Hölder Continuity of Solutions of 2D Navier-Stokes Equations with Singular Forcing

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Dedicated to Nina Nikolaevna Uraltseva

Abstract We discuss the regularity of solutions of 2D incompressible Navier-Stokes equations forced by singular forces. The problem is motivated by the study of complex fluids modeled by the Navier-Stokes equations coupled to a nonlinear Fokker-Planck equation describing microscopic corpora embedded in the fluid. This leads naturally to bounded added stress and hence to $W^{-1,\infty}$ forcing of the Navier-Stokes equations.

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1 Introduction

We discuss the regularity of solutions of 2D incompressible Navier-Stokes equations forced by singular forces. The problem is motivated by the study of complex fluids modeled by the Navier-Stokes equations coupled to a non-linear Fokker-Planck equation describing microscopic corpora embedded in

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the fluid. This leads naturally to bounded added stress and hence to $W^{-1,\infty}$ forcing of the Navier-Stokes equations. A more detailed description of the problem in question, together with an application of the results in the present paper can be found in our forthcoming paper [2].

In this paper we focus on the 2D Navier-Stokes issues. The global existence of energy solutions and their uniqueness are well known as classical results of J. Leray for the Cauchy problem and O. Ladyzhenskaya for initial boundary value problems in bounded domains. These results remain to be true for singular forces as well.

The regularity of energy solutions with relatively smooth forces is also known. Regularity can be established, for instance, by scalar multiplication of the Navier-Stokes equation by the Stokes operator of the velocity field, integration by parts, and application of Ladyzhenskay's inequality

$$\|u\|^2_{L^4(\mathbb{R}^2)} \leq \sqrt{2}\|u\|_{L^2(\mathbb{R}^2)}\|\nabla u\|_{L^2(\mathbb{R}^2)}, \qquad \forall u \in C_0^\infty(\mathbb{R}^2).$$

This procedure yields summability of the second spatial derivatives. Further regularity can be obtained perturbatively, with the help of the linear theory.

The regularity of energy solutions with singular forcing is limited. The best one can expect is Hölder continuity of the velocity field. We prove Hölder continuity at a local level, in both space and in time. We assume that our local solution has finite energy and the pressure field is in L^2. This latter assumption seems restrictive: we are not able to justify it for general initial boundary value problems with reasonable singular forcing. The assumption is however satisfied in the absence of boundaries, i.e., for the Cauchy problem in the whole space and for the initial value problem on the torus. We briefly explain in this paper how the local regularity results can be applied to the Cauchy problem in the whole space.

In our proof, the Hölder continuity of the velocity field depends quantitatively on the modulus of continuity of the function $\omega \mapsto \int\limits_\omega |u|^4 dz$. In order to be able to apply this regularity result to coupled systems or to families of Navier-Stokes systems, this modulus of continuity needs to be a priori uniformly controlled. We achieve this in the absence of boundaries by obtaining higher integrability of the velocity, $u \in L^\infty(dt; L^r(dx))$, $r \geq 4$. In order to obtain the higher integrability we prove the generalized Ladyzhenskaya inequality that reads

$$\|u\|^2_{L^{2r}(\mathbb{R}^2)} \leq \frac{r}{\sqrt{2}}\|u\|_{L^r(\mathbb{R}^2)}\|\nabla u\|_{L^2(\mathbb{R}^2)}, \qquad \forall u \in C_0^\infty(\mathbb{R}^2)$$

for $r \geq 2$. The proof is elementary and can be found in the Appendix.

