Note on Lagrangian-Eulerian Methods for Uniqueness in Hydrodynamic Systems

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Abstract

We discuss the Lagrangian-Eulerian framework for hydrodynamic models and provide a proof of Lipschitz dependence of solutions on initial data in path space. The paper presents a corrected version of the result in [1].

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1 Introduction

Many hydrodynamical systems consist of evolution equations for fluid velocities forced by external stresses, coupled to evolution equations for the external stresses. In the simplest cases, the Eulerian velocity u can be recovered from the stresses σ via a linear operator

$$u = \mathbb{U}(\sigma) \tag{1}$$

and the stress matrix σ obeys a transport and stretching equation of the form

$$\partial_t \sigma + u \cdot \nabla \sigma = F(\nabla u, \sigma),$$

where F is a nonlinear coupling depending on the model. The Eulerian velocity gradient is obtained in terms of the operator

$$\nabla_x u = \mathbb{G}(\sigma), \tag{2}$$

and, in many cases, $\mathbb{G}$ is bounded in Hölder spaces of low regularity. Then, passing to Lagrangian variables,

$$\tau = \sigma \circ X$$

where X is the particle path transformation $X(\cdot, t) : \mathbb{R}^d \to \mathbb{R}^d$, a volume preserving diffeomorphism, the system becomes

$$\begin{cases} \partial_t X = \mathcal{U}(X, \tau), \\ \partial_t \tau = \mathcal{T}(X, \tau). \end{cases} \tag{3}$$

with

$$\begin{aligned} \mathcal{U}(X, \tau) &= \mathbb{U}(\tau \circ X^{-1}) \circ X, \\ \mathcal{T}(X, \tau) &= F(\mathbb{G}(\tau \circ X^{-1}) \circ X, \tau). \end{aligned} \tag{4}$$

In particular, τ solves an ODE

$$\frac{d}{dt}\tau = F(g, \tau) \tag{5}$$

where $g = \nabla_x u \circ X$ is of the same order of magnitude as τ in appropriate spaces, and so the size of τ is readily estimated from the information provided by the ODE model, analysis of $\mathbb{G}$ and of the

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operation of composition with X. The main additional observation that leads to Lipschitz dependence in path space is that derivatives with respect to parameters of expressions of the type encountered in the Lagrangian evolution (4),

$$\mathbb{U}(\tau \circ X^{-1}) \circ X, \quad \mathbb{G}(\tau \circ X^{-1}) \circ X,$$

introduce commutators, and these are well behaved in spaces of relatively low regularity. The Lagrangian-Eulerian method of [2] formalized these considerations leading to uniqueness and Lipschitz dependence on initial data in path space, with application to several examples including incompressible 2D and 3D Euler equations, the surface quasi-geostrophic equation (SQG), the incompressible porous medium equation, the incompressible Boussinesq system, and the Oldroyd-B system coupled with the steady Stokes system. In all these examples the operators $\mathbb{U}$ and $\mathbb{G}$ are time-independent.

The paper [1] considered time-dependent cases. When the operators $\mathbb{U}$ and $\mathbb{G}$ are time-dependent, in contrast to the time-independent cases studied in [2], $\mathbb{G}$ is not necessarily bounded in $L^\infty(0,T;C^\alpha)$. This was addressed in [1] by using a Hölder continuity $\sigma \in C^\beta(0,T;C^\alpha)$. While this treated the Eulerian issue, it was tacitly used but never explicitly stated in [1] that this kind of Hölder continuity is transferred to σ from τ by composition with a smooth time-depending diffeomorphism close to the identity. This is false. In fact, we can easily give examples of C^α functions τ which are time-independent (hence analytic in time with values in C^α) and diffeomorphisms $X(t)(a) = a + vt$ with constant v, such that $\sigma = \tau \circ X^{-1}$ is not continuous in C^α as a function of time. In this paper we present a correct version of the results in [1]. Instead of relying on the time regularity of τ alone, we also use the fact that $\mathbb{G}$ is composed from a time-independent bounded operator and an operator whose kernel is smooth and rapidly decaying in space. Then the time singularity is resolved by using the Lipschitz dependence in L^1 of Schwartz functions composed with smoothly varying diffeomorphisms near the identity.

A typical example of the systems we can treat is the Oldroyd-B system coupled with Navier-Stokes equations:

$$\begin{cases} \partial_t u - \nu\Delta u = \mathbb{H}\left(\text{div }(\sigma - u\otimes u)\right), \\ \nabla\cdot u = 0, \\ \partial_t\sigma + u\cdot\nabla\sigma = (\nabla u)\sigma + \sigma(\nabla u)^T - 2k\sigma + 2\rho K((\nabla u) + (\nabla u)^T), \\ u(x,0) = u_0(x), \sigma(x,0) = \sigma_0(x). \end{cases} \tag{6}$$

Here $(x,t) \in \mathbb{R}^d \times [0,T)$. The Leray-Hodge projector $\mathbb{H} = \mathbb{I} + R\otimes R$ is given in terms of the Riesz transforms $R = (R_1, \ldots, R_d)$, and $\nu, \rho K, k$ are fixed positive constants. This system is viscoelastic, and the behavior of the solution depends on the history of its deformation.
The non-resistive MHD system

$$\begin{cases} \partial_t u - \nu\Delta u = \mathbb{H}\left(\text{div }(b\otimes b - u\otimes u)\right), \\ \nabla\cdot u = 0, \\ \nabla\cdot b = 0, \\ \partial_t b + u\cdot\nabla b = (\nabla u)b, \\ u(x,0) = u_0(x), b(x,0) = b_0(x). \end{cases} \tag{7}$$

can also be treated by this method. The systems (6) and (7) have been studied extensively, and a review of the literature is beyond the scope of this paper.

2 The Lagrangian-Eulerian formulation

We show calculations for (6) in order to be explicit, and because the calculations for (7) are entirely similar. The solution map for $u(x,t)$ of (6) is

$$u(x,t) = \mathbb{L}_\nu(u_0)(x,t) + \int_0^t g_{\nu(t-s)} * \left(\mathbb{H}\left(\text{div }(\sigma - u\otimes u)\right)\right)(x,s)ds. \tag{8}$$

where

$$\mathbb{L}_\nu(u_0)(x,t) = g_{\nu t} * u_0(x) = \int_{\mathbb{R}^d} \frac{1}{(4\pi\nu t)^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{4\nu t}} u_0(y)dy. \tag{9}$$

Thoroughout the paper we use

$$g_{\nu t}(x) = \frac{1}{(4\pi\nu t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4\nu t}}.$$

The velocity gradient satisfies

$$(\nabla u)(x,t) = \mathbb{L}_\nu(\nabla u_0)(x,t) + \int_0^t \left(g_{\nu(t-s)} * (\mathbb{H}\nabla \mathrm{div}\ (\sigma - u\otimes u))\right)(x,s)ds. \tag{10}$$

We denote the Eulerian velocity and gradient operators

$$\begin{cases} \mathbb{U}(f)(x,t) = \displaystyle\int_0^t (g_{\nu(t-s)} * \mathbb{H}\mathrm{div}\ f)(x,s)ds, \\ \mathbb{G}(f)(x,t) = \displaystyle\int_0^t (g_{\nu(t-s)} * \mathbb{H}\nabla\mathrm{div}\ f)(x,s)ds. \end{cases} \tag{11}$$

Note that for a second order tensor f, $\mathbb{G}(f) = \nabla_x \mathbb{U}(f) = R\otimes R\left(\mathbb{U}(\nabla_x f)\right)$. Let X be the Lagrangian path diffeomorphism, v the Lagrangian velocity, and τ the Lagrangian added stress,

$$\begin{aligned} v = \frac{\partial X}{\partial t} &= u\circ X, \\ \tau &= \sigma\circ X. \end{aligned} \tag{12}$$

We also set

$$\begin{aligned} g(a,t) = (\nabla u)(X(a,t),t) = \mathbb{L}_\nu(\nabla u_0)\circ X(a,t) \\ +\mathbb{G}\left(\tau\circ X^{-1}\right)\circ X(a,t) - \mathbb{U}\left(\nabla_x\left((v\otimes v)\circ X^{-1}\right)\right)\circ X(a,t). \end{aligned} \tag{13}$$

In Lagrangian variables the system is

$$\begin{cases} X(a,t) = a + \displaystyle\int_0^t \mathcal{V}(X,\tau,a,s)ds, \\ \tau(a,t) = \sigma_0(a) + \displaystyle\int_0^t \mathcal{T}(X,\tau,a,s)ds, \\ v(a,t) = \mathcal{V}(X,\tau,t) \end{cases} \tag{14}$$

where the Lagrangian nonlinearities $\mathcal{V},\mathcal{T}$ are

$$\begin{cases} \mathcal{V}(X,\tau,a,s) = \mathbb{L}_\nu(u_0)\circ X(a,s) + (\mathbb{U}\left((\tau - v\otimes v)\circ X^{-1}\right))\circ X(a,s), \\ \mathcal{T}(X,\tau,a,s) = \left(g\tau + \tau g^T - 2k\tau + 2\rho K(g+g^T)\right)(a,s), \end{cases} \tag{15}$$

and g is defined above in (13). The main result of the paper is

Theorem 1. *Let $0<\alpha<1$ and $1<p<\infty$, be given. Let also $v_1(0) = u_1(0) \in C^{1+\alpha,p}$ and $v_2(0) = u_2(0) \in C^{1+\alpha,p}$ be given divergence-free initial velocities, and $\sigma_1(0),\sigma_2(0)\in C^{\alpha,p}$ be given initial stresses. Then there exists $T_0>0$ and $C>0$ depending on the norms of the initial data such that $(X_1,\tau_1,v_1),(X_2,\tau_2,v_2)$, with initial data $(Id,\sigma_1(0),u_1(0)),(Id,\sigma_2(0),u_2(0))$, are bounded in $Id + Lip(0,T_0;C^{1+\alpha,p})\times Lip(0,T_0;C^{\alpha,p})\times L^\infty(0,T_0;C^{1+\alpha,p})$ and solve the Lagrangian form (14) of (6). Moreover,*

$$\begin{aligned} \|X_2 - X_1\|_{Lip(0,T_0;C^{1+\alpha,p})} + \|\tau_2-\tau_1\|_{Lip(0,T_0;C^{\alpha,p})} + \|v_2-v_1\|_{L^\infty(0,T_0,C^{1+\alpha,p})} \\ \le C(\|u_2(0)-u_1(0)\|_{1+\alpha,p} + \|\tau_2(0)-\tau_1(0)\|_{\alpha,p}) \end{aligned} \tag{16}$$

