Systems of Conservation Laws

Intoduction to PDE

1 Description

The equations of motion of compressible fluids and gases are obtained from the laws of conservation of mass, momentum and energy for arbitrary volumes of the liquid. One basic assumption is that a small volume of the fluid still contains a very large number of molecules, and in the limit of small volumes only macroscopic averages of molecular activities survive.

The physical quantities required to describe the mass flow are the density $\rho(x,t)$, a non-negative function of position $x \in \mathbb{R}^3$ and time t, the velocities $u = u(x,t)$, and the internal energy $e = e(x,t)$. In order to describe momentum conservation we need to consider what forces act on the fluid. There will be external forces $F(x,t)$ per unit mass, acting on each volume element, and internal forces acting at the interface between volume elements. The internal forces are given by a stress tensor describing the components of the internal forces in terms of the direction normal to the interface on which they act. The internal force per unit area is thus a function

$$G = (G_1, G_2, G_3) = G(x, t, \ell)$$

with $|\ell| = 1$, the normal. There will be heat transfer across the boundary of volume elements, and this will be given by a function $q(x,t)$ denoting heat flow per unit area.

The conservation of mass in a volume V says that the rate of change of mass in V is given by the mass flux through ∂V:

$$\frac{d}{dt}\int_V \rho dx + \int_{\partial V} \rho(\ell \cdot u) dS = 0$$

where ℓ is the outward normal to ∂V. This is a consequence of the transport lemma.

Lemma 1. *Let $f(x,t)$ be a C^1 function, let u be a ($C^{1,\alpha}$) smooth velocity field, and let B be an open bounded set with smooth boundary, and let $B_t = X(t,B)$ the image of B under the flow generated by the particle trajectories*

$$\frac{dX(a,t)}{dt} = u(X(a,t),t), \quad X(a,0) = a.$$

Then

$$\frac{d}{dt}\int_{B_t} f(x,t)dx = \int_{B_t} \left(\partial_t f + \partial_j(u_j f)\right) dx.$$

Proof Let $\widetilde{f}(a,t) = f(X(a,t),t)$. Then

$$\int_{B_t} f(x,t)dx = \int_{B_0} \widetilde{f}(a,t) \left|\det \nabla_a X(a,t)\right| da$$

We have that $\det(\nabla_a X) > 0$. We differentiate with respect to time. Using chain rule

$$\frac{d}{dt}(\nabla_a X) = (\nabla_x u)(X(a,t),t)(\nabla_a X)$$

we obtain

$$\frac{d}{dt}(\det \nabla_a X) = (\nabla_x \cdot u)(X(a,t),t)(\det \nabla_a X)$$

and thus

$$\begin{aligned}
&\tfrac{d}{dt}\textstyle\int_{B_t} f(x,t)dx \\
&= \textstyle\int_{B_0}\left(\partial_t \widetilde{f}(a,t) + \widetilde{f}(a,t)(\nabla_x \cdot u)(X(a,t),t)\right) \det \nabla_a X(a,t)da \\
&= \textstyle\int_{B_t}(\partial_t f(x,t) + u(x,t)\cdot\nabla_x f(x,t) + (\nabla_x \cdot u(x,t))f(x,t))dx
\end{aligned}$$

Using the transport lemma we have

$$\begin{aligned}
\tfrac{d}{dt}\textstyle\int_V \rho dx &= \textstyle\int_V \partial_t \rho dx + \int_V \operatorname{div}(u\rho)dx \\
&= \textstyle\int_V \partial_t\rho dx + \int_{\partial V} \rho(u\cdot\ell)dS
\end{aligned}$$

If the mass is conserved then it follows that

$$\int_V \partial_t \rho dx + \int_{\partial V} \rho(u\cdot\ell)dS = 0.$$

Of course, the same thing is expressed by

$$\int_V \partial_t\rho dx + \int_V \operatorname{div}(u\rho)dx = 0.$$

Because V is arbitrary we deduce

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{1}$$

the mass conservation equation. The equation of conservation of momentum is obtained in a similar way. The total (linear) momentum in a volume is $\int_V \rho u dx$. The rate of change of this is given by the total forces acting on the fluid in V:

$$\frac{d}{dt}\int_V \rho u dx = \int_V \rho F dx + \int_{\partial V} G dS$$

Using the transport lemma we obtain

$$\int_V (\partial_t(\rho u) + u \cdot \nabla_x(\rho u) + (\operatorname{div} u)(\rho u))dx = \int_V \rho F dx + \int_{\partial V} G dS$$