2 Notation and Local Regularity Result

We assume that Ω and Ω_1 are domains in $\mathbb{R}^2$ such that $\Omega_1 \Subset \Omega$ and $0 < T_1 < T$, and let

$$Q = \Omega \times (-T, 0), \qquad Q_1 = \Omega_1 \times (-T_1, 0).$$

Parabolic balls will be denoted as $Q(z_0, R) = B(x_0, R) \times (t_0 - R^2, t_0)$, where $z_0 = (x_0, t_0)$, $x_0 \in \mathbb{R}^2$, $t_0 \in \mathbb{R}$, and $B(x_0, R)$ is an open disk in $\mathbb{R}^2$ having radius R and centered at the point x_0.
We use the following notation for mean values:

$$(f)_{z_0,R} = \frac{1}{|Q(z_0,R)|} \int\limits_{Q(z_0,R)} f(z)dz, \qquad [p]_{x_0,R} = \frac{1}{|B(x_0,R)|} \int\limits_{B(x_0,R)} p(x)dx.$$

$L^p(\Omega)$ and $W^{l,p}(\Omega)$ stand for usual Lebesgues and Sobolev spaces of functions defined Ω, and the norm of the Lebesgues space is denoted by $\|\cdot\|_{m,\Omega}$. For the forcing we are going use a functional space $M_{2,\gamma}(Q)$ with parameter $0 \leq \gamma < 1$ and seminorm

$$\|f\|_{M_{2,\gamma}(Q)} = \sup_{Q(z_0,R) \subset Q} R^{1-\gamma} \Big(\frac{1}{|Q(z_0,R)|} \int\limits_{Q(z_0,R)} |f(z) - (f)_{z_0,R}|^2 dz \Big)^{\frac{1}{2}} < \infty.$$

We denote by c all positive universal constants. Our regularity result can be formulated as follows.

Theorem 2.1. *Assume that we are given functions*

$$u \in L^4(Q; \mathbb{R}^2), \quad p \in L^2(Q), \quad F \in M_{2,\gamma}(Q; \mathbb{M}^{2\times 2}) \tag{2.1}$$

with $0 \leq \gamma < 1$*, satisfying the Navier-Stokes equations*

$$\partial_t u + u \cdot \nabla u - \Delta u + \nabla p = -\operatorname{div} F, \quad \operatorname{div} u = 0 \tag{2.2}$$

in Q *in the sense of distributions.*

Then

$$u \in C^\gamma(\overline{Q}_1) \tag{2.3}$$

if $0 < \gamma < 1$ *and*

$$u \in BMO(Q_1) \tag{2.4}$$

if $\gamma = 0$.

Remark 2.2. *The Hölder continuity and the BMO space are defined with respect to parabolic metrics.*

Remark 2.3. *The corresponding norms are estimated in terms of the quantities* $\|u\|_{4,Q}$, $\|p\|_{2,Q}$, $\|F\|_{M_{2,\gamma}(Q)}$, $\mathrm{dist}(\Omega_1, \partial\Omega)$, $T - T_1$, *and the modulus of continuity of the function* $\omega \mapsto \int\limits_{\omega} |u|^4 dz$.

Several additional results can be proved by means of minor modifications of the proof of Theorem 2.1. Before stating one them, we define usual energy spaces for the 2D Navier-Stokes equations. Let H and V be completions of the set of all divergence-free vector fields from $C_0^\infty(\mathbb{R}^2; \mathbb{R}^2)$ with respect to the L^2 norm and the Dirichlet integral, respectively.

Proposition 2.4. *Let* $u \in L^\infty(0, T; H) \cap L^2(0, T; V)$, $p \in L^2(0, T; L^2(\mathbb{R}^2))$ *be a solution of the Cauchy problem*

$$\partial_t u + u \cdot \nabla u - \Delta u + \nabla p = -\mathrm{div}\, F, \qquad \mathrm{div}\, u = 0, \tag{2.5}$$

$$v(\cdot, 0) = a(\cdot) \in H, \tag{2.6}$$

where $F \in L^q(Q_T; \mathbb{M}^{2\times 2}) \cap L^2(Q_T; \mathbb{M}^{2\times 2})$ *with* $q > 4$ *and* $Q_T = \mathbb{R}^2 \times (0, T)$.