Remark 1. *The solutions' Lagrangian stresses τ are Lipschitz in time with values in C^α. Their Lagrangian counterparts $\sigma = \tau\circ X^{-1}$ are bounded in time with values in C^α and space-time Hölder continuous with exponent α. The Eulerian version of the equations (6) is satisfied in the sense of distributions, and solutions are unique in this class.*

The spaces $C^{\alpha,p}$ are defined in the next section. The proof of the theorem occupies the rest of the paper. We start by considering variations of Lagrangian variables. We take a family $(X_\epsilon, \tau_\epsilon)$ of flow maps depending smoothly on a parameter $\epsilon \in [1,2]$, with initial data $u_{\epsilon,0}$ and $\sigma_{\epsilon,0}$. Note that $v_\epsilon = \partial_t X_\epsilon$. We use the following notations

$$\begin{cases} u_\epsilon = \partial_t X_\epsilon \circ X_\epsilon^{-1}, g'_\epsilon = \dfrac{d}{d\epsilon} g_\epsilon, \\ X'_\epsilon = \dfrac{d}{d\epsilon} X_\epsilon,\ \eta_\epsilon = X'_\epsilon \circ X_\epsilon^{-1}, \\ v'_\epsilon = \dfrac{d}{d\epsilon} v_\epsilon, \\ \sigma_\epsilon = \tau_\epsilon \circ X_\epsilon^{-1}, \\ \tau'_\epsilon = \dfrac{d}{d\epsilon}\tau_\epsilon, \delta_\epsilon = \tau'_\epsilon \circ X_\epsilon^{-1}, \end{cases} \tag{17}$$

and

$$u'_{\epsilon,0} = \frac{d}{d\epsilon} u_\epsilon(0), \sigma'_{\epsilon,0} = \frac{d}{d\epsilon}\sigma_\epsilon(0). \tag{18}$$

We represent

$$\begin{cases} X_2(a,t) - X_1(a,t) = \displaystyle\int_1^2 \mathcal{X}'_\epsilon d\epsilon, \\ \tau_2(a,t) - \tau_1(a,t) = \displaystyle\int_1^2 \pi_\epsilon d\epsilon, \\ v_2(a,t) - v_1(a,t) = \displaystyle\int_1^2 \frac{d}{d\epsilon}\mathcal{V}_\epsilon d\epsilon, \end{cases} \tag{19}$$

where

$$\begin{gathered} \mathcal{X}'_\epsilon = \int_0^t \frac{d}{d\epsilon}\mathcal{V}_\epsilon ds,\ \pi_\epsilon = \int_0^t \frac{d}{d\epsilon}\mathcal{T}_\epsilon ds + \sigma'_{\epsilon,0}, \\ \mathcal{V}_\epsilon = \mathcal{V}(X_\epsilon, \tau_\epsilon),\ \mathcal{T}_\epsilon = \mathcal{T}(X_\epsilon, \tau_\epsilon). \end{gathered} \tag{20}$$

We have the following commutator expressions arising by differentiating in ϵ ([1], [2])):

$$\left(\frac{d}{d\epsilon}\left(\mathbb{U}(\tau_\epsilon \circ X_\epsilon^{-1}) \circ X_\epsilon\right)\right) \circ X_\epsilon^{-1} = [\eta_\epsilon \cdot \nabla_x, \mathbb{U}](\sigma_\epsilon) + \mathbb{U}(\delta_\epsilon), \tag{21}$$

where

$$[\eta_\epsilon \cdot \nabla_x, \mathbb{U}](\sigma_\epsilon) = \eta_\epsilon \cdot \nabla_x \left(\mathbb{U}(\sigma_\epsilon)\right) - \mathbb{U}\left(\eta_\epsilon \cdot \nabla_x \sigma_\epsilon\right) \tag{22}$$

and

$$\begin{aligned} &\left(\frac{d}{d\epsilon}\mathbb{U}(v_\epsilon \otimes v_\epsilon \circ X_\epsilon^{-1}) \circ X_\epsilon\right) \circ X_\epsilon^{-1} \\ &= [\eta_\epsilon \cdot \nabla_x, \mathbb{U}](u_\epsilon \otimes u_\epsilon) + \mathbb{U}((v'_\epsilon \otimes v_\epsilon + v_\epsilon \otimes v'_\epsilon) \circ X_\epsilon^{-1}). \end{aligned} \tag{23}$$

We note, by the chain rule,

$$\nabla_a \mathcal{V} = (\nabla_a X)\, g. \tag{24}$$

Consequently, differentiating $\mathcal{V}_\epsilon$, g_ϵ and the relation (24) we have

$$\begin{cases} \left(\dfrac{d}{d\epsilon}\mathcal{V}_\epsilon\right) \circ X_\epsilon^{-1} = \eta_\epsilon \cdot (\mathbb{L}_\nu(\nabla_x u_{\epsilon,0})) + \mathbb{L}_\nu(u'_{\epsilon,0}) \\ +[\eta_\epsilon \cdot \nabla_x, \mathbb{U}](\sigma_\epsilon - u_\epsilon \otimes u_\epsilon) + \mathbb{U}(\delta_\epsilon - (v'_\epsilon \otimes v_\epsilon + v_\epsilon \otimes v'_\epsilon) \circ X_\epsilon^{-1}), \\ g_\epsilon = \mathbb{L}(\nabla_x u_{\epsilon,0}) \circ X_\epsilon + \mathbb{G}(\sigma_\epsilon) \circ X_\epsilon - \mathbb{U}(\nabla_x (u_\epsilon \otimes u_\epsilon)) \circ X_\epsilon, \\ g'_\epsilon \circ X_\epsilon^{-1} = \eta_\epsilon \cdot \mathbb{L}_\nu(\nabla_x \nabla_x u_{\epsilon,0}) + \mathbb{L}_\nu(\nabla_x u'_{\epsilon,0}) + [\eta_\epsilon \cdot \nabla_x, \mathbb{G}](\sigma_\epsilon) + \mathbb{G}(\delta_\epsilon) \\ -[\eta_\epsilon \cdot \nabla_x, \mathbb{U}]\left(\nabla_x (u_\epsilon \otimes u_\epsilon)\right) - \mathbb{U}\left(\nabla_x\left((v'_\epsilon \otimes v_\epsilon + v_\epsilon \otimes v'_\epsilon) \circ X_\epsilon^{-1}\right)\right), \\ \dfrac{d}{d\epsilon}(\nabla_a \mathcal{V}_\epsilon) = (\nabla_a X'_\epsilon) g_\epsilon + (\nabla_a X_\epsilon) g'_\epsilon, \\ \dfrac{d}{d\epsilon}\mathcal{T}_\epsilon = g'_\epsilon \tau_\epsilon + g_\epsilon \tau'_\epsilon + \tau'_\epsilon g_\epsilon^T + \tau_\epsilon (g'_\epsilon)^T - 2k\tau'_\epsilon + 2\rho K(g'_\epsilon + (g'_\epsilon)^T). \end{cases} \tag{25}$$

3 Functions, operators, commutators

We consider function spaces

$$C^{\alpha,p} = C^{\alpha}(\mathbb{R}^d) \cap L^p(\mathbb{R}^d) \tag{26}$$

with norm

$$\|f\|_{\alpha,p} = \|f\|_{C^{\alpha}(\mathbb{R}^d)} + \|f\|_{L^p(\mathbb{R}^d)} \tag{27}$$

for $\alpha \in (0,1), p \in (1,\infty)$, $C^{1+\alpha}(\mathbb{R}^d)$ with norm

$$\|f\|_{C^{1+\alpha}(\mathbb{R}^d)} = \|f\|_{L^\infty(\mathbb{R}^d)} + \|\nabla f\|_{C^{\alpha}(\mathbb{R}^d)}, \tag{28}$$

and

$$C^{1+\alpha,p} = C^{1+\alpha}(\mathbb{R}^d) \cap W^{1,p}(\mathbb{R}^d) \tag{29}$$

with norm

$$\|f\|_{1+\alpha,p} = \|f\|_{C^{1+\alpha}(\mathbb{R}^d)} + \|f\|_{W^{1,p}(\mathbb{R}^d)}. \tag{30}$$

We also use spaces of paths, $L^\infty(0,T;Y)$ with the usual norm,

$$\|f\|_{L^\infty(0,T;Y)} = \sup_{t\in[0,T]} \|f(t)\|_Y, \tag{31}$$

spaces $Lip(0,T;Y)$ with norm

$$\|f\|_{Lip(0,T;Y)} = \sup_{t\neq s, t,s\in[0,T]} \frac{\|f(t)-f(s)\|_Y}{|t-s|} + \|f\|_{L^\infty(0,T;Y)} \tag{32}$$

where Y is $C^{\alpha,p}$ or $C^{1+\alpha,p}$ in the following. We use the following lemmas.

Lemma 1 ([2]). *Let $0<\alpha<1$, $1<p<\infty$. Let $\eta \in C^{1+\alpha}(\mathbb{R}^d)$ and let*

$$(\mathbb{K}\sigma)(x) = P.V.\int_{\mathbb{R}^d} k(x-y)\sigma(y)dy \tag{33}$$

be a classical Calderon-Zygmund operator with kernel k which is smooth away from the origin, homogeneous of degree $-d$ and with mean zero on spheres about the origin. Then the commutator $[\eta\cdot\nabla, \mathbb{K}]$ can be defined as a bounded linear operator in $C^{\alpha,p}$ and

$$\|[\eta\cdot\nabla,\mathbb{K}]\sigma\|_{C^{\alpha,p}} \le C\,\|\eta\|_{C^{1+\alpha}(\mathbb{R}^d)}\,\|\sigma\|_{C^{\alpha,p}}. \tag{34}$$

Lemma 2 (Generalized Young's inequality). *Let $1\le q\le\infty$ and $C>0$. Suppose K is a measurable function on $\mathbb{R}^d\times\mathbb{R}^d$ such that*

$$\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}|K(x,y)|dy \le C,\ \ \sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}|K(x,y)|dx \le C. \tag{35}$$

If $f\in L^q(\mathbb{R}^d)$, the function Tf defined by

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y)f(y)dy \tag{36}$$

is well defined almost everywhere and is in L^q, and $\|Tf\|_{L^q} \le C\,\|f\|_{L^q}$.

The proof of this lemma for $1<q<\infty$ is done using duality, a straightforward application of Young's inequality and changing order of integration. The extreme cases $q=1$ and $q=\infty$ are proved directly by inspection.