Now we have Cauchy's theorem: The function $G(x, t, \ell)$ is linear in ℓ

$$G_i(x, t, \ell) = T_{ij}(x, t)\ell_j$$

and is given by a stress tensor. Here is why Cauchy's theorem is true (under our smoothness assumptions). Looking at the balance above in a small volume of diameter ϵ we see that

$$\int_{\partial V} G dS = O(\epsilon^3)$$

We pick V to be a tetrahedron with 3 faces perpendicular to axes and one perpendicular to ℓ. We denote T_{ij} the i component of the force acting on the surface perpendicular to the j-th coordinate direction. We have

$$AG_i(\ell) = T_{i1}A_1 + T_{i2}A_2 + T_{i3}A_3 + O(\epsilon^3)$$

where A, A_i are the areas of the corresponding faces. But $A_i = \ell_i A$, and $A = O(\epsilon^2)$, so

$$G_i - T_{ij}\ell_j = O(\epsilon).$$

Thus, the momentum conservation becomes

$$\partial_t(\rho u_i) + (u \cdot \nabla)(\rho u_i) + (\operatorname{div} u)\rho u_i = \rho F_i + \partial_j(T_{ij}) \tag{2}$$

The conservation of angular momentum

$$\int_V \rho(x \times u)dx$$

leads to

$$T_{ij} = T_{ji}. \tag{3}$$

Finally, considering the total energy per unit mass $E = \frac{1}{2}|u|^2 + e$, we have the rate of change of energy given by the sum of the work of the forces and heat losses:

$$\frac{d}{dt}\int_V (\rho E)dx = \int_{\partial V} (\ell_j T_{ij} u_i - \ell_j q_j)dS + \int_V \rho F_i u_i dx,$$

resulting in

$$\begin{array}{l} \int_V (\partial_t(\rho(\frac{|u|^2}{2} + e)) + \mathrm{div}(\rho(\frac{|u|^2}{2} + e) - Tu)dx \\ = \int_V (\rho(F \cdot u) - \mathrm{div} q)dx \end{array}$$

which gives

$$\partial_t \left(\rho\left(\frac{|u|^2}{2} + e\right)\right) + \partial_j \left(u_j \rho \left(\frac{|u|^2}{2} + e\right) + q_j - T_{jk} u_k\right) = \rho F_i u_i \tag{4}$$

We have exhausted our principles, and do not have enough equations. Even if the heat transfer flux q is known, and the body forces F are also given, we still have eleven unknowns ρ, u_i, e, T_{ij}. The rest is modeling. A first assumption is isotropy of stresses,

$$T_{ij} = -p\delta_{ij},$$

with $p = p(x,t)$ the pressure. If viscosity is allowed and Newtonian fluids are considered, then

$$T_{ij} = -(p + \frac{2}{3}\mu(\mathrm{div} u))\delta_{ij} + \mu\left(\partial_i u_j + \partial_j u_i\right)$$

The internal energy is postulated to be a function of density and pressure

$$e = e(p, \rho).$$

Finally, either

$$q = 0$$

(no heat flux) or the Fourier law

$$q_i = \kappa \partial_i T$$

where T is the temperature (say, it is given at this point), and κ is the heat conduction coefficient. Now the unknowns are u, p, ρ and we have enough equations:

$$\begin{cases} \partial_t \rho + \partial_j(\rho u_j) = 0, \\ \partial_t(\rho u_i) + \partial_j(\rho u_j u_i) = \partial_j T_{ij} + \rho F_i \\ \partial_t \left(\rho \left(\frac{|u|^2}{2} + e\right)\right) + \partial_j \left(\left(\rho \left(\frac{|u|^2}{2} + e\right)\right) + q_j - T_{jk} u_k\right) = \rho F_i u_i \end{cases} \tag{5}$$

