

Degree Theory

1 Degree theory in finite dimensions

This is adapted from [1] Recall the local inverse thm:

Theorem 1. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^1 function and assume that $\nabla f(x_0)$ is invertible. Then there exists an open neighborhood U of x_0 and an open neighborhood V of $f(x_0)$ such that the inverse function $f^{-1} : V \rightarrow U$ exists and belongs to C^1 .*

Let X, Y be open, paracompact (separable, all covers admit a locally finite refinement) smooth manifolds of dimensions n and k , $n \geq k$ respectively. Let f be a map $f : X \rightarrow Y$ which is C^{n-k+1} .

Definition 1. *A point $x_0 \in X$ is a regular point for f if $(\nabla f(x_0))$ has maximal rank k . A point which is not regular is called a critical point. A point $y \in Y$ is a critical value if the preimage $f^{-1}(\{y\})$ contains a critical point. Otherwise, y is called a regular value.*

Theorem 2. *Sard's theorem: If $f \in C^{n-k+1}$, $f : X \rightarrow Y$ like above, then the set of its critical values has measure zero in Y .*

Proof. “Measure zero” in Y is well defined in a chart. We only give the proof for $n = k$. Enough to prove when X is a closed cube with sides parallel to the axes in \mathbb{R}^n and with side of size L . We subdivide the cube in small cubes of side $\frac{L}{N}$, with sides parallel to the axes. If x and x_0 belong to the same small cube Q then

$$f(x) = f(x_0) + (\nabla f(x_0))(x - x_0) + o\left(\frac{L}{N}\right)$$

because the first derivatives of f are continuous in X . If the point x_0 is critical for f then $\det(\nabla f(x_0)) = 0$, and therefore the image of Q lies in a

cylinder of base in a plane of dimension $n - 1$ and base area $\leq C \left(\frac{L}{N}\right)^{n-1}$ and height $o\left(\frac{L}{N}\right)$. As there are at most N^n cubes containing critical points, their image under f is contained in a set whose volume is of the order $No\left(\frac{1}{N}\right)$. This converges to zero as $N \rightarrow \infty$.

Now we recall some notation from differential geometry. If we have an n form in \mathbb{R}^n it is given locally by

$$\mu = f dy$$

where $dy = dy^1 \wedge \cdots \wedge dy^n$ is the volume form and f is a real valued function. The pull back of μ under a change of variables ϕ is

$$\phi^*(\mu) = (f \circ \phi)(x) \det J_\phi(x) dx$$

where J_ϕ is the Jacobian of ϕ . By the change of variables formula

$$\int_Y \mu = \text{sgn } J_\phi \int_X \phi^* \mu \quad (1)$$

Let $X \subset X_0$ where X_0 is a smooth paracompact manifold of dimension n and X is an open subset with compact closure $\bar{X} = X \cup \partial X$ in X_0 . Let $\phi : \bar{X} \rightarrow Y$ be a continuous map which is C^1 in X , to the smooth n dimensional paracompact manifold Y . Let $y_0 \in Y \setminus \phi(\partial X)$. From the local inverse theorem, the preimage

$$\phi^{-1}(\{y_0\}) = \{x \in \bar{X} \mid \phi(x) = y_0\}$$

is a discrete set (consists of isolated points). Because \bar{X} is compact, this set is finite.

Definition 2. *If y_0 is a regular value of ϕ then*

$$d(y_0) = \sum_{j=1}^k \text{sgn } J_\phi(x_j) \quad (2)$$

where

$$\phi^{-1}(\{y_0\}) = \{x_1, \dots, x_k\}$$

We say that a coordinate patch Ω of a point $y_0 \in Y$ is “nice” if there are suitable coordinates $g : \Omega \rightarrow \mathbb{R}^n$ so that $g(\Omega)$ is a cube.

Definition 3. Let $\mu = f(y)dy$ be a smooth n form on Y with support contained in a nice coordinate patch Ω of $y_0 \in Y$, with $\Omega \subset Y \setminus \phi(\partial X)$ and $\int_Y \mu = 1$. Then we set

$$\deg(\phi, X, y_0) = \int_X \phi^* \mu \quad (3)$$

Differential forms of the kind above will be called “admissible”. The fact that $\deg(\phi, X, y_0)$ is well defined is a consequence of the following lemma.