For simplicity of notation, let us denote

$$M_X = 1 + \|X - \mathrm{Id}\|_{L^\infty(0,T;C^{1+\alpha})}. \tag{37}$$

Theorem 2. *Let $0<\alpha<1, 1<p<\infty$ and let $T>0$. Also let X be a volume preserving diffeomorphism such that $X-\mathrm{Id}\in Lip(0,T;C^{1+\alpha})$. Then*

$$\left\|\tau\circ X^{-1}\right\|_{L^\infty(0,T;C^{\alpha,p})}\le\|\tau\|_{L^\infty(0,T;C^{\alpha,p})}M_X^\alpha. \tag{38}$$

If $X'\in Lip(0,T;C^{1+\alpha})$, then

$$\left\|X'\circ X^{-1}\right\|_{L^\infty(0,T;C^{1+\alpha})}\le\|X'\|_{L^\infty(0,T;C^{1+\alpha})}M_X^{1+2\alpha}. \tag{39}$$

If $v\in Lip(0,T;W^{1,p})$, then

$$\left\|v\circ X^{-1}\right\|_{L^\infty(0,T;W^{1,p})}\le\|v\|_{L^\infty(0,T;W^{1,p})}M_X. \tag{40}$$

If in addition ∂_tX', ∂_tX exist in $L^\infty(0,T;C^{1+\alpha})$, then

$$\left\|X'\circ X^{-1}\right\|_{Lip(0,T;C^\alpha)}\le\|X'\|_{Lip(0,T;C^{1+\alpha})}\|X-\mathrm{Id}\|_{Lip(0,T;C^{1+\alpha})}M_X^{1+3\alpha}. \tag{41}$$

Proof.

$$\left\|\tau\circ X^{-1}\right\|_{L^p\cap L^\infty}=\|\tau\|_{L^p\cap L^\infty}, \tag{42}$$

and, denoting the seminorm

$$[\tau]_\alpha=\sup_{a\neq b,a,b\in\mathbb{R}^2}\frac{|\tau(a)-\tau(b)|}{|a-b|^\alpha}$$

we have

$$\left[\tau\circ X^{-1}(t)\right]_\alpha\le[\tau(t)]_\alpha\left\|\nabla_xX^{-1}(t)\right\|_{L^\infty}^\alpha\le[\tau(t)]_\alpha\left(1+\|X-\mathrm{Id}\|_{L^\infty(0,T;C^{1+\alpha})}\right)^\alpha. \tag{43}$$

Note that this shows that the same bound holds when we replace X^{-1} by X. For the second and third part, it suffices to remark that

$$\nabla_x(X'\circ X^{-1})=\left((\nabla_aX)\circ X^{-1}\right)^{-1}\left((\nabla_aX')\circ X^{-1}\right) \tag{44}$$

and the previous part gives the bound in terms of Lagrangian variables. For the last part, we note that

$$\begin{aligned}&\frac{1}{t-s}\left(X'\left(X^{-1}(x,t),t\right)-X'\left(X^{-1}(x,s),s\right)\right)\\&=\int_0^1\left((\partial_tX')\left(X^{-1}(x,\beta_\tau),\beta_\tau\right)+\left(\partial_tX^{-1}\right)(x,\beta_\tau)(\nabla_aX')\left(X^{-1}(x,\beta_\tau),\beta_\tau\right)\right)d\tau,\end{aligned} \tag{45}$$

where

$$\beta_\tau=\tau t+(1-\tau)s. \tag{46}$$

Now noting that

$$\partial_tX^{-1}=-\left((\partial_tX)\circ X^{-1}\right)\left((\nabla_aX)^{-1}\circ X^{-1}\right) \tag{47}$$

we have

$$\begin{aligned}&\frac{1}{t-s}\left\|X'\circ X^{-1}(t)-X'\circ X^{-1}(s)\right\|_{C^\alpha}\\&\le\left(\|\partial_tX'\|_{L^\infty(0,T;C^\alpha)}+\|\partial_tX\|_{L^\infty(0,T;C^\alpha)}\|X'\|_{L^\infty(0,T;C^{1+\alpha})}\right)\left(1+\|X-\mathrm{Id}\|_{L^\infty(0,T;C^{1+\alpha})}\right)^{1+3\alpha}\end{aligned} \tag{48}$$

so that

$$\left\|X'\circ X^{-1}\right\|_{Lip(0,T;C^\alpha)}\le\|X'\|_{Lip(0,T;C^{1+\alpha})}\|X-\mathrm{Id}\|_{Lip(0,T;C^{1+\alpha})}\left(1+\|X-\mathrm{Id}\|_{L^\infty(0,T;C^{1+\alpha})}\right)^{1+3\alpha}. \tag{49}$$

□

Theorem 3. *Let $0 < \alpha < 1, 1 < p < \infty$ and let $T > 0$. There exists a constant C independent of T and ν such that for any $0 < t < T$,*

$$\begin{aligned}
\|\mathbb{L}_\nu(u_0)\|_{L^\infty(0,T;C^{\alpha,p})} &\le C\,\|u_0\|_{\alpha,p}\,,\\
\|\mathbb{L}_\nu(u_0)\|_{L^\infty(0,T;C^{1+\alpha,p})} &\le C\,\|u_0\|_{1+\alpha,p}\,,\\
\|\mathbb{L}_\nu(\nabla u_0)(t)\|_{\alpha,p} &\le \frac{C}{(\nu t)^{\frac12}}\,\|u_0\|_{\alpha,p}\,,\\
\|\mathbb{L}_\nu(\nabla u_0)\|_{L^\infty(0,T;C^{\alpha,p})} &\le C\,\|u_0\|_{1+\alpha,p}
\end{aligned} \tag{50}$$

hold.

Proof.

$$\begin{aligned}
\|\mathbb{L}_\nu(u_0)(t)\|_{\alpha,p} &\le \|g_{\nu t}\|_{L^1}\,\|u_0\|_{\alpha,p} = \|u_0\|_{\alpha,p}\,,\\
\|\mathbb{L}_\nu(u_0)(t)\|_{1+\alpha,p} &\le \|g_{\nu t}\|_{L^1}\,\|u_0\|_{1+\alpha,p} = \|u_0\|_{1+\alpha,p}\,,\\
\|\mathbb{L}_\nu(\nabla u_0)(t)\|_{\alpha,p} &\le \|\nabla g_{\nu t}\|_{L^1}\,\|u_0\|_{1+\alpha,p} = \frac{C}{(\nu t)^{\frac12}}\,\|u_0\|_{\alpha,p}\,,\\
\|\mathbb{L}_\nu(\nabla u_0)(t)\|_{\alpha,p} &\le \|g_{\nu t}\|_{L^1}\,\|\nabla u_0\|_{\alpha,p} \le \|u_0\|_{1+\alpha,p}\,.
\end{aligned} \tag{51}$$

□

Theorem 4. *Let $0 < \alpha < 1, 1 < p < \infty$ and let $T > 0$. There exists a constant C such that*

$$\|\mathbb{U}(\sigma)\|_{L^\infty(0,T;C^{\alpha,p})} \le C\left(\frac{T}{\nu}\right)^{\frac12}\|\sigma\|_{L^\infty(0,T;C^{\alpha,p})}\,. \tag{52}$$

Proof.

$$\begin{aligned}
\|\mathbb{U}(\sigma)(t)\|_{C^{\alpha,p}} &\le C\int_0^t \left\|\nabla g_{\nu(t-s)}\right\|_{L^1}\|\sigma(s)\|_{\alpha,p}\,ds\\
\le \frac{C}{\nu^{\frac12}}\int_0^t \frac{1}{(t-s)^{\frac12}}ds\,\|\sigma\|_{L^\infty(0,T;C^{\alpha,p})} &\le \frac{C}{\nu^{\frac12}}\sqrt{T}\,\|\sigma\|_{L^\infty(0,T;C^{\alpha,p})}\,.
\end{aligned} \tag{53}$$

□

Theorem 5. *Let $0 < \alpha < 1, 1 < p < \infty$ and let $T > 0$. There exist constants C_1, C_2 depending only on α and ν, and $C_3(T,X), C_4(T,X)$ such that*

$$\begin{aligned}
\left\|\mathbb{G}(\tau\circ X^{-1})\right\|_{L^\infty(0,T;C^{\alpha,p})} \le C_1\,&\|X-\mathrm{Id}\|^\alpha_{Lip(0,T;C^{1+\alpha})}\,\|\tau(0)\|_{\alpha,p}\,(1+C_3(T,X))\\
&+C_2\,\|\tau\|_{Lip(0,T;C^{\alpha,p})}\,C_4(T,X)
\end{aligned} \tag{54}$$

where $C_3(T,X)$ and $C_4(T,X)$ are of the form $CT^{\frac12}\left(\|X-\mathrm{Id}\|^\alpha_{Lip(0,T;C^{1+\alpha})} + \|X-\mathrm{Id}\|^4_{Lip(0,T;C^{1+\alpha})}\right)$.

Proof. Since $\mathbb{G} = (R\otimes R)\mathbb{H}\Gamma$ where

$$\Gamma(\tau\circ X^{-1}) = \int_0^t \Delta g_{\nu(t-s)} * (\tau\circ X^{-1}(s))ds, \tag{55}$$

we can replace $\mathbb{G}$ by Γ. Then $\Gamma(\tau\circ X^{-1})$ can be written as

$$\begin{aligned}
\Gamma(\tau\circ X^{-1})(t) = &\int_0^t \Delta g_{\nu(t-s)} * \left(\left(\tau\circ X^{-1}\right)(s) - \left(\tau\circ X^{-1}\right)(t)\right)ds\\
&+\int_0^t \Delta g_{\nu(t-s)} * \left(\tau\circ X^{-1}\right)(t)ds.
\end{aligned} \tag{56}$$

But

$$\int_0^t \Delta g_{\nu(t-s)} * (\tau\circ X^{-1})(t)ds = \tau\circ X^{-1}(t) - g_{\nu t} * (\tau\circ X^{-1})(t) \tag{57}$$

so the second term is bounded by $2\|\tau\|_{L^\infty(0,T;C^{\alpha,p})} M_X^\alpha$ by Theorem 2. Now we let

$$\tau \circ X^{-1}(x,s) - \tau \circ X^{-1}(x,t) = \Delta_1\tau(x,s,t) + \Delta_2\tau(x,s,t), \tag{58}$$

where

$$\begin{aligned} \Delta_1\tau(x,s,t) &= \tau(X^{-1}(x,s),s) - \tau(X^{-1}(x,s),t), \\ \Delta_2\tau(x,s,t) &= \tau(X^{-1}(x,s),t) - \tau(X^{-1}(x,t),t). \end{aligned} \tag{59}$$

But since

$$\|\Delta_1\tau(s,t)\|_{C^{\alpha,p}} \le |t-s| M_X^\alpha \|\tau\|_{Lip(0,T;C^{\alpha,p})}, \tag{60}$$

by the proof of Theorem 2 we get

$$\left\|\int_0^t \Delta g_{\nu(t-s)} * \Delta_1\tau(s,t)ds\right\|_{\alpha,p} \le \frac{Ct}{\nu}\|\tau\|_{Lip(0,T;C^{\alpha,p})} M_X^\alpha, \tag{61}$$

On the other hand,

$$\int_0^t \Delta g_{\nu(t-s)} * \Delta_2\tau(s,t)ds = \int_0^t \int_{\mathbb{R}^d} K(x,z,t,s)\tau(z,t)dzds, \tag{62}$$

where

$$K(x,z,t,s) = \Delta g_{\nu(t-s)}(x - X(z,s)) - \Delta g_{\nu(t-s)}(x - X(z,t)). \tag{63}$$

We use the following lemma.