These are the Navier-Stokes equations of Newtonian fluid dynamics (or if $\mu = 0$ the Euler equations). One may think that T and $e(p, \rho)$ are given empirically. There are some thermodynamical considerations to make. The one-form

$$de + pd\left(\frac{1}{\rho}\right)$$

plays an important role. The main thermodynamic relation is

$$TdS = de + pd\left(\frac{1}{\rho}\right) \tag{6}$$

T is temperature and S is entropy. The ideal gas law is

$$p = R\rho T$$

with R constant. From (6) we have

$$dS = \frac{1}{T}de - d(R \log \rho)$$

so $\frac{1}{T}de$ is an exact form. Then it follows $e = e(T)$. (But how T might depend on ρ and p is left unsaid...) The quantity

$$h = e + \frac{p}{\rho}$$

is called enthalpy. Under the assumption of constant specific heat at constant volume c_v and constant specific heat at constant pressure c_p we are given the relations

$$\begin{aligned} e &= c_v T, \\ h &= c_p T \end{aligned}$$

and because $h - e = \frac{p}{\rho}$ we obtain the constant

$$R = c_p - c_v.$$

Defining the ratio of specific heats

$$\gamma = \frac{c_p}{c_v} \tag{7}$$

we have now

$$c_p = \gamma c_v, \qquad R = (\gamma - 1)c_v,$$
$$e = \frac{1}{\gamma - 1}\frac{p}{\rho},\ h = \frac{\gamma}{\gamma - 1}\frac{p}{\rho},\ T = \frac{1}{R}\frac{p}{\rho}$$

From (6) we obtain

$$dS = c_v\left(\frac{dp}{p} - \gamma\frac{d\rho}{\rho}\right)$$

so

$$S = c_v \log\frac{p}{\rho^\gamma} + \text{constant} \tag{8}$$

or

$$p = k\rho^\gamma e^{\frac{S}{c_v}} \tag{9}$$

An ideal gas with constant specific heat is called a polytropic gas. The Euler equations are then

$$\begin{cases} D_t\rho + (\text{div}u)\rho = 0, \\ \rho D_t u + \nabla p = \rho F, \\ \rho D_t e + (\text{div}u)p = 0 \end{cases} \tag{10}$$

with $D_t = \partial_t + u\cdot\nabla$. Combining the first and third one we have

$$D_t e - \frac{p}{\rho^2}D_t\rho = 0$$

and using (6) we obtain

$$TD_tS = 0 \tag{11}$$

Thus, entropy is conserved along particle trajectories. A fluid with this property is called adiabatic. Note that if we consider $p = p(\rho, S)$ then

$$D_t p = c^2 D_t\rho$$

with $c^2 = \partial_\rho p$. The function c is called the sound speed. A fluid with constant entropy is called isentropic. For such fluids

$$p = k\rho^\gamma \tag{12}$$

2 Symmetric hyperbolic systems

The equations of an ideal polytropic gas

$$\begin{cases} \partial_t \rho + \partial_j(\rho u_j) = 0, \\ \partial_t(\rho u_i) + \partial_j(\rho u_i u_j) + \partial_i p = \rho F_i \\ \partial_t(\rho E) + \partial_j(\rho E + p u_j) = \rho F_i u_i \end{cases} \tag{13}$$

for $E = \frac{1}{2}|u|^2 + e$,

$$e = \frac{1}{\gamma - 1}\frac{p}{\rho}$$

can be viewed as a system for the variables $v = (\rho, \rho u, \rho E)$ by expressing p in terms of v. The system looks like

$$\partial_t v + \sum_{j=1}^{n} \partial_j F_j(v) = R \tag{14}$$

where $v : \mathbb{R}^n \times I \to G \subset \mathbb{R}^m$, $F_j : G \to G$, $R : \mathbb{R}^n \to G$. (Of course, new notation, F_j are not the body forces, G is a state domain...). The entropy S is conserved along trajectories, $D_t S = 0$. This gives

$$\partial_t(\rho S) + \partial_j(u_j \rho S) = 0, \tag{15}$$

an additional conservation law. Moreover, expressing $\eta = \rho S$ in terms of the variables v, it turns out that it is sometimes convex. We say that the system (14) admits a convex extension if there exists η convex and q_j so that, on smooth solutions of (14) we have

$$\partial_t(\eta(v)) + \partial_j(q_j(v)) = 0 \tag{16}$$

If this is true then

$$(\partial_{v_k}\eta)\partial_t v_k + (\partial_{v_k} q_j)\partial_j v_k = 0,$$

and so

$$(\partial_{v_k}\eta)(\partial_{v_l} F_j^k)(\partial_j v_l) = (\partial_{v_l} q_j)\partial_j v_l.$$