Then, given $0 < s \leq T$, *there exists a constant* C *depending only on* s, *the norms of* F *in* $L^q(Q_T; \mathbb{M}^{2\times 2})$ *and in* $L^2(Q_T; \mathbb{M}^{2\times 2})$ *and the norm of* a *in* H *such that*

$$\|u\|_{L^\infty(\mathbb{R}^2 \times (s,T))} \leq C. \tag{2.7}$$

Moreover, the function u *is Hölder continuous in* $\mathbb{R}^2 \times [s, T]$ *with exponent* $\gamma = 1 - \frac{4}{q}$.

Remark 2.5. *The existence and uniqueness of a solution to the Cauchy problems (2.5) and (2.6) with above properties is well known, see [5].*

Remark 2.6. *The same statement is valid in the case of periodic boundary conditions. More generally, it is true as long as the pressure field is in* L^2.

3 Proof of Theorem 2.1

We are going to analyze differentiability properties of the velocity field u in terms of the following functionals:

$$\Phi(u; z_0, \varrho) = \Big(\int\limits_{Q(z_0,\varrho)} |u - (u)_{z_0,\varrho}|^4 dz \Big)^{\frac{1}{2}}, \quad \Psi(u; z_0, \varrho) = \Big(\int\limits_{Q(z_0,\varrho)} |u|^4 dz \Big)^{\frac{1}{2}},$$

$$D(p;z_0,\varrho)=\int\limits_{Q(z_0,\varrho)}|p-[p]_{x_0,\varrho}|^2dz.$$

The following two statements are well-known.

Lemma 3.1. *Let the function $v\in L^4(Q(z_0,R))$ satisfy the heat equation*

$$\partial_t v-\Delta v=0$$

in $Q(z_0,R)$. Then

$$\Phi(v;z_0,\varrho)\leq c\Big(\frac{\varrho}{R}\Big)^4\Phi(v;z_0,R) \tag{3.1}$$

for all $0<\varrho\leq R$.

Lemma 3.2. *Given $G\in L^2(Q(z_0,R);\mathbb{M}^{2\times 2})$, there exists a unique function*

$$w\in C([t_0-R^2,t_0];L^2(B(x_0,R);\mathbb{R}^2))\cap L^2([t_0-R^2,t_0];W^{1,2}(B(x_0,R);\mathbb{R}^2))$$

such that

$$\partial_t w-\Delta w=-\operatorname{div}G$$

in $Q(z_0,R)$ and

$$w=0$$

on the parabolic boundary of $Q(z_0,R)$. Moreover, the function w satisfies the estimates:

$$|w|^2_{2,Q(z_0,R)}\equiv\sup_{t_0-R^2<t<t_0}\|w(\cdot,t)\|^2_{2,B(x_0,R)}+\|\nabla w\|^2_{2,Q(z_0,R)}$$

$$\leq 2\|G\|^2_{2,Q(z_0,R)}, \tag{3.2}$$

$$\Phi(w;z_0,R)\leq c|w|^2_{2,Q(z_0,R)}. \tag{3.3}$$

The next couple of statements are about some properties of the solutions of the system (2.2).

Lemma 3.3. *Under the assumptions of Theorem 2.1 we have*

$$\Phi(u;z_0,\varrho)\leq c\Big\{\Big[\Big(\frac{\varrho}{R}\Big)^4+\Psi(u;z_0,R)\Big]\Phi(u;z_0,R)+$$

$$+D(p;z_0,R)+MR^{2+2\gamma}\Big\} \tag{3.4}$$

whenever $Q(z_0,R)\subset Q$ and $0<\varrho\leq R$. Here, $M=\|F\|^2_{M_{2,\gamma}(Q)}$.