Lemma 3. *$K(x,z,t,s)$ is L^1 in both the x variable and the z variable, and*

$$\sup_z \|K(\cdot,z,t,s)\|_{L^1}, \sup_x \|K(x,\cdot,t,s)\|_{L^1} \le \frac{C\|X - \mathrm{Id}\|_{Lip(0,T;L^\infty)}}{|t-s|^{\frac{1}{2}}\nu^{\frac{3}{2}}}. \tag{64}$$

Proof. We define

$$S(x) = 4\pi e^{-|x|^2}\left(|x|^2 - \frac{d}{2}\right) \tag{65}$$

so that

$$(\Delta g_{\nu(t-s)}) = (4\pi\nu(t-s))^{-(\frac{d}{2}+1)} S\left(\frac{x}{(4(t-s))^{\frac{1}{2}}}\right). \tag{66}$$

Then

$$\begin{aligned} \int |K(x,z,t,s)|dz &= \int (4\pi\nu(t-s))^{-(\frac{d}{2}+1)}\left|S(\frac{x-X(z,s)}{(4\nu(t-s))^{\frac{1}{2}}}) - S(\frac{x-X(z,t)}{(4\nu(t-s))^{\frac{1}{2}}})\right| dz \\ &= \int (4\pi\nu(t-s))^{-(\frac{d}{2}+1)}\left|S(\frac{x-y}{(4\nu(t-s))^{\frac{1}{2}}}) - S(\frac{x-X(y,t-s)}{(4\nu(t-s))^{\frac{1}{2}}})\right| dy \\ &= (4\pi\nu(t-s))^{-1}\pi^{-(\frac{d}{2}+1)}\int\left|S(u) - S\left(u - \frac{(X-\mathrm{Id})(x-(4(t-s))^{\frac{1}{2}}u, t-s)}{(4\nu(t-s))^{\frac{1}{2}}}\right)\right| du. \end{aligned} \tag{67}$$

However, for each u

$$\begin{aligned} \left|S(u) - S\left(u - \frac{(X-\mathrm{Id})(x-(4\nu(t-s))^{\frac{1}{2}}u,t-s)}{(4\nu(t-s))^{\frac{1}{2}}}\right)\right| &\le \left|\frac{(X-\mathrm{Id})(x-(4\nu(t-s))^{\frac{1}{2}}u,t-s)}{(4\nu(t-s))^{\frac{1}{2}}}\right| \\ \times \sup\left\{|\nabla S(u-z)| : |z| \le \left|\frac{(X-\mathrm{Id})(x-(4\nu(t-s))^{\frac{1}{2}}u,t-s)}{(4\nu(t-s))^{\frac{1}{2}}}\right|\right\} \end{aligned} \tag{68}$$

and we have

$$\left|\frac{(X-\mathrm{Id})(x-(4\nu(t-s))^{\frac{1}{2}}u,t-s)}{(4\nu(t-s))^{\frac{1}{2}}}\right| \le \|(X-\mathrm{Id})\|_{Lip(0,T;L^\infty)}\frac{|t-s|^{\frac{1}{2}}}{\nu^{\frac{1}{2}}} \le CT^{\frac{1}{2}} \tag{69}$$

and obviously

$$\tilde{S}(u) = \sup_{z \le CT^{\frac{1}{2}}} |(\nabla S)(u-z)| \tag{70}$$

is integrable in $\mathbb{R}^d$; because ∇S is Schwartz,

$$|(\nabla S)(x)| \le \frac{C_d}{(1+2C^2T+|x|^2)^d} \tag{71}$$

for some constant C_d, but if $|z| \le CT^{\frac{1}{2}}$, then $|u-z|^2 \ge |u|^2 - C^2T$ and

$$|(\nabla S)(u-z)| \le \frac{C_d}{(1+C^2T+|u|^2)^d} \tag{72}$$

and the right side of above is clearly integrable with bound depending only on d and T. Therefore, we have

$$\int |K(x,z,t,s)|dz \le |t-s|^{-\frac{1}{2}}\nu^{-\frac{3}{2}} \left\|(X-\mathrm{Id})\right\|_{Lip(0,T;L^\infty)} C(d,T). \tag{73}$$

Similarly,

$$\begin{aligned}\int |K(x,z,t,s)|dx &= \int (4\pi\nu(t-s))^{-(\frac{d}{2}+1)} \left| S(\frac{x-X(z,s)}{(4\nu(t-s))^{\frac{1}{2}}}) - S(\frac{x-X(z,t)}{(4\nu(t-s))^{\frac{1}{2}}}) \right| dx \\ &= \int (4\pi\nu(t-s))^{-1}\pi^{-(\frac{d}{2}+1)} \left| S(y) - S(y+\frac{X(z,s)-X(z,t)}{(4\nu(t-s))^{\frac{1}{2}}}) \right| dy\end{aligned} \tag{74}$$

and again we have

$$\left| \frac{X(z,s)-X(z,t)}{(4\nu(t-s))^{\frac{1}{2}}} \right| \le \left\|(X-\mathrm{Id})\right\|_{Lip(0,T;L^\infty)} |t-s|^{\frac{1}{2}}\nu^{-\frac{1}{2}} \le CT^{\frac{1}{2}}. \tag{75}$$

Therefore, we have the bound

$$\int |K(x,z)|dx \le |t-s|^{-\frac{1}{2}}\nu^{-\frac{3}{2}} \left\|(X-\mathrm{Id})\right\|_{Lip(0,T;L^\infty)} C(d,T). \tag{76}$$

□

From Lemma 3 and generalized Young's inequality, we have

$$\left\| \int_0^t \Delta g_{\nu(t-s)} * \Delta_2\tau(s,t)ds \right\|_{L^p\cap L^\infty} \le \frac{C}{\nu}\left(\left(\frac{t}{\nu}\right)^{\frac{1}{2}} \left\|X-\mathrm{Id}\right\|_{Lip(0,T;C^{1+\alpha})} \right) \left\|\tau\right\|_{L^\infty(0,T;L^p\cap L^\infty)}. \tag{77}$$

For the Hölder seminorm, we measure the finite difference. Let us denote $\delta_h f(x,t) = f(x+h,t) - f(x,t)$. If $|h| < t$, then

$$\delta_h\left(\int_0^t \Delta g_{\nu(t-s)} * \Delta_2\tau(s,t)ds \right) = \int_0^t \delta_h(\Delta g_{\nu(t-s)}) * \Delta_2\tau(s,t)ds. \tag{78}$$

If $0 < t-s < |h|$, then $\left\|\delta_h \Delta g_{\nu(t-s)}\right\|_{L^1} \le 2\left\|\Delta g_{\nu(t-s)}\right\|_{L^1} \le \frac{C}{\nu(t-s)}$ and since

$$\left\|\Delta_2\tau(s,t)\right\|_{L^\infty} \le |t-s|^\alpha \left\|X-\mathrm{Id}\right\|^\alpha_{Lip(0,T;C^{1+\alpha})} \left\|\tau\right\|_{L^\infty(0,T;C^{\alpha,p})} \tag{79}$$

we have

$$\left\| \int_{t-|h|}^t \delta_h(\Delta g_{\nu(t-s)}) * \Delta_2\tau(s,t)ds \right\|_{L^\infty} \le \frac{C}{\nu\alpha}|h|^\alpha \left\|X-\mathrm{Id}\right\|^\alpha_{Lip(0,T;C^{1+\alpha})} \left\|\tau\right\|_{L^\infty(0,T;C^{\alpha,p})}. \tag{80}$$

If $|h| < t-s < t$, then following lines of Lemma 3 $\delta_h(\Delta g_{\nu(t-s)})$ is a L^1 function with

$$\left\|\delta_h(\Delta g_{\nu(t-s)})\right\|_{L^1} \le \frac{C|h|}{(\nu(t-s))^{\frac{3}{2}}} \tag{81}$$

and we have

$$
\begin{aligned}
&\left\| \int_0^{t-|h|} \delta_h(\Delta g_{\nu(t-s)}) * \Delta_2 \tau(s,t) ds \right\|_{L^\infty} \\
&\le \begin{cases} \frac{C}{\nu^{\frac{3}{2}}} \left\| X - \mathrm{Id} \right\|^{\alpha}_{Lip(0,T;C^{1+\alpha})} \left\| \tau \right\|_{L^\infty(0,T;C^{\alpha,p})} |h|^{\frac{1}{2}} \frac{t^\alpha}{\alpha} & \alpha \le \frac{1}{2}, \\ \frac{C}{\nu^{\frac{3}{2}}} \left\| X - \mathrm{Id} \right\|^{\alpha}_{Lip(0,T;C^{1+\alpha})} \left\| \tau \right\|_{L^\infty(0,T;C^{\alpha,p})} |h| \frac{t^{\alpha-\frac{1}{2}}}{\alpha - \frac{1}{2}} & \alpha > \frac{1}{2}. \end{cases}
\end{aligned} \tag{82}
$$

If $|h| \ge t$, then we only have the first term. Therefore, we have

$$
\frac{1}{|h|^\alpha} \left\| \delta_h \left(\int_0^t \Delta g_{\nu(t-s)} * \Delta_2 \tau(s,t) ds \right) \right\|_{L^\infty} \le \frac{C(\alpha)}{\nu} \left\| X - \mathrm{Id} \right\|^{\alpha}_{Lip(0,T;C^{1+\alpha})} \left\| \tau \right\|_{L^\infty(0,T;C^{\alpha,p})}. \tag{83}
$$

We note that

$$
\left\| \tau(t) \right\|_{\alpha,p} \le \left\| \tau(0) \right\|_{\alpha,p} + t \left\| \tau \right\|_{Lip(0,T;C^{\alpha,p})}. \tag{84}
$$