Because this holds at all states v, it follows that

$$\partial_{v_l} q_j = \partial_{v_l} F_j^k (\partial_{v_k}\eta)$$

Denoting

$$F_j' = \partial_{v_l} F_j^k$$

we have

$$\nabla_v q_j = (F_j')^* \nabla_v \eta$$

Differentiationg in some v_k direction we get

$$\partial^2_{v_k v_l} q_j = (\partial^2_{v_k v_p}\eta)\partial_{v_l} F_j^p + (\partial^2_{v_l v_k} F_j^p)\partial_{v_p}\eta$$

Denoting

$$A_0(v) = \partial^2_{v_k v_p}\eta$$

we obtain

$$A_0(v) F_j' = q_j'' - F_j'' \cdot \nabla_v \eta = A_j$$

where

$$q_j'' = \partial^2_{v_k v_l} q_j$$

and

$$F_j'' = \partial^2_{v_k v_l} F^p$$

The main point of all this is: given a convex extension, we can rewrite the system as

$$A_0(v)\partial_t v + A_j(v)\partial_j v = 0, \tag{17}$$

with A_0 positive definite and A_j symmetric.

3 Energy estimates, uniqueness

We consider the system

$$\partial_t v + B_j(v)\partial_j v = 0 \tag{18}$$

We say that (18) is hyperbolic if the symbol matrix

$$B(v,\omega) = \sum_{j=1}^{n} \omega_j B_j(v) \tag{19}$$

has real eigenvalues

$$\lambda_1(v,\omega) \le \dots \lambda_m(v,\omega)$$

for any $\omega \in \mathbb{S}^{n-1}$. A symmetrizable system is one for which there exists a smooth (in v) matrix $A_0(v)$ that is uniformly positive definite and such that $A_j = A_0 B_j$ are all symmetric. Symmetrizable systems are hyperbolic. The system can then be written as

$$A_0(v)\partial_t v + A_j(v)\partial_j v = 0 \tag{20}$$

Proposition 1. *Two bounded $C^1 \cap L^2$ solutions of (20) with the same initial data coincide.*

Proof. The difference $w = v_1 - v_2$ of the two solutions obeys an equation

$$A_0(x,t)\partial_t w + A_j(x,t)\partial_j w + L(x,t)w = 0 \tag{21}$$

with initial data $w(x,0) = 0$. Denoting by $\bar{v} = \frac{1}{2}(v_1 + v_2)$ and $\partial_t = \partial_0$, we have

$$A_0(x,t) = \frac{1}{2}\left(A_0(v_1) + A_0(v_2)\right),$$

$$A_j(x,t) = \frac{1}{2}\left(A_j(v_1) + A_j(v_2)\right),$$

and

$$L(x,t)w = \sum_{k=1}^{m} w_k \sum_{\alpha=0}^{n} \left[\int_0^1 \frac{\partial A_\alpha}{\partial v_k}((1-\lambda)v_2 + \lambda v_1)d\lambda\right]\partial_\alpha \bar{v}$$

Because the system is symmetrizable, we have

$$\delta\mathbb{I} \leq A_0(x,t) \leq \frac{1}{\delta}\mathbb{I}$$

with $\delta > 0$. This follows from the assumption that $A_0(v)$ is uniformly positive definite on a domain G and the assumption that $v_1(x,t) \in G,\ v_2(x,t) \in G$. The matrices $A_\alpha(x,t)$ are symmetric and bounded in C^1. We introduce the energy norm

$$\|w\|_E^2 = \int_{\mathbb{R}^n} \langle A_0(x,t)w(x,t), w(x,t)\rangle dx$$

with $\langle u, v\rangle$ scalar product in $\mathbb{R}^m$. Note that

$$\delta\|w\|_{L^2}^2 \leq \|w\|_E^2 \leq \frac{1}{\delta}\|w\|_{L^2}^2.$$

Let us denote

$$\mathrm{div}A = \sum_{\alpha=0}^{n} \partial_\alpha A_\alpha(x,t)$$

and

$$\|L\| = \sup_{x,t,\ \langle w,w\rangle \leq 1} \left|\langle L(x,t)w, w\rangle\right|.$$

