C_0^1(\Omega)$, $v \geq 0$. We say that u is a supersolution if the inequality is reversed

$$\int_\Omega (\partial_i v) A_i(x, u, \nabla u) + v B(x, u, \nabla u) dx \geq 0. \tag{7}$$

We will quote theorems in full generality but we will present the ideas of proofs only, and for that purpose we will take $b = c = d = f = g = 0$.

1 Local boundedness, Harnack inequality

We denote

$$k(R) = \lambda^{-1} \left(R^{1-\frac{n}{q}} \|f\|_{L^q} + R^{2(1-\frac{n}{q})} \|g\|_{L^{\frac{q}{2}}} \right)$$

Theorem 1. *If u is a $W^{1,2}(\Omega)$ subsolution of (1) then there exists a constant $C = C(n, \frac{\Lambda}{\lambda}, q, p, \Gamma R)$ such that, for all balls $B(y, 2R) \subset \Omega$, and $p > 1$ we have*

$$\sup_{B(y,R)} u \leq C \left(R^{-\frac{n}{p}} \|u^+\|_{L^p(B(y,2R))} + k(R) \right). \tag{8}$$

If u is a $W^{1,2}(\Omega)$ supersolution then

$$\sup_{B(y,R)} (-u) \leq C \left(R^{-\frac{n}{p}} \|u^-\|_{L^p(B(y,2R))} + k(R) \right). \tag{9}$$

The idea of proof is to use $v = \eta^2 u^\beta$ as test function, where η is a cutoff function, β is arbitrary, positive, and deduce bounds of the type

$$\int |\nabla u|^2 u^{\beta-1} \eta^2 dx \leq C \int |\nabla \eta|^2 u^{\beta+1} dx$$

This, together with a Sobolev embedding produces bounds for higher L^p norms on the left hand side, depending on lower L^p norms on the right hand side on larger domains. An iteration, due to Moser, finishes the proof. Unfortunately, because u^β is not an admissible test function we have to trim it first. We consider $k > 0$ a small positive constant, M a large positive constant. We take $\overline{u} = u^+ + k$ and set $\overline{u}_M = \overline{u}$ if $u < M$, $\overline{u}_M = M + k$ if $u \geq M$. We take now

$$v = \eta^2 (\overline{u}_M^{\beta-1} \overline{u} - k^\beta).$$

where η is a nonnegative smooth cutoff function and $\beta > 0$. We take without loss of generality $y = 0$, $R = 4$ and $\eta \in C_0^1(B_4)$. We denote $B_R = B(0,R)$. Now v is a legitimate test function and

$$\nabla v = \eta^2 \overline{u}_M^{\beta-1}[(\beta-1)\nabla\overline{u}_M + \nabla\overline{u}] + 2\eta\nabla\eta(\overline{u}_M^{\beta-1}\overline{u} - k^\beta)$$

Here we used the fact that $\overline{u} = \overline{u}_M$ when the gradient of the latter is nonzero. Later we will use also that $\nabla\overline{u} = \nabla\overline{u}_M$ when the latter is nonzero. Because u is a subsolution we obtain (remember, $b = c = d = f = g = 0$ in our proof)

$$\int \eta^2 \overline{u}_M^{\beta-1}\left[(\beta-1)|\nabla\overline{u}_M|^2 + |\nabla\overline{u}|^2\right]dx \le C\int |\nabla\eta|\eta(|\nabla\overline{u}|\overline{u}_M^{\beta-1}\overline{u})dx.$$

Here we used the fact that $\overline{u}_M^{\beta-1}\overline{u} - k^\beta \ge 0$ to drop the k^β term in the right hand side. After hiding the term involving $\nabla\overline{u}$ from the right hand side in the left hand side, we have (new constant C!)

$$\int \eta^2 \overline{u}_M^{\beta-1}\left[(\beta-1)|\nabla\overline{u}_M|^2 + |\nabla\overline{u}|^2\right]dx \le C\int |\nabla\eta|^2\overline{u}^2\overline{u}_M^{\beta-1}dx$$