To summarize, we have

$$
\begin{aligned}
&\left\| \Gamma(\tau \circ X^{-1}) \right\|_{L^\infty(0,T;C^{\alpha,p})} \\
&\le C(\alpha) \left(1 + \frac{1}{\nu} \right) \left\| X - \mathrm{Id} \right\|^{\alpha}_{Lip(0,T;C^{1+\alpha})} \left\| \tau(0) \right\|_{\alpha,p} + C(\alpha) \left(1 + \frac{1}{\nu} \right) \left\| X - \mathrm{Id} \right\|^{\alpha}_{Lip(0,T;C^{1+\alpha})} T \left\| \tau \right\|_{Lip(0,T;C^{\alpha,p})} \\
&\quad + \frac{C(\alpha)}{\nu} \left(\frac{T}{\nu} \right)^{\frac{1}{2}} \max\{ \left\| X - \mathrm{Id} \right\|^{\alpha}_{Lip(0,T;C^{1+\alpha})}, \left\| X - \mathrm{Id} \right\|^{4}_{Lip(0,T;C^{1+\alpha})} \} (\left\| \tau(0) \right\|_{\alpha,p} + T \left\| \tau \right\|_{Lip(0,T;C^{\alpha,p})}),
\end{aligned} \tag{85}
$$

and this completes the proof. □

Theorem 6. *Let $0 < \alpha < 1, 1 < p < \infty$ and let $T > 0$. Let $X' \in Lip(0,T;C^{1+\alpha})$ with $\partial_t X' \in L^\infty(0,T;C^{1+\alpha})$. There exists a constant C such that*

$$
\begin{aligned}
&\left\| \left[X' \circ X^{-1} \cdot \nabla, \mathbb{U} \right] (\sigma) \right\|_{L^\infty(0,T;C^{\alpha,p})} \\
&\le C \left(\left(\frac{T}{\nu} \right)^{\frac{1}{2}} + \frac{T}{\nu} \left\| X - \mathrm{Id} \right\|_{Lip(0,T;C^{1+\alpha})} \right) M_X^{1+3\alpha} \left\| X' \right\|_{Lip(0,T;C^{1+\alpha})} \left\| \sigma \right\|_{L^\infty(0,T;C^{\alpha,p})}
\end{aligned} \tag{86}
$$

Proof. First, we denote

$$
\eta = X' \circ X^{-1}. \tag{87}
$$

Then we have

$$
\begin{aligned}
&\left[\eta \cdot \nabla, \mathbb{U} \right] (\sigma)(t) \\
&= \eta(t) \cdot \nabla \int_0^t g_{\nu(t-s)} * \mathbb{H} \mathrm{div}\, \sigma(s) ds - \int_0^t g_{\nu(t-s)} * \mathbb{H} \mathrm{div}\, (\eta(s) \cdot \nabla \sigma(s)) ds \\
&= \left[\eta(t) \cdot \nabla, \mathbb{H} \right] \int_0^t g_{\nu(t-s)} * \mathrm{div}\, \sigma(s) ds + \mathbb{H} \int_0^t (\nabla g_{\nu(t-s)}) * (\nabla \cdot \eta(s) \sigma(s)) \, ds \\
&\quad - \mathbb{H} \int_0^t (\nabla \nabla g_{\nu(t-s)}) * (\eta(s) - \eta(t)) \, \sigma(s) ds \\
&\quad + \mathbb{H} \int_0^t \left(\eta(t) \cdot (\nabla \nabla g_{\nu(t-s)}) * \sigma(s) - (\nabla \nabla g_{\nu(t-s)}) * (\eta(t) \sigma(s)) \right) ds,
\end{aligned} \tag{88}
$$

where $(\nabla \nabla g_{\nu(t-s)}) * (\eta(s) - \eta(t)) \sigma(s)$, $\eta(t) \cdot (\nabla \nabla g_{\nu(t-s)}) * \sigma(s)$, and $(\nabla \nabla g_{\nu(t-s)}) * (\eta(s) \sigma(s))$ represent

$$
\begin{aligned}
&\sum_{i,j} (\partial_i \partial_j g_{\nu(t-s)} *) (\eta_i(s) - \eta_i(t)) \sigma_{jk}(s), \\
&\sum_{i,j} \eta_i(t) \left(\partial_i \partial_j g_{\nu(t-s)} \right) * \sigma_{jk}(s), \text{ and respectively} \sum_{i,j} \left(\partial_i \partial_j g_{\nu(t-s)} \right) * (\eta_i(s) \sigma_{jk}(s)).
\end{aligned} \tag{89}
$$

The first term is bounded by Lemma 1 and the second term is estimated directly

$$\begin{aligned}\left\|[\eta(t)\cdot\nabla,\mathbb{H}]\int_0^t g_{\nu(t-s)}*\mathrm{div}\sigma(s)ds\right\|_{\alpha,p} &\le C\,\|\eta(t)\|_{C^{1+\alpha}}\left(\frac{t}{\nu}\right)^{\frac12}\|\sigma\|_{L^\infty(0,T;C^{\alpha,p})}\,,\\ \left\|\mathbb{H}\int_0^t(\nabla g_{\nu(t-s)})*(\nabla\cdot\eta(s)\sigma(s))ds\right\|_{\alpha,p} &\le C\left(\frac{t}{\nu}\right)^{\frac12}\|\eta\|_{L^\infty(0,T;C^{1+\alpha})}\|\sigma\|_{L^\infty(0,T;C^{\alpha,p})}\,.\end{aligned} \tag{90}$$

The third term is bounded by

$$\frac{Ct}{\nu}\|\eta\|_{Lip(0,T;C^\alpha)}\|\sigma\|_{L^\infty(0,T;C^{\alpha,p})} \tag{91}$$

by the virtue of Theorem 2. For the last term, note that

$$\begin{aligned}&\left(\eta(t)\cdot(\nabla\nabla g_{\nu(t-s)})*\sigma(s)-(\nabla\nabla g_{\nu(t-s)})*(\eta(t)\sigma(s))\right)(x)\\ &=\int_{\mathbb{R}^d}\nabla\nabla g_{\nu(t-s)}(z)z\cdot\left(\int_0^1\nabla\eta(x-(1-\lambda)z,t)d\lambda\right)\sigma(x-z,s)dz\end{aligned} \tag{92}$$

and note that $\nabla\nabla g_{\nu(t-s)}(z)z$ is a L^1 function with

$$\left\|\nabla\nabla g_{\nu(t-s)}(z)z\right\|_{L^1}\le\frac{C}{(\nu(t-s))^{\frac12}}. \tag{93}$$

Therefore,

$$\begin{aligned}&\left\|\left(\eta(t)\cdot(\nabla\nabla g_{\nu(t-s)})*\sigma(s)-(\nabla\nabla g_{\nu(t-s)})*(\eta(t)\sigma(s))\right)\right\|_{\alpha,p}\\ &\le\frac{C}{(\nu(t-s))^{\frac12}}\|\eta(t)\|_{C^{1+\alpha}}\|\sigma(s)\|_{\alpha,p}\end{aligned} \tag{94}$$

so that the last term is bounded by

$$C\left(\frac{t}{\nu}\right)^{\frac12}\|\eta(t)\|_{C^{1+\alpha}}\|\sigma\|_{L^\infty(0,T;C^{\alpha,p})}\,. \tag{95}$$

We finish the proof by replacing η by X' using Theorem 2. □

Theorem 7. *Let $0<\alpha<1$, $1<p<\infty$ and let $T>0$. Let $X'\in Lip(0,T;C^{1+\alpha})$ with $\partial_t X'\in L^\infty(0,T;C^{1+\alpha})$. There exists a constant $C(\alpha)$ depending only on α such that*

$$\begin{aligned}&\left\|\left[X'\circ X^{-1}\cdot\nabla,\mathbb{G}\right](\tau\circ X^{-1})\right\|_{L^\infty(0,T;C^{\alpha,p})}\\ &\le(\|X'\|_{L^\infty(0,T;C^{1+\alpha})}+\|X'\|_{Lip(0,T;C^{1+\alpha})}T^{\frac12})R\end{aligned} \tag{96}$$

where R is a polynomial function on $\|\tau\|_{Lip(0,T;C^{\alpha,p})}$, $\|X-\mathrm{Id}\|_{Lip(0,T;C^{1+\alpha})}$, whose coefficients depend on α, ν, and T, and in particular it grows polynomially in T and bounded below.

Proof. Again we denote $\eta=X'\circ X^{-1}$. Also it suffices to bound

$$[\eta\cdot\nabla,\Gamma]\left(\tau\circ X^{-1}\right)=\eta(t)\cdot\nabla\Gamma\left(\tau\circ X^{-1}\right)-\Gamma\left(\eta\cdot\nabla\left(\tau\circ X^{-1}\right)\right) \tag{97}$$

where Γ is as defined in (55), since

$$[\eta\cdot\nabla,\mathbb{G}]=(R\otimes R)\mathbb{H}\,[\eta\cdot\nabla,\Gamma]+[\eta(t)\cdot\nabla,(R\otimes R)\mathbb{H}]\,\Gamma \tag{98}$$

and the second term is bounded by Lemma 1. For the first term, we have

$$[\eta\cdot\nabla,\Gamma]\,(\tau\circ X^{-1})(t)=I_1+I_2+I_3+I_4+I_5+I_6, \tag{99}$$

where

$$I_1 = \int_0^t \eta(t) \cdot \left(\nabla\Delta g_{\nu(t-s)} * \left(\tau \circ X^{-1}(t)\right)\right) - \nabla\Delta g_{\nu(t-s)} * \left(\eta(t)\tau \circ X^{-1}(t)\right) ds,$$

$$\begin{aligned} I_2 &= \int_0^t \eta(t) \cdot \left(\nabla\Delta g_{\nu(t-s)} * \left(\tau \circ X^{-1}(s) - \tau \circ X^{-1}(t)\right)\right) \\ &\quad -\nabla\Delta g_{\nu(t-s)} * \left(\eta(t)\left(\tau \circ X^{-1}(s) - \tau \circ X^{-1}(t)\right)\right) ds, \\ I_3 &= -\int_0^t \nabla\Delta g_{\nu(t-s)} * \left((\eta(s) - \eta(t))\left(\tau \circ X^{-1}(s)\right)\right) ds, \\ I_4 &= \int_0^t \Delta g_{\nu(t-s)} * \left(\nabla \cdot (\eta(s) - \eta(t))\, \tau \circ X^{-1}(s)\right) ds, \\ I_5 &= \int_0^t \Delta g_{\nu(t-s)} * \left(\nabla \cdot \eta(t)\left(\tau \circ X^{-1}(s) - \tau \circ X^{-1}(t)\right)\right) ds, \\ I_6 &= -\frac{1}{\nu}\left(\nabla \cdot \eta(t)\tau \circ X^{-1}(t) - g_{\nu t} * \left(\nabla \cdot \eta(t)\tau \circ X^{-1}(t)\right)\right). \end{aligned} \tag{100}$$