PROOF. Setting

$$G = F - (F)_{z_0,R} + (p - [p]_{x_0,R})\mathbb{I} + (u - (u)_{z_0,R}) \otimes u$$

in Lemma 3.2, we get the following estimate for w

$$\Phi(w; z_0, R) \leq$$

$$\leq c \int\limits_{Q(z_0,R)} \Big[|F - (F)_{z_0,R}|^2 + |p - [p]_{x_0,R}|^2 + |u - (u)_{z_0,R}|^2|u|^2\Big] dz \leq$$

$$\leq c\Big[MR^{2+2\gamma} + D(p; z_0, R) + \Psi(u; z_0, R)\Phi(u; z_0, R)\Big]. \tag{3.5}$$

Obviously, the function $v = u - w$ satisfies the heat equation. Then, applying Lemma 3.1, we find

$$\Phi(u - w; z_0, \varrho) \leq c\Big(\frac{\varrho}{R}\Big)^4 \Phi(u - w; z_0, R).$$

The latter inequality gives us:

$$\Phi(u; z_0, \varrho) \leq c\Big[\Big(\frac{\varrho}{R}\Big)^4 \Phi(u; z_0, R) + \Phi(w; z_0, R)\Big]. \tag{3.6}$$

Combining (3.5) and (3.6), we arrive at (3.4) and thus Lemma 3.3 is proved.

Lemma 3.4. *Under the assumptions of Theorem 2.1, we have the estimate*

$$D(p; z_0, \varrho) \leq c\Big[\Big(\frac{\varrho}{R}\Big)^4 D(p; z_0, R) + \Psi(u; z_0, R)\Phi(u; z_0, R) + MR^{2+2\gamma}\Big] \tag{3.7}$$

whenever $Q(z_0, R) \subset Q$ *and* $0 < \varrho \leq R$.

PROOF. The crucial part of the proof is the pressure decomposition

$$p = p_1 + p_2,$$

where the first component p_1 satisfies the identity

$$\int\limits_{B(x_0,R)} p_1 \Delta\varphi dx = - \int\limits_{B(x_0,R)} \Big((u - (u)_{z_0,R}) \otimes u + F - (F)_{z_0,R}\Big) : \nabla^2\varphi dx$$

where the test function $\varphi \in W^2_2(B(x_0,R))$ is subject to the Dirichlet boundary condition: $\varphi = 0$ on $\partial B(x_0,R)$. It is not difficult to show that such a function p_1 exists and obeys the estimate

$$\int\limits_{Q(z_0,R)} |p_1 - [p_1]_{x_0,R}|^2 dz \leq c \int\limits_{Q(z_0,R)} |p_1|^2 dz \leq$$

$$\leq c\Big[\Psi(u;z_0,R)\Phi(u;z_0,R) + MR^{2+2\gamma}\Big]. \tag{3.8}$$

The second counterpart of the pressure p_2 is a harmonic function and thus satisfies the estimate

$$\int\limits_{B(x_0,\varrho)} |p_2 - [p_2]_{x_0,\varrho}|^2 dx \leq c\Big(\frac{\varrho}{R}\Big)^4 \int\limits_{B(x_0,R)} |p_2 - [p_2]_{x_0,R}|^2 dx$$

$$\leq c\Big(\frac{\varrho}{R}\Big)^4 \int\limits_{B(x_0,R)} |p - [p]_{x_0,R}|^2 dx + c \int\limits_{B(x_0,R)} |p_1|^2 dx$$

for any $0 < \varrho \leq R$. Hence, by (3.8), we show

$$D(p_2;z_0,\varrho) \leq c\Big[\Big(\frac{\varrho}{R}\Big)^4 D(p,z_0,R) + \Psi(u;z_0,R)\Phi(u;z_0,R) + MR^{2+2\gamma}\Big]. \tag{3.9}$$

Taking into account simple inequality

$$D(p;z_0,\varrho) \leq 2D(p_1;z_0,\varrho) + 2D(p_2;z_0,\varrho) \leq$$

$$\leq \int\limits_{Q(z_0,R)} |p_1|^2 dz + 2D(p_2;z_0,\varrho)$$

we deduce the estimate (3.7) from (3.8) and (3.9). Lemma 3.4 is proved.