Lemma 1. Let $\mu = f(y)dy$ be a smooth form on Y with $\int_Y \mu = 0$ and with $\text{supp } \mu$ contained in a nice coordinate patch Ω . Then there exists an $n-1$ -form ω whose support is included in Ω and such that $\mu = d\omega$.

Indeed, given the lemma, if ν and μ are admissible for y_0 and ϕ in X then, because $\nu - \mu = d\omega$ and because $\phi^*(\nu - \mu) = \phi^*(d\omega) = d\phi^*\omega$, the integrals of $\phi^*\nu$ and $\phi^*\mu$ are equal by Green’s theorem

$$\int_X d(\phi^*\omega) = 0.$$

Proof of Lemma 1 Without loss of generality we may assume that the support of μ is included in a cube Q . We must show that we can find g_j supported in Q such that

$$f = \sum_{j=1}^n \partial_j g_j$$

The proof is by induction. If $n = 1$, then $g_1 = \int_{-\infty}^y f(z)dz$ satisfies $dg_1 = fdy$. Now suppose the lemma is true in n dimensions. Let $y^{n+1} = t$, $(y, t) = (y^1, \dots, y^n, t)$ and set

$$m(y) = \int_{-\infty}^{\infty} f(y, t)dt.$$

Now $\int m(y)dy = 0$, so, by induction, there exist g_1, \dots, g_n such that

$$m(y) = \sum_{j=1}^n \partial_j g_j(y)$$

and g_j are supported in the projection of the cube. Let $\tau(t)$ be a smooth function supported on the corresponding side of the cube, with

$$\int_{-\infty}^{\infty} \tau(t)dt = 1.$$

Consider $f(y, t) - \tau(t)\mu(y)$. Because its integral in t vanishes,

$$g(y, t) = \int_{-\infty}^t (f(y, s) - \tau(s)m(y))ds$$

has support in Q and obeys

$$\partial_t g(y, t) = f(y, t) - \tau(t)m(y).$$

Thus

$$f(y, y^{n+1}) = \partial_{n+1}g(y, y^{n+1}) + \sum_{j=1}^n \partial_j(g_j(y)\tau(y^{n+1}))$$

which finishes the proof.

1.1 Properties of the degree

Proposition 1. *For y_1 close to y_0 ,*

$$\deg(\phi, X, y_0) = \deg(\phi, X, y_1).$$

Proof. Indeed, if μ is admissible for ϕ in X for y_0 , it is also admissible for ϕ in X for y_1 . Because the degree is an integer, it is locally constant and therefore is constant on connected components of $Y \setminus \phi(\partial X)$.

Proposition 2. *If y_0 is a regular point for ϕ then*

$$\deg(\phi, X, y_0) = d(y_0)$$

Proof. There are disjoint neighborhoods V_j of x_j , the points which comprise $\phi^{-1}(\{y_0\})$, such that ϕ is one-to-one on them. Then if $N = \cap_{j=1}^k \phi(V_j)$, then N is a neighborhood of y_0 , and if μ is admissible with support in N then

$$\begin{aligned} \deg(\phi, X, y_0) &= \int \phi^* \mu = \sum_{j=1}^k \int_{V_j} \phi^* \mu = \sum_{j=1}^k \operatorname{sgn} J_\phi(x_j) \int_{\phi(V_j)} \mu \\ &= \sum_{j=1}^k \operatorname{sgn} J_\phi(x_j) \int_Y \mu = \sum_{j=1}^k \operatorname{sgn} J_\phi(x_j) = d(y_0). \end{aligned}$$

It follows that $\deg(\phi, X, y_0)$ is an integer equal to $d(y)$ for any regular value y belonging to the same connected component of $Y \setminus \phi(\partial X)$ as y_0 .

Proposition 3. *Homotopy invariance. Consider a one parameter family of maps $\phi_t : \bar{X} \rightarrow Y$, continuous on $\bar{X} \times [0, 1]$ and with $\phi_t \in C^1(X)$ for each $t \in [0, 1]$. Assume that $y_0 \notin \phi_t(\partial X)$ holds for each $t \in [0, 1]$. Then $\deg(\phi_t, X, y_0)$ does not depend on t .*

Proof. We take a small neighborhood of y_0 which avoids the compact set $\phi(\partial X \times [0, 1])$. Let μ be admissible for all ϕ_t , $t \in [0, 1]$ and y_0 in X . Then

$$\deg(\phi_t, X, y_0) = \int \phi_t^*(\mu)$$

is continuous and integer valued, so it is constant.