First, $I_1 + I_6$ can be bounded:

$$\begin{aligned} I_1 + I_6 = \frac{1}{\nu}&\left(\eta(t) \cdot \nabla\left(g_{\nu t} * \left(\tau \circ X^{-1}(t)\right)\right) - \nabla\left(g_{\nu t} * \left(\eta(t)\tau \circ X^{-1}(t)\right)\right)\right) \\ &-\frac{1}{\nu} g_{\nu t} * \left(\nabla \cdot \eta(t)\left(\tau \circ X^{-1}(t)\right)\right) \end{aligned} \tag{101}$$

and the first term is treated in the same way as (92). Since the first term is

$$\frac{1}{\nu}\left(\int_{\mathbb{R}^d} \nabla g_{\nu t}(y) y \cdot \int_0^1 \nabla\eta(x - (1-\lambda)y, t) d\lambda \left(\tau \circ X^{-1}\right)(x - y, t) dy\right) \tag{102}$$

and

$$\left\|\nabla g_{\nu t}(y) y\right\|_{L^1} \le C, \tag{103}$$

the $C^{\alpha,p}$-norm of the first term is bounded by

$$\frac{C}{\nu}\left\|\eta(t)\right\|_{C^{1+\alpha}}\left\|\tau \circ X^{-1}(t)\right\|_{\alpha,p}. \tag{104}$$

The $C^{\alpha,p}$-norm of the second term is also bounded by the same bound. Therefore,

$$\left\|I_1 + I_6\right\|_{L^\infty(0,T;C^{\alpha,p})} \le \frac{C}{\nu} M_X^{1+3\alpha}\left\|X'\right\|_{L^\infty(0,T;C^{1+\alpha})}\left\|\tau\right\|_{L^\infty(0,T;C^{\alpha,p})}. \tag{105}$$

The term I_3 is bounded due to Theorem 2. Since $\eta \in Lip(0, T; C^\alpha)$ we have

$$\begin{aligned} \left\|I_3\right\|_{L^\infty(0,T;C^{\alpha,p})} \le \frac{C}{\nu}\left(\frac{T}{\nu}\right)^{\frac{1}{2}} M_X^{1+4\alpha}\left\|X - \mathrm{Id}\right\|_{Lip(0,T;C^{1+\alpha})} \\ \left\|X'\right\|_{Lip(0,T;C^{1+\alpha})}\left\|\tau\right\|_{L^\infty(0,T;C^{\alpha,p})}. \end{aligned} \tag{106}$$

The terms I_4, and I_5 are treated in the spirit of Theorem 5. We treat $L^p \cap L^\infty$ norm and Hölder seminorm separately. For the term I_5, we have

$$I_5 = \int_0^t \Delta g_{\nu(t-s)} * \left(\nabla \cdot \eta(t)\left(\Delta_1\tau(s,t) + \Delta_2\tau(s,t)\right)\right) ds \tag{107}$$

where $\Delta_1\tau$ and $\Delta_2\tau$ are the same as (59). From the same arguments from the above,

$$\begin{aligned} &\left\|\int_0^t \Delta g_{\nu(t-s)} * \left(\nabla \cdot \eta(t)\Delta_1\tau(s,t)\right) ds\right\|_{\alpha,p} \\ &\le \frac{Ct}{\nu}\left\|\eta\right\|_{L^\infty(0,T;C^{1+\alpha})}\left\|\tau\right\|_{Lip(0,T;C^{\alpha,p})} M_X^\alpha. \end{aligned} \tag{108}$$

On the other hand,

$$\begin{aligned}\Delta g_{\nu(t-s)} * (\nabla\cdot\eta(t)\Delta_2\tau(s,t))(x) = \int_{\mathbb{R}^d} &(K(x,z,t,s)(\nabla\cdot\eta)(X(z,t),t) \\ &+\Delta g_{\nu(t-s)}(x-X(z,t))((\nabla\cdot\eta)(X(z,s),t)-(\nabla\cdot\eta)(X(z,t),t)))dz,\end{aligned} \tag{109}$$

where K is as in (63). Then as in the proof of Lemma 3, by the generalized Young's inequality we have

$$\begin{aligned}\left\|\int_0^t \Delta g_{\nu(t-s)} * (\nabla\cdot\eta(t)\Delta_2\tau(s,t))\,ds\right\|_{L^p\cap L^\infty} &\leq C\,\|\tau(t)\|_{L^p\cap L^\infty}\|\eta\|_{L^\infty(0,T;C^{1+\alpha})} \\ \|X-\mathrm{Id}\|_{Lip(0,T;C^{1+\alpha})}&\left(\frac{t^\alpha}{\nu\alpha}+\left(\frac{t}{\nu^3}\right)^{\frac{1}{2}}+\frac{t^2}{\nu^3}\|X-\mathrm{Id}\|^3_{Lip(0,T;C^{1+\alpha})}\right).\end{aligned} \tag{110}$$

For the Hölder seminorm, we repeat the same argument in the proof of Theorem 5, using the bound (81). Then we obtain

$$\begin{aligned}&\frac{1}{|h|^\alpha}\left\|\delta_h\left(\int_0^t \Delta g_{\nu(t-s)} * \Delta_2\tau(s,t)ds\right)\right\|_{L^\infty} \\ &\leq \frac{C(\alpha)}{\nu}\left(1+\left(\frac{t}{\nu}\right)^{\frac{1}{2}}+\left(\frac{t}{\nu}\right)^2\right)\|X-\mathrm{Id}\|^\alpha_{Lip(0,T;C^{1+\alpha})}\|\tau\|_{L^\infty(0,T;C^{\alpha,p})}\|\eta\|_{L^\infty(0,T;C^{1+\alpha})}.\end{aligned} \tag{111}$$

Therefore,

$$\begin{aligned}\|I_5\|_{L^\infty(0,T;C^{\alpha,p})} \leq \frac{C(\alpha)}{\nu}&\left(1+t+\left(\frac{t}{\nu}\right)^2\right)\left(1+\|X-\mathrm{Id}\|_{Lip(0,T;C^{1+\alpha})}\right)^3 M_X^{1+2\alpha} \\ &\|X'\|_{L^\infty(0,T;C^{1+\alpha})}\|\tau\|_{Lip(0,T;C^{\alpha,p})}.\end{aligned} \tag{112}$$

The term $I_4(t)$ is treated in the exactly same way, by noting that

$$\begin{aligned}\nabla\cdot(\eta(s)-\eta(t)) &= \nabla_x X^{-1}(s) : (\Delta_1\nabla_a X'(s,t)) + \nabla_x X^{-1}(s) : (\Delta_2\nabla_a X'(s,t)) \\ &+\left(\nabla_x X^{-1}(s)-\nabla_x X^{-1}(t)\right) : \left(\nabla_a X'\circ X^{-1}\right)(t),\end{aligned} \tag{113}$$

where as in (59)

$$\begin{aligned}\Delta_1\nabla_a X'(x,s,t) &= \nabla_a X'(X^{-1}(x,s),s)-\nabla_a X'(X^{-1}(x,s),t), \\ \Delta_2\nabla_a X'(x,s,t) &= \nabla_a X'(X^{-1}(x,s),t)-\nabla_a X'(X^{-1}(x,t),t),\end{aligned} \tag{114}$$

and

$$\nabla_x\left(X^{-1}(x,s)-X^{-1}(x,t)\right) = \left(\nabla_a X\circ X^{-1}\right)(x,t)\left(\nabla_a(X-\mathrm{Id})\right)\left(X^{-1}(x,t),t-s\right) \tag{115}$$

so that

$$\left\|\nabla_x X^{-1}(s)-\nabla_x X^{-1}(t)\right\|_{C^\alpha} \leq |t-s|\,\|X-\mathrm{Id}\|_{Lip(0,T;C^{1+\alpha})} M_X^{1+2\alpha}. \tag{116}$$

Also note that

$$\|\Delta_2\nabla_a X'(s,t)\|_{L^\infty} \leq \|\nabla_a X'(t)\|_{C^\alpha}\|X-\mathrm{Id}\|^\alpha_{Lip(0,T;L^\infty)}|t-s|^\alpha \tag{117}$$

so that

$$\begin{aligned}&\left\|\int_0^t \Delta g_{\nu(t-s)} * \left(\nabla_x X^{-1}(s) : (\Delta_2\nabla_a X'(s,t))\tau\circ X^{-1}(s)\right)ds\right\|_{C^{\alpha,p}} \\ &\leq \frac{C(\alpha)}{\nu}\left(1+t^\alpha+\left(\frac{t}{\nu}\right)^2\right)M_X^{1+2\alpha}\|X-\mathrm{Id}\|^\alpha_{Lip(0,T;C^{1+\alpha})} \\ &\qquad\|X'\|_{L^\infty(0,T;C^{1+\alpha})}\|\tau\|_{L^\infty(0,T;C^{\alpha,p})}.\end{aligned} \tag{118}$$

The final result is

$$\begin{aligned}\|I_4(t)\|_{\alpha,p} \leq \frac{C(\alpha)}{\nu}&\left(1+t+\left(\frac{t}{\nu}\right)^2\right)M_X^{2+4\alpha}\|X'\|_{L^\infty(0,T;C^{1+\alpha})}\|\tau\|_{L^\infty(0,T;C^{\alpha,p})} \\ &+C\frac{t}{\nu}M_X^{1+3\alpha}\|X'\|_{Lip(0,T;C^{1+\alpha})}\|\tau\|_{L^\infty(0,T;C^{\alpha,p})}.\end{aligned} \tag{119}$$

Finally, I_2 can be bounded using the combination of the technique in Theorem 5 and Theorem 6. First, we have

$$
\begin{aligned}
&I_2(x,t) = \\
&\int_0^t \int_{\mathbb{R}^d} \nabla\Delta g_{\nu(t-s)}(y) \cdot y \cdot \left(\int_0^1 \nabla\eta(x-(1-\lambda)y,t) d\lambda \left(\Delta_1 \tau(x-y,s,t)\right)\right) dyds \\
&+ \int_0^t \int_{\mathbb{R}^d} \nabla\Delta g_{\nu(t-s)}(x-z) \cdot (x-z) \cdot \left(\int_0^1 \nabla\eta(\lambda x+(1-\lambda)z,t) d\lambda \left(\Delta_2 \tau(z,s,t)\right)\right) dzds.
\end{aligned}
\tag{120}
$$