Now, we pass to the proof of Theorem 2.1. Assuming that $Q(z_0,R) \subset Q$ and $\tau \in (0,1)$, we find from (3.4) and (3.7) two inequalities:

$$\Phi(u;z_0,\tau^2 R) \leq c(\tau^4 + \Psi(u;z_0,\tau R))\Phi(u;z_0,\tau R) + cD(p;z_0,\tau R)+$$

$$+cM(\tau R)^{2+2\gamma}$$

and

$$(1+c)D(p;z_0,\tau R) \leq (1+c)c\Big\{\tau^4 D(p;bz_0,R) + \Psi(u;z_0,R)\Phi(u;z_0,R)+$$

$$+MR^{2+2\gamma}\Big\}.$$

Adding the latter inequalities and introducing the new functional

$$\Theta(z_0,R)=\Phi(u;z_0,\tau R)+D(p;z_0,R),$$

we arrive at the basic estimate

$$\Theta(z_0,\tau R)\leq c(\tau^4+\Psi(u;z_0,\tau R))\Theta(z_0,R)+$$

$$+c\Psi(u;z_0,R)\Phi(u;z_0,R)+cMR^{2+2\gamma}. \tag{3.10}$$

It is not difficult to check validity of the following inequality

$$\Phi(u;z_0,\tau R)\leq c\Phi(u;z_0,R)$$

for any $\tau\in(0,1)$ and any $R>0$. Then from (3.10) it follows that

$$\Theta(z_0,\tau R)\leq c(\tau^4+\Psi(u;z_0,R))\Theta(z_0,R/\tau)+cMR^{2+2\gamma} \tag{3.11}$$

under assumption that $Q(z_0,R/\tau)\subset Q$. Letting $\gamma_1=(1+\gamma)/2$ and choosing $\tau=\tau(\gamma)\in(0,1)$ so that

$$c\tau^{2-2\gamma_1}\leq 1/2, \tag{3.12}$$

we can choose $R_0<\tau\min\{\mathrm{dist}(\Omega_1,\partial\Omega),\sqrt{T-T_1}\}$ so that

$$\Psi(u;z_0,R)<\tau^4 \tag{3.13}$$

for all $z_0\in Q_1$ and all $0<R\leq R_0$. It is here that the modulus of continuity of $\omega\mapsto\int_\omega|u|^4dxdt$ is used. So, summarizing all the above, we have

$$\Theta(z_0,\tau R)\leq\tau^{2+2\gamma_1}\Theta(z_0,R/\tau)+cMR^{2+2\gamma} \tag{3.14}$$

for all $z_0\in Q_1$ and all $0<R\leq R_0$. To reduce (3.14) to a known iterative procedure, we let $\varrho=R/\tau$ and $\vartheta=\tau^2$. As a result, we find

$$\Theta(z_0,\vartheta\varrho)\leq\vartheta^{1+\gamma_1}\Theta(z_0,\varrho)+cM\vartheta^{1+\gamma}\varrho^{2+2\gamma} \tag{3.15}$$

for all $z_0\in Q_1$ and all $0<\rho\leq R_0/\tau$. The inequality (3.15) can be easily iterated, see [1],

$$\Theta(z_0,\vartheta^kR_0/\tau)\leq\vartheta^{k(1+\gamma)}\Big(\Theta(z_0,R_0/\tau)+c_1MR_0^{2+2\gamma}\Big)$$

for any $k \in \mathbb{N}$. Here and in what follows, all positive constants depending on γ only are denoted by c_1. The latter inequality implies

$$\Phi(u; z_0, \vartheta^k R_0) \leq \vartheta^{k(1+\gamma)}\Big(\Theta(z_0, R_0/\tau) + c_1 M R_0^{2+2\gamma}\Big)$$

for any $k \in \mathbb{N}$ and, hence,

$$\Phi(u; z_0, \varrho) \leq \varrho^{1+\gamma} H \tag{3.16}$$

for any $0 < \varrho \leq R_0$, where

$$H = c_1\Big(\frac{1}{R_0}\Big)^{1+\gamma}\Big[\Theta(z_0, R_0/\tau) + M R_0^{2+2\gamma}\Big].$$

Obviously, H is a function of R_0, $\|u\|_{4,Q}$, $\|p\|_{2,Q}$, M, $\mathrm{dist}(\Omega_1, \partial\Omega)$, $T - T_1$, and γ.