We let now $w = \overline{u}_M^{\frac{\beta-1}{2}}\overline{u}$. The inequality above implies

$$\int \eta^2|\nabla w|^2 dx \le C(\beta+1)\int |\nabla\eta|^2 w^2 dx$$

In view of the fact that $\nabla(\eta w) = \eta\nabla w + w\nabla\eta$ and the estimate above we have, using the Sobolev embedding,

$$\left(\int (\eta w)^{\frac{2n}{n-2}}dx\right)^{\frac{n-2}{n}} \le C(\beta+1)\int |\nabla\eta|^2 w^2 dx$$

if $n > 2$. (If $n = 2$ we replace $\frac{2n}{n-2}$ in the left hand side by any $q < \infty$). We choose appropriately now the test function η so that, for balls $B_r \subset B_R$ we obtain

$$\left(\int_{B_r} w^{\frac{2n}{n-2}}dx\right)^{\frac{n-2}{n}} \le C(\beta+1)\left(\frac{1}{R-r}\right)^2\int_{B_R}\overline{u}^{\beta+1}dx$$

Writing $q = \beta + 1$, we have

$$\left(\int_{B_r} \overline{u}_M^{\frac{nq}{n-2}}dx\right)^{\frac{n-2}{n}} \le Cq\left(\frac{1}{R-r}\right)^2\int_{B_R}\overline{u}^q dx$$

for any $q > 1$ and $r < R$. Letting $M \to \infty$ and then $k \to 0$ we obtain

$$\|u^+\|_{L^{\frac{nq}{n-2}}(B_r)} \le \left(C \frac{q}{(R-r)^2} \right)^{\frac{1}{q}} \|u^+\|_{L^q(B_R)}$$

Denoting the amplification factor by $a = \frac{n}{n-2} > 1$, and choosing for $i = 0, 1, \ldots$, $q_i = pa^i$, $r_i = 2 + 2^{1-i}$, we obtain

$$\|u^+\|_{L^{q_{i+1}}(B_{r_{i+1}})} \le C^{\frac{i}{a^i}} \|u^+\|_{L^{q_i}(B_{r_i})}$$

and so

$$\|u^+\|_{L^{q_i}(B_{r_i})} \le C^{\sum \frac{i}{a^i}} \|u^+\|_{L^p(B_4)}$$

and thus

$$\|u^+\|_{L^\infty(B_2)} \le C \|u^+\|_{L^p(B_4)}$$

proving the theorem for subsolutions. The same idea of proof works for supersolutions. Now we prove a lemma, a weak Harnack inequality.

Theorem 2. *If u is a supersolution of (1) that is nonnegative in a ball $B(y, 4R) \subset \Omega$ then*

$$R^{-\frac{n}{p}} \|u\|_{L^p(B(y,2R))} \le C \left(\inf_{B(y,R)} u + k(R) \right)$$

holds for $1 \le p < \frac{n}{n-2}$ with $C = C(n, \frac{\Lambda}{\lambda}, q, p, R\Gamma)$.

The idea of proof is similar to the one for L^∞ bounds except we are going to use negative powers, and a result of John-Nirenberg, of independent interest. We start by assuming without loss of generality that $y = 0$, $R = 1$, and so on. Also, we may assume u is bounded (either by the previous result, or by a trimming procedure like in the proof above). We set

$$v = \eta^2 \bar{u}^\beta$$

with $\eta \in C_0^1(B_4)$ a nonnegative cutoff function,

$$\bar{u} = u + k$$

with $k > 0$ and $\beta \neq 0$. Using the fact that u is a supersolution we have, after hiding one term,

$$\int \eta^2 \bar{u}^{\beta-1} |\nabla u|^2 dx \le C(|\beta|) \int |\nabla \eta|^2 \bar{u}^{\beta+1} dx$$