We can generalize this by allowing y_0 to depend continuously on t and having a relatively open set $A \subset X \times [0, 1]$ with compact closure. If y_t does not belong to $\phi_t((\partial A)_t)$ where $A_t = \{x \in X; (x, t) \in A\}$ and $(\partial A)_t = \{x \in X; (x, t) \in \partial A\}$, then $\deg(\phi_t, A_t, y_t)$ is constant.

Proposition 4. *Let X_i be a sequence of disjoint open sets contained in the interior of X . Let $y_0 \notin \phi(\bar{X} \setminus \cup_i X_i)$. Then $\deg(\phi, X_i, y_0) = 0$ for all but finitely many i , and*

$$\deg(\phi, X, y_0) = \sum_i \deg(\phi, X_i, y_0).$$

Proof. Let N be an open neighborhood of y_0 not intersecting $\phi(\bar{X} \setminus \cup_i X_i)$ (because the latter is compact, hence closed). Then we take a regular value $y \in N$. The degrees are computed at y , and y has a finite number of preimages. A particular case is

Proposition 5. *Excision. Let $K \subset \bar{X}$ be closed. If $y_0 \notin \phi(K) \cup \phi(\partial X)$ then*

$$\deg(\phi, X, y_0) = \deg(\phi, X \setminus K, y_0).$$

Proof. We apply the previous proposition with $X_1 = X \setminus K$.

Proposition 6. *Let X, Y be manifolds of dimension n and X', Y' of dimension m and $\phi : X \rightarrow Y$ and $\phi' : X' \rightarrow Y'$ be such that the degrees are defined at y and y' respectively. Then*

$$\deg(\phi \times \phi', X \times X', (y, y')) = \deg(\phi, X, y) \times \deg(\phi', X', y')$$

Proof. If μ and μ' are admissible for ϕ and ϕ' and y and y' then $\mu \times \mu'$ is admissible for $\phi \times \phi'$ and (y, y') at $X \times X'$ and

$$\int (\phi \times \phi')^*(\mu \times \mu') = \int \phi^* \mu \cdot \int \phi'^* \mu'$$

A few remarks about the degree. First, if the map ϕ is one-to-one and preserving the orientation and if $y_0 \in \phi(X) \cap (Y \setminus \phi(\partial X))$ then $\deg(\phi, X, y_0) = 1$. If $y_0 \notin \phi(\bar{X})$ then $\deg(\phi, X, y_0) = 0$. If $\partial X = \emptyset$, X is compact and Y is connected and not compact, then the degree vanishes at any $y \in Y$.

Extension to continuous maps. If $\phi_n \rightarrow \phi$ uniformly in \bar{X} , then for large enough n , the degrees $\deg(\phi_n, X, y_0)$ are independent of n . Indeed, the property $y_0 \notin \phi(\partial X)$ implies that there exists a neighborhood N of y_0 such that $\phi_n(\partial X) \cap N = \emptyset$ for large enough n . If $\text{dist}(\phi_i(\partial X), y_0) \geq \delta > 0$, $i = 1, 2$, then $(1-t)\phi_1(x) + t\phi_2(x) = \phi_1(x) + t\psi(x)$ with $\psi(x)$ uniformly small on ∂X , and therefore the homotopy cannot touch ∂X . Note that the convergence in C^0 does not imply continuity of the degree, but the homotopy invariance does. Note also that the degree depends only on values of ϕ on ∂X : all continuous extensions of ϕ to the whole \bar{X} have the same degree. (same proof: if we have two continuous extensions, then the homotopy described above does not touch the boundary).

Theorem 3. *Let $\phi : X \rightarrow Y$, $\phi \in C(\bar{X})$. Let Ω be a connected component of $Y \setminus \phi(\partial X)$ and μ a smooth n -form in Y with compact support in Ω and with $\int_Y \mu \neq 0$. Then*

$$\deg(\phi, X, \Omega) = \frac{\int_X \phi^* \mu}{\int_Y \mu}$$

The proof follows by establishing the relation first for measures supported in nice coordinate patches, then using a partition of unity, cross multiplying (using Lemma 1) and summing.