Now, our next step is to figure out how does $\Psi(u; z_0, R)$ depend on R. By (3.16), we have

$$|(u)_{z_0,\varrho/2} - (u)_{z_0,\varrho}| \leq \frac{c}{\varrho}\Phi^{1/2}(u; z_0, \varrho) \leq c H^{1/2} \varrho^{-(1-\gamma)/2}$$

for any $0 < \varrho \leq R_0$. Therefore,

$$|(u)_{z_0,R_0/2^k} - (u)_{z_0,R_0}| \leq c H^{1/2} \sum_{i=0}^{k-1} \Big(\frac{R_0}{2^i}\Big)^{-(1-\gamma)/2} =$$

$$= c H^{1/2}\Big(\frac{R_0}{2^k}\Big)^{-(1-\gamma)/2} \sum_{i=0}^{k-1} \Big(\frac{1}{2^{k-i}}\Big)^{(1-\gamma)/2} \leq c_1 H^{1/2}\Big(\frac{R_0}{2^k}\Big)^{-(1-\gamma)/2}$$

for any $k \in \mathbb{N}$, or

$$|(u)_{z_0,\varrho} - (u)_{z_0,R_0}| \leq c_1 H^{1/2} \frac{1}{\varrho^{(1-\gamma)/2}} \tag{3.17}$$

for any $0 < \varrho \leq R_0$. Proceeding and making use of (3.17), we find

$$\Psi^{1/2}(u; z_0, \varrho) \leq \Phi^{1/2}(u; z_0, \varrho) + c\varrho|(u)_{z_0,\varrho}| \leq c_1 \varrho^{(1+\gamma)/2} H^{1/2} +$$

$$+ c\varrho|(u)_{z_0,R_0}|.$$

The latter implies

$$\Psi(u; z_0, \varrho) \leq \varrho^{(1+\gamma)} H_1 \tag{3.18}$$

for any $0 < \varrho \leq R_0$, where H_1 depends on the same arguments as H.

Coming back to the basic estimate (3.10) and taking into account (3.12), (3.13), (3.16), and (3.18),

$$\Theta(z_0, \tau R) \leq \tau^{2+2\gamma}\Theta(z_0, R) + c(M + HH_1)R^{2+2\gamma}$$

for any $0 < R \leq R_0$. After iterations of the latter inequality, we show

$$\Theta(z_0, \tau^k R) \leq \tau^{k(2+2\gamma)}\Big(\Theta(z_0, R_0) + c(M + HH_1)R_0^{2+2\gamma}\Big)$$

for any $k \in \mathbb{N}$. Consequently, we obtain:

$$\Phi(u; z_0, \varrho) \leq \varrho^{2+2\gamma} H_2 \tag{3.19}$$

for any $z_0 \in Q_1$ and for any $0 < \varrho \leq R_0$ with H_2 depending on the same arguments as H and H_1. Finally, the Hölder continuity of u subject to (3.19) follows from known considerations, see, for example, [3] or [4]. Theorem 2.1 is proved.

Proof of Proposition 2.4 Our first remark is that

$$L^q(Q_T) \subset M_{2,\gamma}(Q_T) \tag{3.20}$$

if $\gamma = 1 - \frac{4}{q}$. The second remark is that

$$u \in L^4(Q_T) \tag{3.21}$$

and the corresponding norm is bounded by $\|a\|_{2,\mathbb{R}^2} + \|F\|_{2,Q_T}$.