The constant $C(|\beta|)$ is bounded when $|\beta| > \epsilon > 0$. We set $w = \bar{u}^{\frac{\beta+1}{2}}$ if $\beta \neq -1$ and $w = \log \bar{u}$ if $\beta = -1$. Letting $\gamma = \beta + 1$ we have

$$\int |\eta \nabla w|^2 \leq C(|\beta|)\gamma^2 \int |\nabla \eta|^2 w^2 dx$$

if $\beta \neq -1$, and

$$\int |\eta \nabla w|^2 \leq C \int |\nabla \eta|^2 dx \quad (*)$$

if $\beta = -1$. We will refer to this inequality at the end of the proof. Thus, for $n > 2$, from the Sobolev inequality we obtain the inequality

$$\|\eta w\|_{L^{\frac{2n}{n-2}}} \leq C\left(1 + |\gamma|\right) \||\nabla \eta| w\|_{L^2}$$

Choosing η like before, we have

$$\|w\|_{L^{2a}(B_{r_1})} \leq C(1 + |\gamma|)(r_2 - r_1)^{-1} \|w\|_{L^2(B_{r_2})}$$

with $a = \frac{n}{n-2}$ as before, and $r_2 > r_1$. Denote

$$\Phi(p, r) = \left(\int_{B_r} |\bar{u}|^p dx \right)^{\frac{1}{p}}$$

We have the inequalities

$$\begin{aligned} \Phi(a\gamma, r_1) &\leq \left(\tfrac{C(1+|\gamma|)}{r_2 - r_1}\right)^{\frac{2}{|\gamma|}} \Phi(\gamma, r_2), \quad \text{if } \gamma > 0 \\ \Phi(\gamma, r_2) &\leq \left(\tfrac{C(1+|\gamma|)}{r_2 - r_1}\right)^{\frac{2}{|\gamma|}} \Phi(a\gamma, r_1), \quad \text{if } \gamma < 0. \end{aligned}$$

We take any $0 < p_0 < p < a$ and we have, with $r_m = 1 + 2^{-m}$, $m = 0, \ldots,$ and appropriate $\gamma_m < 0$,

$$\Phi(-p_0, 3) \leq C\Phi(-\infty, 1)$$

We can check that $\Phi(-\infty, r) = \inf_{B_r} \bar{u}$. We also can prove one step, using $\gamma > 0$,

$$\Phi(p, 2) \leq C\Phi(p_0, 3)$$

We would be therefore done if we could find some $p_0 > 0$ such that

$$\Phi(p_0, 3) \leq C\Phi(-p_0, 3)$$

This follows from a result of F. John and Nirenberg.

Theorem 3. *(F. John-L. Nirenberg) Let $u \in W^{1,1}(U)$ where U is convex. Suppose that there exists a constant K so that*

$$\int_{U \cap B_r} |\nabla u| dx \leq K r^{n-1}$$

holds for all balls B_r. Then there exists $\sigma_0 > 0$ and C depending only on n such that

$$\int_\Omega \exp\left(\frac{\sigma}{K}|u - u_U|\right) dx \leq C(\text{diam } U)^n$$

holds with $\sigma = \sigma_0 |U| (\text{diam } U)^{-n}$ and $u_U = \fint_U u$.

Let us note that, from the inequality $(*)$, we obtain, via Schwartz

$$\int_{B_r} |\nabla w| dx \leq C r^{\frac{n}{2}} \left(\int_{B_r} |\nabla w|^2 dx \right)^{\frac{1}{2}} \leq C r^{n-1}$$

Therefore, by the theorem of John-Nirenberg, there exists a constant $p_0 > 0$ such that

$$\int_{B_3} \exp\left(p_0 |w - w_3|\right) dx \leq C$$

where $w_3 = \fint_{B_3} w dx$. Therefore

$$\left(\int_{B_3} e^{p_0 w} dx\right)\left(\int_{B_3} e^{-p_0 w} dx\right) \leq C e^{p_o w_3} e^{-p_0 w_3} = C$$

This concludes the proof of the weak Harnack inequality, modulo the John-Nirenberg result. Putting together Theorem 1 and Theorem 2 we have the full Harnack inequality for weak solutions:

Theorem 4. *If u is a weak $W^{1,2}(\Omega)$ solution with $u \geq 0$ then there exists a constant C so that*

$$\sup_{B(y,R)} u \leq C \inf_{B(y,R)} u$$

holds for any $y \in \Omega$ so that $B(y, 4R) \subset \Omega$. Moreover, for any $U \subset\subset \Omega$ there exists a constant, depending on U and Ω so that

$$\sup_U u \leq C \inf_U u$$

2 Hölder continuity

The weak Harnack inequality is sufficient to prove the Hölder continuity of weak solutions.

Theorem 5. *Let u be a weak $W^{1,2}(\Omega)$ solution of (1). Then u is locally Hölder continuous in Ω. For every ball $B_0 = B(y, R_0) \subset \Omega$ there exists a constant $C = C(n, \frac{\Lambda}{\lambda}, \Gamma, q, R_0)$ such that*

$$\mathrm{osc}_{B(y,R)} u \leq C R^{\alpha} (R_0^{-\alpha} \sup_{B_0} |u| + k)$$

where $\alpha = \alpha(n, \frac{\Lambda}{\lambda}, \Gamma R_0, q) > 0$ and $k = \lambda^{-1}(\|f\|_{L^q} + \|g\|_{L^{\frac{q}{2}}})$.

Moreover, for every $U \subset\subset \Omega$, there exists $\alpha = \alpha(n, \frac{\Lambda}{\lambda}, \Gamma d) > 0$ where $d = dist(U, \partial\Omega)$, so that

$$\|u\|_{C^{\alpha}(U)} \leq C(\|u\|_{L^2(\Omega)} + k)$$

We provide the proof for the case $b = c = d = f = g = 0$. Let $M_0 = \sup_{B_0} |u|$, $M_4 = \sup_{B(y,4R)} u$, $m_4 = \inf_{B(y,4R)} u$, $M_1 = \sup_{B(y,R)} u$, $m_1 = \inf_{B(y,R)} u$. We have

$$L(M_4 - u) = 0, \quad L(u - m_4) = 0$$

so, we can apply the weak Harnack inequality of Theorem 2 with $p = 1$. We obtain

$$R^{-n} \int_{B(y,2R)} (M_4 - u) dx \leq C(M_4 - M_1)$$

and

$$R^{-n} \int_{B(y,2R)} (u - m_4) \leq C(m_1 - m_4)$$

Adding, we obtain

$$(M_4 - m_4) \leq C[M_4 - m_4 - (M_1 - m_1)]$$

and so

$$M_1 - m_1 \leq \gamma (M_4 - m_4)$$

with $\gamma = \frac{C-1}{C} < 1$. We have thus, for $\omega(R) = \mathrm{osc}_{B(y,R)} u$

$$\omega(R) \leq \gamma \omega(4R)$$

It follows by iteration (exercise!) that

$$\omega(R) \leq C R^{\alpha} \omega(R_0)$$

3 Riesz potentials and the John-Nirenberg inequality

We use the notation of Gilbarg and Trudinger

$$(V_\mu f)(x) = \int_\Omega |x-y|^{n(\mu-1)} f(y)dy$$

with $\mu \in (0,1]$. This is the same as the Riesz potential of order $n\mu$ of $f\chi_\Omega$, where Ω is a bounded open set and χ_Ω its characteristic (indicator) function. Note that

$$V_\mu 1 \le \mu^{-1}\omega_n^{1-n}|\Omega|^\mu$$

Indeed, choosing R so that $|\Omega| = \omega_n R^n$, we have

$$(V_\mu 1)(x) = \int_\Omega |x-y|^{n(\mu-1)}dy \le \int_{B(x,R)} |x-y|^{n(\mu-1)}dy$$

because $|\Omega \setminus B(x,R)| = |B(x,R)\setminus\Omega|$ and the points in $\Omega\setminus B(x,R)$ are farther away and hence have smaller potential the points in $B(x,R)\setminus\Omega$.