Then applying the argument of the proof of Theorem 6, the first term is bounded by

$$
\frac{C}{\nu} t M_X^\alpha \|\eta\|_{L^\infty(0,T;C^{1+\alpha})} \|\tau\|_{Lip(0,T;C^{\alpha,p})}. \tag{121}
$$

The second term is treated using the method used in Theorem 5. By changing variables to form a kernel similar to (63), and applying generalized Young's inequality, the $L^p \cap L^\infty$ norm of the second term is bounded by

$$
\frac{C(\alpha)}{\nu}\left(t^\alpha + \left(\frac{t}{\nu}\right)^{\frac{1}{2}} + \left(\frac{t}{\nu}\right)^2\right)\left(1+\|X-\mathrm{Id}\|_{Lip(0,T;C^{1+\alpha})}\right)^4 \|\eta\|_{L^\infty(0,T;C^{1+\alpha})} \|\tau\|_{L^\infty(0,T;L^p\cap L^\infty)}. \tag{122}
$$

Finally, the Hölder seminorm of the second term is bounded by the same method as Theorem 5. The only additional point is the finite difference of $\nabla\eta$ term, but this term is bounded by a straightforward estimate. The bound for the Hölder seminorm of the second term is

$$
\frac{C(\alpha)}{\nu}\left(1+t^\alpha + \left(\frac{t}{\nu}\right)^{\frac{1}{2}} + \left(\frac{t}{\nu}\right)^2\right) \|X-\mathrm{Id}\|^\alpha_{Lip(0,T;C^{1+\alpha})} \|\eta\|_{L^\infty(0,T;C^{1+\alpha})} \|\tau\|_{L^\infty(0,T;C^{\alpha,p})}. \tag{123}
$$

To sum up, we have

$$
\begin{aligned}
\|I_2(t)\|_{\alpha,p} \le \frac{C(\alpha)}{\nu}\left(1+t+\left(\frac{t}{\nu}\right)^2\right)\left(1+\|X-\mathrm{Id}\|_{Lip(0,T;C^{1+\alpha})}\right)^4 M_X^{1+3\alpha} \\
\|X'\|_{L^\infty(0,T;C^{1+\alpha})} \|\tau\|_{Lip(0,T;C^{\alpha,p})}.
\end{aligned}
\tag{124}
$$

If we put this together,

$$
\begin{aligned}
&\left\|\left[X'\circ X^{-1}\cdot\nabla, \mathbb{G}\right](\tau\circ X^{-1})\right\|_{L^\infty(0,T;C^{\alpha,p})} \\
&\le C\|X'\|_{L^\infty(0,T;C^{1+\alpha})} M_X^{1+2\alpha} \left\|\Gamma(\tau\circ X^{-1})\right\|_{L^\infty(0,T;C^{\alpha,p})} \\
&+(\|X'\|_{L^\infty(0,T;C^{1+\alpha})} + \|X'\|_{Lip(0,T;C^{1+\alpha})} T^{\frac{1}{2}}) F_1(\nu,\alpha,X,\|\tau\|_{Lip(0,T;C^{\alpha,p})},T)
\end{aligned}
\tag{125}
$$

where F_1 depends on the written variables and grows like polynomial in $T, \|\tau\|_{Lip(0,T;C^{\alpha,p})}$, and $\|X-\mathrm{Id}\|_{Lip(0,T;C^{1+\alpha})}$. The bound on $\Gamma(\tau\circ X^{-1})$ is given by Theorem 5. □

4 Bounds on variations and variables

Using the results from the previous section we find bounds for variations and variables. For simplicity, we adopt the notation

$$
M_\epsilon = 1 + \|X_\epsilon - \mathrm{Id}\|_{L^\infty(0,T;C^{1+\alpha})}. \tag{126}
$$

First, we bound $\frac{d}{d\epsilon}\mathcal{V}_\epsilon$. Note that $X_\epsilon(0) = \mathrm{Id}$, so $X'_\epsilon(0) = 0$ and by Theorem 2 and since $X'_\epsilon \in Lip(0,T;C^{1+\alpha,p})$ we have

$$
\begin{aligned}
\|X'_\epsilon\|_{L^\infty(0,T;C^{1+\alpha})} \le T\|X'_\epsilon\|_{Lip(0,T;C^{1+\alpha,p})}, \\
\|\eta_\epsilon(t)\|_{C^\alpha} \le t\|X'\|_{Lip(0,T;C^{1+\alpha,p})} M_\epsilon^\alpha.
\end{aligned}
\tag{127}
$$

Then by the Theorem 3, we have

$$\begin{aligned}\|\eta_\epsilon\cdot\mathbb{L}_\nu(\nabla_x u_{\epsilon,0})\|_{L^\infty(0,T;C^{\alpha,p})}\le C\left(\frac{T}{\nu}\right)^{\frac{1}{2}}M_\epsilon^\alpha\left\|X'_\epsilon\right\|_{Lip(0,T;C^{1+\alpha,p})}\left\|u_{\epsilon,0}\right\|_{1+\alpha,p},\\ \left\|\mathbb{L}_\nu(u'_{\epsilon,0})\right\|_{L^\infty(0,T;C^{\alpha,p})}\le C\left\|u'_{\epsilon,0}\right\|_{\alpha,p}.\end{aligned}\tag{128}$$

By Theorem 6, we have

$$\begin{aligned}\|[\eta_\epsilon\cdot\nabla_x,\mathbb{U}]\,(\sigma_\epsilon-u_\epsilon\otimes u_\epsilon)\|_{L^\infty(0,T;C^{\alpha,p})}\le C\left(\left(\frac{T}{\nu}\right)^{\frac{1}{2}}+\left(\frac{T}{\nu}\right)\right)M_\epsilon^{2+4\alpha}\\ \|X'_\epsilon\|_{Lip(0,T;C^{1+\alpha})}\left\|\tau_\epsilon-v_\epsilon\otimes v_\epsilon\right\|_{L^\infty(0,T;C^{\alpha,p})},\end{aligned}\tag{129}$$

and by Theorem 4, we have

$$\begin{aligned}\left\|\mathbb{U}\left(\delta_\epsilon-(v'_\epsilon\otimes v_\epsilon+v_\epsilon\otimes v'_\epsilon)\circ X_\epsilon^{-1}\right)\right\|_{L^\infty(0,T;C^{\alpha,p})}\le C\left(\frac{T}{\nu}\right)^{\frac{1}{2}}M_\epsilon^\alpha\\ \|\tau'_\epsilon-(v'_\epsilon\otimes v_\epsilon+v_\epsilon\otimes v'_\epsilon)\|_{L^\infty(0,T;C^{\alpha,p})}.\end{aligned}\tag{130}$$

Therefore,

$$\begin{aligned}\left\|\frac{d}{d\epsilon}\mathcal{V}_\epsilon\right\|_{L^\infty(0,T;C^{\alpha,p})}\le C\left\|u'_{\epsilon,0}\right\|_{\alpha,p}\\ +S_1(T)(\|X'_\epsilon\|_{Lip(0,T;C^{1+\alpha,p})}+\|v'_\epsilon\|_{L^\infty(0,T;C^{\alpha,p})}+\left\|\sigma'_{\epsilon,0}\right\|_{\alpha,p}+\|\tau'_\epsilon\|_{Lip(0,T;C^{\alpha,p})})Q_1\end{aligned}\tag{131}$$

where $S_1(T)$ vanishes as $T^{\frac{1}{2}}$ as $T\to 0$ and Q_1 is a polynomial in $\|u_{\epsilon,0}\|_{1+\alpha,p}$, $\|X_\epsilon-\mathrm{Id}\|_{Lip(0,T;C^{1+\alpha,p})}$, $\|\tau_\epsilon\|_{L^\infty(0,T;C^{\alpha,p})}$, and $\|v_\epsilon\|_{L^\infty(0,T;C^{\alpha,p})}$, whose coefficients depend on ν. Similarly,

$$\|g_\epsilon\|_{L^\infty(0,T;C^{\alpha,p})}\le M_X^\alpha\left\|u_0\right\|_{1+\alpha,p}+C_1\left\|X-\mathrm{Id}\right\|^\alpha_{Lip(0,T;C^{1+\alpha})}\left\|\sigma_{\epsilon,0}\right\|_{\alpha,p}+S_2(T)Q_2,\tag{132}$$

where $S_2(T)$ vanishes as $T^{\frac{1}{2}}$ as $T\to 0$ and Q_2 is polynomial in $\|\tau\|_{Lip(0,T;C^{\alpha,p})}$ and $\|X-\mathrm{Id}\|_{Lip(0,T;C^{1+\alpha})}$, whose coefficients depend on α and ν. Also

$$\begin{aligned}\|g'_\epsilon\|_{L^\infty(0,T;C^{\alpha,p})}\le C(\left\|u'_{\epsilon,0}\right\|_{1+\alpha,p}+\|X-\mathrm{Id}\|^\alpha_{Lip(0,T;C^{1+\alpha})}\left\|\tau'_{\epsilon,0}\right\|_{\alpha,p})\\ +S_3(T)(\|X'_\epsilon\|_{Lip(0,T;C^{1+\alpha,p})}+\left\|\sigma'_{\epsilon,0}\right\|_{\alpha,p}+\|\tau'_\epsilon\|_{Lip(0,T;C^{\alpha,p})}+\|v'_\epsilon\|_{L^\infty(0,T;C^{1+\alpha,p})})Q_3,\end{aligned}\tag{133}$$

where $S_3(T)$ vanishes as $T^{\frac{1}{2}}$ as $T\to 0$ and Q_3 is polynomial in $\|u_{\epsilon,0}\|_{1+\alpha,p}$, $\|X-\mathrm{Id}\|_{Lip(0,T;C^{1+\alpha,p}}$, $\|\tau\|_{Lip(0,T;C^{\alpha,p})}$, and $\|v_\epsilon\|_{L^\infty(0,T;C^{1+\alpha,p})}$, whose coefficients depend on ν and α. Then we have

$$\left\|\nabla_a\frac{d}{d\epsilon}\mathcal{V}_\epsilon\right\|_{L^\infty(0,T;C^{\alpha,p})}\le T\left\|X'_\epsilon\right\|_{Lip(0,T;C^{1+\alpha})}\|g_\epsilon\|_{L^\infty(0,T;C^{\alpha,p})}+M_\epsilon\left\|g'_\epsilon\right\|_{L^\infty(0,T;C^{\alpha,p})}\tag{134}$$

and

$$\begin{aligned}\left\|\frac{d}{d\epsilon}\mathcal{T}_\epsilon\right\|_{L^\infty(0,T;C^{\alpha,p})}\le 2\left\|g'_\epsilon\right\|_{L^\infty(0,T;C^{\alpha,p})}\left(\|\tau_\epsilon\|_{L^\infty(0,T;C^{\alpha,p})}+2\rho K\right)\\ +\|\tau'_\epsilon\|_{L^\infty(0,T;C^{\alpha,p})}\left(\|g_\epsilon\|_{L^\infty(0,T;C^{\alpha,p})}+2k\right).\end{aligned}\tag{135}$$