Theorem 4. *Let $\phi : \bar{X} \rightarrow Y$, $\psi : Y \rightarrow Z$ be continuous. Let Ω_i be the connected components of $Y \setminus \phi(\partial X)$ having compact closure in Y . Then, for $z \notin \psi \circ \phi(\partial X)$ we have*

$$\deg(\psi \circ \phi, X, z) = \sum_i \deg(\phi, X, \Omega_i) \deg(\psi, \Omega_i, z)$$

and the sum on the right hand side is finite.

Proof. WLOG: $\phi, \psi \in C^1$ and z is a regular value for both $\psi \circ \phi$ and for ψ . Then,

$$\begin{aligned} \deg(\psi \circ \phi, X, z) &= \sum_{\psi \circ \phi(x)=z} \text{sign } J_{\psi \circ \phi}(x) \\ &= \sum_{\psi(\phi(x))=z} \text{sign } J_{\psi(\phi(x))} \text{sign } J_{\phi(x)} \\ &= \sum_{\psi(y)=z} \text{sign } J_{\psi(y)} \sum_{\phi(x)=y} \text{sign } J_{\phi}(x) \\ &= \sum_{\psi(y)=z} \text{sign } J_{\psi(y)} \deg(\phi, X, y). \end{aligned}$$

Note that if y belongs to a connected component of $Y \setminus \phi(\partial X)$ whose closure is not compact, then $\deg(\phi, X, y) = 0$ so the sum is restricted to the connected components whose closure is compact. Then

$$\begin{aligned} \deg(\psi \circ \phi, X, z) &= \sum_{\psi(y)=z} \text{sign } J_{\psi(y)} \sum_i \deg(\phi, X, \Omega_i) \\ &= \sum_i \deg(\phi, X, \Omega_i) \deg(\psi, \Omega_i, z). \end{aligned}$$

2 Applications

Let B be the closed unit ball in \mathbb{R}^n .

Proposition 7. *Let $\phi : B \rightarrow \mathbb{R}^n$ be continuous and such that $\phi(x)$ never points opposite to x on ∂B , i.e.,*

$$\phi(x) + tx \neq 0, \quad \forall t \geq 0, x \in \partial B.$$

Then $\phi(x) = 0$ has a solution inside B .

Proof. Indeed $t\phi(x) + (1-t)x$ does not vanish for any $t \in [0, 1]$ and $x \in \partial B$. Therefore $\deg(\phi, B, 0) = 1$.

Note that the same result holds for $-\phi$, i.e. if $\phi(x)$ never points in the same direction as x on ∂B . In particular, if $(\phi(x), x) \leq 0$ on ∂B then ϕ has a fixed point in B .

Proposition 8. *Let $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be continuous and satisfy*

$$\lim_{x \rightarrow \infty} \frac{(\phi(x), x)}{|x|} = \infty.$$

Then ϕ is onto.

Indeed, because $\phi(x) - y$ still satisfies the assumption, it is enough to prove that $\exists x, \phi(x) = 0$. But because $(\phi(x), x) \geq 0$ for $|x| \geq R$ we see that, if $\phi(x) \neq 0$ on $|x| = R$, we obtain a function which never points in the opposite direction of $\phi(x)$ on $|x| = R$, and we may use Proposition 7.

Theorem 5. *If $F : B \rightarrow \mathbb{R}^n$ is continuous and $F(\partial B) \subset B$ then F has a fixed point.*

Proof. Assume that there is no fixed point on the boundary. Let $\phi = x - F(x)$. Then, $0 = \phi(x) + tx = (1+t)x - F(x)$ is impossible for $x \in \partial X$, $t \geq 0$. (If $t > 0$ this would send $F(x)$ outside B .) We apply Prop 7.

A variant of Brouwer's fixed point:

Theorem 6. (*Brouwer fixed point*) *A continuous map f from a closed convex set in \mathbb{R}^n to itself has a fixed point.*

Proof. We first prove the result in the case when K is the closure of an open bounded convex set Ω . In that case, WLOG 0 is in the interior of the open set. We consider $\phi(x) = x - f(x)$. If we assume that $0 \notin \phi(\partial\Omega)$ then $0 \notin \phi_t(\partial\Omega)$ where $\phi_t(x) = x - tf(x)$. Indeed, if $x = tf(x)$ for $0 \leq t < 1$ and $x \in \partial\Omega$ then $tf(x)$ is on one hand in $\partial\Omega$ and on the other hand $(1-t)0 + tf(x) \in \Omega$ for $t < 1$ because 0 is in the interior and $f(x) \in K$. This is easily seen by taking a tiny ball B_r around zero so that its dilate by $\frac{1}{t}$ is still included in Ω . That produces $tf(x) + z \in K$ for $|z| < r$. We conclude by degree theory ϕ has a zero in Ω .