In order to handle the modulus of continuity of the function $\omega \mapsto \int\limits_{\omega} |u|^4 dz$, we are going to show that, for any $0 < s < T$,

$$\sup_{s<t<T} \|u(\cdot, t)\|_{4,\mathbb{R}^2} \leq C(s, \|F\|_{4,Q_T}, \|a\|_{2,\mathbb{R}^2}). \tag{3.22}$$

Assume that (3.22) has been already proved. Let the number $\tau \in (0,1)$ be defined by (3.12). Then by (3.22) we can find a number $0 < R_1 < \frac{1}{2}\sqrt{s}$ such that

$$\Big(\int\limits_{Q(z_0,R_1)} |u|^4 dz\Big)^{\frac{1}{2}} < R_1 \sup_{s/2<t<T} \|u(\cdot, t)\|_{4,\mathbb{R}^2} <$$

$$< R_1 C(s/2, \|F\|_{4,Q_T}, \|a\|_{2,\mathbb{R}^2}) < \tau^4 \tag{3.23}$$

for any $z_0 = (x_0, t_0)$ such that $t_0 > s$. Obviously, R_1 depends only on s, $\|F\|_{q,Q_T}$, $\|F\|_{2,Q_T}$, and $\|a\|_{2,\mathbb{R}^2}$ as, by interpolation, $\|F\|_{4,Q_T}$ is estimated by $\|F\|_{q,Q_T}$ and $\|F\|_{2,Q_T}$. Then we repeat the proof of Theorem 2.1 replacing Q with $Q(z_0, R_1)$ and Q_1 with $Q(z_0, R_1/2)$ and establish $u \in L^\infty(Q(z_0, R_1/2))$ with a uniform estimate with respect to z_0 having $t_0 > s$. Thus, the solution is Hölder continuous.

So, let us prove (3.22). To this end, we test the Navier-Stokes by $|u|^2 u$, as a result we have

$$\frac{1}{4}\partial_t \int\limits_{\mathbb{R}^2} |u|^4 dx + \int\limits_{\mathbb{R}^2} \Big(|u|^2 |\nabla u|^2 dx + 2|u|^2 |\nabla |u||^2\Big) dx =$$

$$= \int\limits_{\mathbb{R}^2} \Big(2|u| p u \cdot \nabla |u| + F : (|u|^2 \nabla u + 2|u| u \otimes \nabla |u|)\Big) dx.$$

After application of the Cauchy inequality with a suitable weight, we find the following estimate

$$\partial_t \|u\|^4_{4,\mathbb{R}^2} \leq c \int\limits_{\mathbb{R}^2} \Big(p^2 |u|^2 + |F|^2 |u|^2\Big) dx \leq$$

$$\leq c\Big(\|p\|^2_{4,\mathbb{R}^2} + \|F\|^2_{4,\mathbb{R}^2}\Big)\|u\|^2_{4,\mathbb{R}^2}.$$

It remains to make use the pressure equation which, in the case of the Cauchy problem, gives us

$$\partial_t \|u(\cdot,t)\|^4_{4,\mathbb{R}^2} \leq c\Big(\|u(\cdot,t)\|^4_{8,\mathbb{R}^2} + \|F(\cdot,t)\|^2_{4,\mathbb{R}^2}\Big)\|u(\cdot,t)\|^2_{4,\mathbb{R}^2}. \tag{3.24}$$

To evaluate the right hand side of (3.24), we are going to use a particular case of the generalized Ladyzhenskaya inequality

$$\|u(\cdot,t)\|^2_{8,\mathbb{R}^2} \leq c\|u(\cdot,t)\|_{4,\mathbb{R}^2} \|\nabla u\|_{2,\mathbb{R}^2}.$$