5 Local existence

We define the function space $\mathcal{P}_1$ and the set $\mathcal{I}$,

$$\begin{aligned}\mathcal{P}_1=Lip(0,T;C^{1+\alpha,p})\times Lip(0,T;C^{\alpha,p})\times L^\infty(0,T;C^{1+\alpha,p})\\ \mathcal{I}=\{(X,\tau,v):\|(X-\mathrm{Id},\tau,v)\|_{\mathcal{P}_1}\le\Gamma,v=\frac{dX}{dt}\},\end{aligned}\tag{136}$$

where $\Gamma > 0$ and $T > 0$ are to be determined. Now, for given $u_0 \in C^{1+\alpha,p}$ divergence free and $\sigma_0 \in C^{\alpha,p}$ we define the map

$$(X, \tau, v) \to \mathcal{S}(X, \tau, v) = (X^{new}, \tau^{new}, v^{new}) \tag{137}$$

where

$$\begin{cases} X^{new}(t) = \mathrm{Id} + \displaystyle\int_0^t \mathcal{V}(X(s), \tau(s), v(s))ds, \\ \tau^{new}(t) = \sigma_0 + \displaystyle\int_0^t \mathcal{T}(X(s), \tau(s), v(s))ds, \\ \qquad v^{new}(t) = \mathcal{V}(X, \tau, v). \end{cases} \tag{138}$$

If $(X - \mathrm{Id}, \tau, v) \in \mathcal{P}_1$, then $(X^{new} - \mathrm{Id}, \tau^{new}, v^{new}) \in \mathcal{P}_1$ for any choice of $T > 0$. Moreover, we have the following:

Theorem 8. *For given $u_0 \in C^{1+\alpha,p}$ divergence free and $\sigma_0 \in C^{\alpha,p}$, there is a $\Gamma > 0$ and $T > 0$ such that the map $\mathcal{S}$ of (138) maps $\mathcal{I}$ to itself.*

Proof. It is obvious that $\frac{d}{dt}X^{new} = v^{new}$. For the size of $\mathcal{S}(X, \tau, v)$, first note that if $(X - \mathrm{Id}, \tau, v)_{\mathcal{P}_1} \leq \Gamma$, then

$$M_X = 1 + \|X - \mathrm{Id}\|_{L^\infty(0,T;C^{1+\alpha})} \leq 1 + T\Gamma. \tag{139}$$

Applying Theorem 3 and Theorem 4, we know that

$$\|\mathcal{V}\|_{L^\infty(0,T;C^{\alpha,p})} \leq \|u_0\|_{\alpha,p} + A_1(T)B_1(\Gamma, \|u_0\|_{\alpha,p}, \|\sigma_0\|_{\alpha,p}), \tag{140}$$

where $A_1(T)$ vanishes like $T^{\frac{1}{2}}$ for small $T > 0$ and B_1 is a polynomial in its arguments, and some coefficients depend on ν. We estimate

$$\|g\|_{L^\infty(0,T;C^{\alpha,p})} \leq \|u_0\|_{1+\alpha,p} + C_1\Gamma^\alpha \|\sigma_0\|_{\alpha,p} + A_2(T)B_2(\Gamma, \|u_0\|_{1+\alpha,p}, \|\sigma_0\|_{\alpha,p}), \tag{141}$$

where C_1 is as in Theorem 5, depending only on α and ν, $A_2(T)$ vanishes in the same order as $A_1(T)$ as $T \to 0$, and B_2 is a polynomial in its arguments, and some coefficients depend on ν and α. From (24) we conclude

$$\|\mathcal{V}\|_{L^\infty(0,T;C^{1+\alpha,p})} \leq K_1(\|u_0\|_{1+\alpha,p} + \Gamma^\alpha \|\sigma_0\|_{\alpha,p}) + A_3(T)B_3(\Gamma, \|u_0\|_{1+\alpha,p}, \|\sigma_0\|_{\alpha,p}), \tag{142}$$

where K_1 is a constant depending only on ν and α, and A_3 and B_3 have the same properties as previous A_is and B_is. Now we measure $\mathcal{T}$. From (84) and the previous estimate on g we have

$$\begin{aligned} \|\mathcal{T}\|_{L^\infty(0,T;C^{\alpha,p})} \leq K_2(&\|u_0\|_{1+\alpha,p}(\rho K + \|\sigma_0\|_{\alpha,p}) + \|\sigma_0\|_{\alpha,p}(\Gamma^\alpha \|\sigma_0\|_{\alpha,p} + \rho K\Gamma^\alpha + k)) \\ &+ A_4B_4, \end{aligned} \tag{143}$$

where K_2 is a constant depending on ν and α, and A_4 and B_4 are as before. Since $\alpha < 1$, we can appropriately choose large $\Gamma > \|\sigma_0\|_{\alpha,p} + \|u_0\|_{1+\alpha,p}$ and correspondingly small $\frac{1}{6} > T > 0$ so that the right side of (142) and (143) are bounded by $\frac{\Gamma}{6}$. Then $\|(X^{new} - \mathrm{Id}, \tau^{new}, v^{new})\|_{\mathcal{P}_1} \leq \Gamma$. □

We show now that $\mathcal{S}$ is a contraction mapping on $\mathcal{I}$ for a short time.

Theorem 9. *For given $u_0 \in C^{1+\alpha,p}$ divergence free and $\sigma_0 \in C^{\alpha,p}$, there is a Γ and $T > 0$, depending only on $\|u_0\|_{1+\alpha,p}$ and $\|\sigma_0\|_{\alpha,p}$, such that the map $\mathcal{S}$ is a contraction mapping on $\mathcal{I} = \mathcal{I}(\Gamma, T)$, that is*

$$\|\mathcal{S}(X_2, \tau_2, v_2) - \mathcal{S}(X_1, \tau_1, v_1)\|_{\mathcal{P}_1} \leq \frac{1}{2}\|(X_2 - X_1, \tau_2 - \tau_1, v_2 - v_1)\|_{\mathcal{P}_1}. \tag{144}$$

Proof. First from Theorem 8 we can find a Γ and $T_0 > 0$, depending only on the size of initial data, say

$$N = \max\{\|u_0\|_{1+\alpha,p}, \|\sigma_0\|_{\alpha,p}\}, \tag{145}$$

which guarantees that $\mathcal{S}$ maps $\mathcal{I}$ to itself. This property still holds if we replace T_0 by any smaller $T > 0$. In view of the fact that $\mathcal{I}$ is convex, we put

$$\begin{aligned} X_\epsilon &= (2-\epsilon)X_1 + (\epsilon-1)X_2, \\ \tau_\epsilon &= (2-\epsilon)\tau_1 + (\epsilon-1)\tau_2, 1 \le \epsilon \le 2. \end{aligned} \tag{146}$$

Then $(X_\epsilon, \tau_\epsilon, v_\epsilon) \in \mathcal{I}$, $v_\epsilon = (2-\epsilon)v_1 + (\epsilon-1)v_2$, $u_{\epsilon,0} = u_0$, and $\sigma_{\epsilon,0} = \sigma_0$. This means that

$$X'_\epsilon = X_2 - X_1, v'_\epsilon = v_2 - v_1, u'_{\epsilon,0} = 0, \sigma'_{\epsilon,0} = 0. \tag{147}$$

Then from the results of Section 4, we see that

$$\begin{aligned} \left\| \frac{d}{d\epsilon}\mathcal{V}_\epsilon \right\|_{L^\infty(0,T;C^{1+\alpha,p})} &\le (\|X_2 - X_1\|_{Lip(0,T;C^{1+\alpha,p})} + \|v_2 - v_1\|_{L^\infty(0,T;C^{\alpha,p})} \\ &\quad + \|\tau_2 - \tau_1\|_{Lip(0,T;C^{\alpha,p})}) S'_1(T)Q'_1(\Gamma), \\ \|\mathcal{X}'_\epsilon\|_{Lip(0,T;C^{1+\alpha,p})} &\le (\|X_2 - X_1\|_{Lip(0,T;C^{1+\alpha,p})} + \|v_2 - v_1\|_{L^\infty(0,T;C^{\alpha,p})} \\ &\quad + \|\tau_2 - \tau_1\|_{Lip(0,T;C^{\alpha,p})}) S'_2(T)Q'_2(\Gamma), \\ \\ \|\pi_\epsilon\|_{Lip(0,T;C^{\alpha,p})} &\le (\|X_2 - X_1\|_{Lip(0,T;C^{1+\alpha,p})} + \|v_2 - v_1\|_{L^\infty(0,T;C^{\alpha,p})} \\ &\quad + \|\tau_2 - \tau_1\|_{Lip(0,T;C^{\alpha,p})}) S'_3(T)Q'_3(\Gamma), \end{aligned} \tag{148}$$

where $\mathcal{X}'_\epsilon$ and π_ϵ are defined in (20), $S'_1(T), S'_2(T), S'_3(T)$ vanish at the rate of $T^{\frac{1}{2}}$ as $T \to 0$, and $Q'_1(\Gamma), Q'_2(\Gamma), Q'_3(\Gamma)$ are polynomials in Γ, whose coefficients depend only on ν and α. By choosing $0 < T < T_0$ small enough, depending on the size of $Q'_i(\Gamma)$s, we conclude the proof. □

We have obtained a solution to the system (6) in the path space $\mathcal{P}_1$ for a short time, that is, we have (X, τ, v) satisfying $v = \frac{dX}{dt}$ and satisfying (14). We also have Lipschitz dependence on initial data, Theorem 1.

Proof. We repeat the calculation of the Theorem 9, but this time $u'_{\epsilon,0} = u_1(0) - u_2(0)$ and $\sigma'_{\epsilon,0} = \sigma_1(0) - \sigma_2(0)$. Then we choose T_0 small enough that $S'_i(T_0)Q'_1(\Gamma) < \frac{1}{2}$. □

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