Lemma 1. *The operator V_μ maps $L^p(\Omega)$ continuously into $L^q(\Omega)$, $1 \le q \le \infty$*

$$0 \le \delta = \delta(p,q) = p^{-1} - q^{-1} < \mu$$

Moreover

$$\|V_\mu f\|_{L^q} \le \left(\frac{1-\delta}{\mu-\delta}\right)^{1-\delta} \omega_n^{1-\mu}|\Omega|^{\mu-\delta}\|f\|_{L^p}$$

The proof: choose r so that

$$r^{-1} = 1 + q^{-1} - p^{-1} = 1-\delta$$

Then $h(x-y) = |x-y|^{n(\mu-1)}$ is in $L^r(\Omega)$ and, using the same trick as above

$$\|h\|_{L^r} \le \left(\frac{1-\delta}{\mu-\delta}\right)^{1-\delta} \omega_n^{1-\mu}|\Omega|^{\mu-\delta}$$

Now we write

$$h|f| = h^{\frac{r}{q}} h^{r(1-p^{-1})}|f|^{\frac{p}{q}}|f|^{p\delta}$$

and using Hölder we get

$$|V_\mu(x)| \leq \left[\int_\Omega h^r(x-y)|f(y)|^p dy\right]^{\frac{1}{q}} \left[\int_\Omega h^r(x-y)dy\right]^{1-p^{-1}} \|f\|_{L^p}^{p\delta}$$

But $\sup_{x\in\Omega}\left[\int_\Omega h^r(x-y)dy\right]^{\frac{1}{r}}$ is finite and raising the inequality above to power q and integrating we obtain the desired result.

Lemma 2. *Let Ω be convex and $u \in W^{1,1}(\Omega)$. Let S be any measurable set. Then*

$$|u(x) - u_S| \leq \frac{d^n}{n|S|}\int_\Omega |x-y|^{1-n}|\nabla u(y)|dy$$

a.e. in Ω, where

$$u_S = \frac{1}{|S|}\int_S u dy$$

and $d = \operatorname{diam}\Omega$.

It is enough to establish the inequality for $C^1(\Omega)$ functions. Then

$$u(x) - u(y) = -\int_0^{|x-y|} \partial_r u(x + r\omega)dr$$

where $\omega = \frac{y-x}{|x-y|}$ and $\partial_r = \omega\cdot\nabla$. Integrating in y over S:

$$|S|(u(x) - u_S) = -\int_S dy \int_0^{|x-y|} \partial_r u(x + r\omega)dr$$

We define

$$V(x) = \begin{cases} |\partial_r u(x)|, & \text{if } x \in \Omega \\ 0, & \text{if } x \notin \Omega \end{cases}$$

and thus we have

$$\begin{aligned} |u(x) - u_S| &\leq \tfrac{1}{|S|}\textstyle\int_{|x-y|\leq d} dy \int_0^\infty V(x+r\omega)dr \\ &= \tfrac{1}{|S|}\textstyle\int_0^\infty dr \int_{|\omega|=1} d\omega \int_0^d V(x+r\omega)\rho^{n-1}d\rho \\ &= \tfrac{d^n}{n|S|}\textstyle\int_0^\infty \int_{|\omega|=1} V(x+r\omega)drd\omega \\ &= \tfrac{d^n}{n|S|}\textstyle\int_\Omega |x-y|^{1-n}|\partial_r u(y)|dy \end{aligned}$$

We introduce now Morrey spaces: We say that $f \in M^p(\Omega)$ if there exists a constant K so that

$$\int_{B_r\cap\Omega} |f|dx \leq K r^{n(1-\frac{1}{p})}$$

holds for all balls $B_r = B(x_0, r)$. The norm $\|f\|_{M^p(\Omega)}$ is the smallest such constant K.