The proof of the generalized Ladyzhenskaya inequality is given in the Appendix. But then (3.24) can be reduced to the form

$$\partial_t \|u(\cdot,t)\|^4_{4,\mathbb{R}^2} \leq c\Big(\|u(\cdot,t)\|^4_{4,\mathbb{R}^2} \|\nabla u\|^2_{2,\mathbb{R}^2} + \|F(\cdot,t)\|^4_{4,\mathbb{R}^2} + \|u(\cdot,t)\|^4_{4,\mathbb{R}^2}\Big).$$

Multiplying the last inequality by a suitable cut-off function in t, keeping in mind that our solution has the finite energy bounded by $\|a\|^2_{2,\mathbb{R}^2} + \|F\|^2_{2,Q_T}$ from above, and using Gronwall's lemma, we prove (3.22). Proposition 2.4 is proved.

4 Appendix: Generalized Ladyzhenskaya Inequality

The inequality

$$\|u\|^2_{4,\mathbb{R}^2} \le \sqrt{2}\|u\|_{2,\mathbb{R}^2}\|\nabla u\|_{2,\mathbb{R}^2}, \qquad \forall u \in C_0^\infty(\mathbb{R}^2), \tag{4.1}$$

was used by Ladyzhenskaya in [5] to prove unique global solvability initial boundary value problem for the Navier-Stokes equations in bounded domains of $\mathbb{R}^2$. The generalized version of the Ladyzhenskaya inequality is as follows:

$$\|u\|^2_{2r,\mathbb{R}^2} \le \frac{r}{\sqrt{2}}\|u\|_{r,\mathbb{R}^2}\|\nabla u\|_{2,\mathbb{R}^2}, \qquad \forall u \in C_0^\infty(\mathbb{R}^2) \tag{4.2}$$

for $r \ge 2$. The proof of (4.2) is essentially the same as the proof of (4.1). The main ingredient of it the following identity

$$|u|^r(x_1,x_2) = r\int\limits_{-\infty}^{x_1} |u|^{r-1}(t,x_2)(|u|)_{,1}(t,x_2)dtdx_2.$$

Using the same identity with respect to x_2, we find

$$\int\limits_{\mathbb{R}^2} |u|^{2r}dx_1dx_2 \le r^2\int\limits_{\mathbb{R}^2} |u|^{r-1}|u_{,1}|dx_1dx_2\int\limits_{\mathbb{R}^2} |u|^{r-1}|u_{,2}|dx_1dx_2 \le$$

$$\le \frac{r^2}{2}\int\limits_{\mathbb{R}^2} |u|^{2(r-1)}dx_1dx_2\int\limits_{\mathbb{R}^2} |\nabla u|^2dx_1dx_2.$$

By interpolation,

$$\int\limits_{\mathbb{R}^2} |u|^{2(r-1)}dx_1dx_2 \le \Big(\int\limits_{\mathbb{R}^2} |u|^{2r}dx_1dx_2\Big)^{\frac{r-2}{r}}\Big(\int\limits_{\mathbb{R}^2} |u|^{r}dx_1dx_2\Big)^{\frac{2}{r}}.$$

Now we deduce (4.2) from the latter inequalities.

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References

[1] Campanato, S., Equazioni paraboliche del secondo ordine e spazi $\mathcal{L}^{2,\theta}(\Omega, \delta)$. Ann. Mat. Pura Appl. 73(1966), 55-102.

[2] Constantin, P., Seregin, G. Global regularity of solutions of copuled Navier-Stokes equations and nonlinear Fokker-Planck equations, to appear.

[3] Da Prato, G. Spazi $\mathcal{L}^{2,\theta}(\Omega, \delta)$ e loro proprieta. Ann. Mat. Pura Appl. 69(1965), 383-392.

[4] Giaquinta, M., Struwe, M., On the Partial Regularity of Weak Solutions of Nonlinear Parabolic Systems, Math. Z. 179(1982), 437-451.

[5] Ladyzhenskaya, O. A., Global solvability of a boundary value roblem for the Navier-Stokes equations in the case of two spatial variables, Doklady of the USSR, 123(1958), 427–429.