We have
$$\frac{d}{dt}\|w\|_E^2 \le \left(\frac{\|\mathrm{div}A\|_{L^\infty} + 2\|L\|}{\delta}\right)\|w\|_E^2$$
Because $w(x,0) = 0$ it follows that $w(x,t) = 0$.

Energy estimates for derivatives of solutions of linear symmetric hyperbolic systems
$$A_\alpha(x,t)\partial_\alpha w = f \tag{22}$$
are obtained using commutators. They can be used to prove short time existence of smooth solutions. We will not pursue this here.

Hyperbolic systems have finer uniqueness properties.

Definition 1. *We say that an oriented hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ is spacelike with respect to (20) if the symbol matrix*
$$A(x,t,\nu) = \sum_{\alpha=0}^{n} \nu_\alpha A_\alpha(x,t)$$
is positive definite for any $(x,t) \in \Sigma$, when ν is the external normal to Σ.

We consider lens-shaped domains
$$D = D_{\Omega,h} = \{(x,t)|\ x \in \Omega, 0 < t < h(x)\}$$
where h is a nonnegative smooth function compactly supported in $\Omega \subset \mathbb{R}^n$. We say that D is space-like if ∂D is space-like. We say that D is normal space-like if there exists a smooth family of positive functions $h(x,\lambda)$, $\lambda \in [0,1)$ so that $h(x,0) = 0$, $h(x,1) = h(x)$ and
$$h(x,\lambda) \le c,$$
$$|\partial_\lambda h(x,\lambda)| \le c(1 + |\nabla_x h(x,\lambda)|,$$
$$A_\alpha(x,t)\nu_\alpha \ge m_\lambda \mathbb{I}$$
holds for $(x,t) \in \Sigma_\lambda = \{(x,t)|\ x \in \Omega, t = h(x,\lambda)\}$ with ν external normal to Σ_λ, $m_\lambda > 0$ locally uniformly bounded from below in $[0,1)$.

Theorem 1. *Two bounded C^1 solutions of (20) which coincide at $t = 0$ in Ω, coincide in lens-shaped domains $D_{\Omega,h}$ which are normal space-like for both.*

Proof. We take $w = v_1 - v_2$ as above, and use the same notations. We have

$$\partial_\alpha \langle A_\alpha w, w\rangle = \langle (\mathrm{div} A - 2L)w, w\rangle$$

Integrating in $D_\lambda = \{(x,t)|\ x \in \Omega, t < h(x,\lambda)\}$ we have

$$\int_{\Sigma_\lambda} \langle A(x,t,\nu)w, w\rangle dS \le c_0 \int_{D_\lambda} |w|^2 dxdt$$

We denote

$$I(\mu) = \int_{\Sigma_\mu} |w|^2 dS$$

In view of our assumptions on the domain

$$\int_{D_\lambda} |w|^2 dxdt \le C \int_0^\lambda I(\mu)d\mu,$$

and so, we obtain

$$m_\lambda I(\lambda) \le k \int_0^\lambda I(\mu)d\mu$$

Taking $\lambda_0 < 1$ and $M_{\lambda_0} = k \sup_{0\le\lambda\le\lambda_0} m_\lambda^{-1}$ we have

$$I(\lambda) \le M_{\lambda_0} \int_0^\lambda I(\mu)d\mu$$

for all $0 \le \lambda \le \lambda_0$. Together with $I(0) = 0$, this implies $I(\lambda) = 0$ for all $\lambda \le \lambda_0$, and because λ_0 is arbitrary, $I(\lambda) = 0$ for all $\lambda < 1$. This implies $w = 0$.