Lemma 3. *Let $f \in M^p(\Omega)$, and $\delta = p^{-1} < \mu$. Then*

$$|V_\mu f(x)| \leq \frac{1-\delta}{\mu-\delta} \left(\operatorname{diam} \Omega\right)^{n(\mu-\delta)} \|f\|_{M^p(\Omega)}$$

Proof. We extend f by zero outside Ω and denote

$$m(r) = \int_{B(x,r)} |f| dy$$

Then,

$$\begin{aligned}|V_\mu f(x)| &\leq \textstyle\int_\Omega r^{n(\mu-1)}|f(y)|dy, \quad r = |x-y|, \\ &= \textstyle\int_0^d r^{n(\mu-1)} m'(r)dr, \quad d = \operatorname{diam}\Omega \\ &= d^{n(\mu-1)}m(d) + n(1-\mu)\textstyle\int_0^d r^{n(\mu-1)-1}m(r)dr \\ &\leq C\tfrac{1-\delta}{\mu-\delta} d^{n(\mu-\delta)}\end{aligned}$$

We note here a generalization of Morrey's inequality

Proposition 1. *Let $u \in W^{1,1}(\Omega)$ and assume that there exist $K > 0$ and $0 < \alpha \leq 1$ so that*

$$\int_{B_r} |\nabla u| dx \leq K r^{n-1+\alpha}$$

for all balls $B_r \subset \Omega$. Then $u \in C^\alpha(\Omega)$ and

$$\operatorname{osc}_{B_r} u \leq C K r^\alpha$$

The proof is a direct application of Lemma 2 with $S = \Omega = B_r$ and Lemma 3 with $\Omega = B_r$.

Lemma 4. *Let $f \in M^p(\Omega)$ with $p > 1$ and let $g = V_\mu f$ with $\mu = p^{-1}$. Then there exist constants c_1, c_2 depending only on n and p so that*

$$\int_\Omega \exp\left(\frac{|g|}{c_1 K}\right) dx \leq c_2 \left(\operatorname{diam} \Omega\right)^n$$

where $K = \|f\|_{M^p(\Omega)}$.

Proof: we write, for $q \geq 1$:

$$|x-y|^{n(\mu-1)} = |x-y|^{(\frac{\mu}{q}-1)\frac{n}{q}} |x-y|^{n(1-\frac{1}{q})(\frac{\mu}{q}+\mu-1)}$$

and by Hölder

$$|g(x)| \leq \left(V_{\frac{\mu}{q}}|f|\right)^{\frac{1}{q}} \left(V_{\mu+\frac{\mu}{q}}|f|\right)^{1-\frac{1}{q}}$$

By Lemma 3

$$\begin{aligned} V_{\mu+\frac{\mu}{q}}|f| &\leq \tfrac{(1-\mu)q}{\mu} d^{\frac{n}{pq}} K, \quad d = \operatorname{diam}\Omega \\ &\leq (p-1)q d^{\frac{n}{pq}} K \end{aligned}$$

and by Lemma 1

$$\begin{aligned} \textstyle\int_\Omega V_{\frac{\mu}{q}}|f|dx &\leq pq\omega_n^{1-\frac{1}{pq}} |\Omega|^{\frac{1}{pq}} \|f\|_{L^1} \\ &\leq pq\omega_n K d^{n(1-\frac{1}{p}+\frac{1}{pq})} \end{aligned}$$

Therefore

$$\begin{aligned} \textstyle\int_\Omega |g|^q dx &\leq p(p-1)^{q-1}\omega_n q^q d^n K^q \\ &\leq p'\omega_n\{(p-1)qK\}^q d^n \end{aligned}$$

where $p' = \frac{p}{p-1}$. Choosing $c_1 > e(p-1)$ and summing, we have

$$\begin{aligned} \textstyle\int_\Omega \sum \frac{|g|^m}{m!(c_1K)^m} dx &\leq p'\omega_n d^n \sum \left(\tfrac{p-1}{c_1}\right)^m \tfrac{m^m}{m!} \\ &\leq c_2 d^n \end{aligned}$$

Combining Lemma 2 and Lemma 4 we proved Theorem 3.