The general case is done by considering convolution with a mollifier ϕ_ϵ . The function $f_\epsilon = \mathbf{1}_K f * \phi_\epsilon$ is supported in $K_\epsilon = \{x \mid \text{dist}(x, K) \leq \epsilon\}$ which is the closure of the open bounded convex set $\Omega_\epsilon = \{x \mid \text{dist}(x, K) < \epsilon\}$, and f_ϵ maps K_ϵ to itself. A convergent subsequence of fixed points of f_ϵ converges as $\epsilon \rightarrow 0$ to a fixed point of f in K .

Theorem 7. *There is no continuous function $f : B \rightarrow \partial B$ so that $f|_{\partial B} = I$*

Indeed, if there were such a function, then $f_t(x) = (1-t)f(x) + tx$ would be a homotopy to I such that $0 \notin f_t(\partial B)$. Therefore $\deg(f, B, 0) = 1$, but that is impossible because $0 \notin f(B)$, so $\deg(f, B, 0) = 0$.

Theorem 8. *Borsuk's Theorem. Let X be a bounded open subset of \mathbb{R}^n symmetric about the origin and such that $0 \in X$. Let $\psi : \partial X \rightarrow \mathbb{R}^n \setminus \{0\}$ be continuous and odd ($\psi(-x) = -\psi(x)$). Then the $\deg(\psi, X, 0)$ is odd.*

Proof in [1].

The next result is needed for the Leray-Schauder degree.

Proposition 9. *Let Ω be an open bounded set in \mathbb{R}^n and consider \mathbb{R}^n as a direct sum $\mathbb{R}^n = \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}$ with $n = n_1 + n_2$ so that any x in \mathbb{R}^n has a unique decomposition $x = x_1 + x_2$ with $x_i \in \mathbb{R}^{n_i}$, $i = 1, 2$. We consider a map of the form $f = x + \phi(x)$, with $\phi : \bar{\Omega} \rightarrow \mathbb{R}^{n_1}$. Suppose that $y \in \mathbb{R}^{n_1}$ and $y \notin f(\partial\Omega)$. Then*

$$\deg(f, \Omega, y) = \deg(f|_{\Omega_1}, \Omega_1, y)$$

where $\Omega_1 = \Omega \cap \mathbb{R}^{n_1}$.

Proof. We may assume that $f \in C^1(\Omega)$ and $y = 0 \in \mathbb{R}^{n_1}$. Let $\psi_j(x_j)$ be smooth compactly supported functions in \mathbb{R}^{n_j} supported near the origin for $j = 1, 2$ and with normalized integrals $\int_{\mathbb{R}^{n_j}} \psi_j(x_j) dx_j = 1$. Then

$$\deg(f, \Omega, 0) = \int_{\mathbb{R}^n} f^*(\psi_1(x_1)\psi_2(x_2))dx.$$

Now $J_f(x) = \det(I + \nabla_{x_1}\phi(x_1 + x_2))$ so that

$$\deg(f, \Omega, 0) = \int_{\mathbb{R}^{n_2}} \int_{\mathbb{R}^{n_1}} \psi_1(x_1 + \phi(x_1))\psi_2(x_2) \det(I + \nabla_{x_1}\phi(x_1 + x_2)) dx_1 dx_2$$

We replace ψ_2 by a sequence of functions tending to the delta function, without changing the equality. We obtain

$$\begin{aligned} \deg(f, \Omega, 0) &= \int_{\mathbb{R}^{n_1}} \psi_1(x_1 + \phi(x_1)) \det(I + \nabla_{x_1}\phi(x_1)) dx_1 \\ &= \deg(f|_{\Omega_1}, \Omega_1, 0) \end{aligned}$$

References

- [1] L. Nirenberg, Topics in Nonlinear Functional Analysis, CIMS, 1973-1974.