4 Simple waves, Riemann invariants

We seek solutions of (18) of the form

$$v(x,t) = V(f(x\cdot\omega, t))$$

with $\omega \in \mathbb{S}^{n-1}$ a fixed direction, $V : \mathbb{R} \to G \subset \mathbb{R}^m$ and $f = f(y,t)$ a function of $y \in I \subset \mathbb{R}$ and $t \in \mathbb{R}$. Differentiating we see that in order to solve (18) we need

$$(\partial_t f)V' + (\partial_y f)B(V,\omega)V' = 0$$

We try to separate variables. If

$$B(V,\omega)V' = \lambda V'$$

and

$$\partial_t f + \lambda \partial_y f = 0$$

with $\lambda \in \mathbb{R}$ we are in good shape. Recall now that if (18) is hyperbolic, there exist $\lambda_1(v,\omega) \le \cdots \le \lambda_m(v,\omega)$ real eigenvalues of $B(v,\omega)$. Let us assume that for some $k \in \{1,\dots m\}$ the function $\lambda_k(v,\omega)$ is nice for $v \in G_1 \subset G$ and also that the right eigenvector $r_k(v,\omega)$

$$B(v,\omega)r_k(v,\omega) = \lambda_k(v,\omega)r_k(v,\omega)$$

depends in C^1 fashion on $v \in G_1$. Then we may solve the equation

$$V' = r_k(V,\omega)$$

with $V(0) = V_0$ some prescribed vector. The solution $V(\cdot)$ exists for a small interval of parameters $[-\epsilon,\epsilon]$ by ODE theory. Clearly then

$$B(V,\omega)V' = \lambda_k(V,\omega)V'$$

Armed with the function $V(\cdot)$ we solve now

$$\partial_t f + \lambda_k(V(f),\omega)\partial_y f = 0$$

with small initial data $f_0(y) \in [-\epsilon,\epsilon]$. The solution of this first order quasilinear equation is obtained on characteristics. The characteristics are straight lines

$$y = y_0 + \lambda_k(V(f),\omega)t$$

and f is constant on characteristics. This implies that the solution is given implicitly by

$$f(y,t) = f_0(y - t\lambda_k(V(f(y,t)),\omega))$$

and differentiating we obtain

$$\partial_y f(y,t) = \frac{f_0'(y-\lambda_k t)}{1 + t\frac{\partial \lambda_k}{\partial v}\cdot r_k}$$

Therefore f ceases to be C^1 if $1+t\frac{\partial\lambda_k}{\partial v}\cdot r_k = 0$. We say that the k-th eigenvalue is linearly degenerate in the direction ω if

$$\frac{\partial\lambda_k}{\partial v}\cdot r_k = 0$$

holds for all v. We say that it is genuinely nonlinear in the direction ω if

$$\frac{\partial\lambda_k}{\partial v}\cdot r_k \neq 0$$

for all v. If the k-th eigenvalue is genuinely nonlinear, we may normalize

$$\frac{\partial\lambda_k}{\partial v}\cdot r_k = 1.$$

In that case

$$\frac{d}{df}\lambda_k(V(f),\omega) = 1$$

and therefore

$$\lambda_k(V(f),\omega) = \lambda_k(V_0,\omega) + f.$$

Writing $c = \lambda_k(V_0,\omega)$, this means that the equation for f is

$$\partial_t f + (c+f)\partial_y f = 0,$$

the Burgers equation in a moving frame.

We are going to switch now to $n = 1$, $m = 2$ and discuss Riemann invariants. The system is still (18)

$$\partial_t v + B(v)\partial_x v = 0.$$

There is only one two-by-two matrix $B(v)$. We assume that system is strictly hyperbolic, which means that its eigenvalues are real and distinct

$$\lambda_1(v) < \lambda_2(v).$$

We have right and left eigenvectors

$$B(v)r_j(v) = \lambda_j(v)r_j(v), \quad j = 1, 2,$$

and

$$(B(v))^* l_j(v) = \lambda_j(v)l_j(v), \quad j = 1, 2.$$

In view of the fact that $\lambda_1(v) \neq \lambda_2(v)$ we have

$$\langle l_1(v), r_2(v)\rangle = \langle l_2(v), r_1(v)\rangle = 0.$$

We solve the equations

$$\frac{dv_1}{ds} = r_1(v(s))$$

and

$$\frac{dv_2}{ds} = r_2(v(s))$$

We have two families of curves in the state space $G \subset \mathbb{R}^2$. We define the Riemann invariants w_j to be functions that are constant on the v_j curves. This requirement is

$$\langle r_1, \nabla_v w_1\rangle = 0$$

and

$$\langle r_2, \nabla_v w_2\rangle = 0.$$

Because the eigenvalues are distinct, the vectors $r_1(v)$ and $r_2(v)$ are linearly independent. It follows that necessarily

$$\nabla_v w_1 = c(v) l_2(v)$$

and

$$\nabla_v w_2 = d(v) l_1(v).$$

Now we take (18) and multiply from the left by the row vector $\nabla_v w_1$. Using the fact that $\nabla_v w_1$ is proportional to $l_2(v)$ we obtain

$$\partial_t(w_1(v)) + \lambda_2(v)\partial_x(w_1(v)) = 0$$

and similarly

$$\partial_t(w_2(v)) + \lambda_1(v)\partial_x(w_2(v)) = 0.$$

Now we introduce the two families of characteristics

$$\frac{dx}{dt} = \lambda_1(v(x,t))$$

and

$$\frac{dx}{dt} = \lambda_2(v(x,t)).$$

We see that $w_1(v(x,t))$ is constant on the λ_2 characteristics, and $w_2(v(x,t))$ is constant on the λ_1 characteristics. Changing variables in state space to w we have the system

$$\begin{cases} \partial_t w_1 + \lambda_2(w)\partial_x w_1 = 0 \\ \partial_t w_2 + \lambda_1(w)\partial_x w_2 = 0. \end{cases} \tag{23}$$

Differentiating the first equation we obatin

$$(\partial_t + \lambda_2\partial_x)(\partial_x w_1) + \frac{\partial \lambda_2}{\partial w_1}(\partial_x w_1)^2 + \frac{\partial \lambda_2}{\partial w_2}(\partial_x w_2)(\partial_x w_1) = 0$$

This is the ODE

$$D_t p + \left(\frac{\partial \lambda_2}{\partial w_1}\right) p^2 + \left(\frac{\partial \lambda_2}{\partial w_2}\right) pq = 0$$

with

$$p = \partial_x w_1, \quad q = \partial_x w_2, \quad D_t = \partial_t + \lambda_2 \partial_x.$$

We use the second equation in (23) to write

$$D_t w_2 + (\lambda_1 - \lambda_2) q = 0$$

that is

$$q = \frac{D_t w_2}{\lambda_2 - \lambda_1}$$

where we use strict hyperbolicity ($\lambda_2 > \lambda_1$). The mixed derivatives group can be written as

$$\left(\frac{\partial \lambda_2}{\partial w_2}\right) pq = p D_t(w_2) \frac{\frac{\partial \lambda_2}{\partial w_2}}{\lambda_2 - \lambda_1}.$$

Let $h(w_1, w_2)$ be a function that satisfies

$$\frac{\partial h}{\partial w_2} = \frac{1}{\lambda_2 - \lambda_1}\frac{\partial \lambda_2}{\partial w_2}.$$

Then, since $D_t w_1 = 0$, we have

$$D_t h = (D_t w_2)(\partial_{w_2} h) = (D_t w_2) \frac{\frac{\partial \lambda_2}{\partial w_2}}{\lambda_2 - \lambda_1}$$

Thus

$$D_t p + \left(\frac{\partial \lambda_2}{\partial w_1}\right) p^2 + p D_t h = 0$$

Multiplying by e^h we obtain for $g = e^h p$

$$D_t g + \left(\frac{\partial \lambda_2}{\partial w_1}\right) e^{-h} g^2 = 0.$$

Integrating on the λ_2 characteristic we find, with

$$k(t) = \int_0^t e^{-h} \frac{\partial \lambda_2}{\partial w_1} ds$$

$$g(t) \frac{g(0)}{1 + k(t) g(0)}$$

which blows up if $k(t)g(0) = -1$. If $\frac{\partial \lambda_2}{\partial w_1} > 0$ then it is easy to see that $k(t) \geq ct$ and blow up occurs if $\partial_x w_1(0) < 0$.