On putative self-similarity for incompressible 3D Euler

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Abstract. We consider hypothetical solutions of 3D Euler which blow up in finite time in a self-similar fashion. We prove that if the initial data has finite kinetic energy, then the similarity exponent γ which governs the rate of zooming in must be larger than $2/5$. If a smooth globally self-similar blowup profile exists, and this profile satisfies an outgoing property, we prove that $\gamma \geq 1/2$. For axisymmetric solutions, we establish the bound $\gamma \geq 1/2$ in more general settings, including ones in which the outgoing property is not present.

1. Introduction

Self-similarity [1] is a powerful idea: it says that things look the same at different scales, if you just know how to zoom in or out. It is a particularly useful tool for identifying singular solutions of nonlinear PDE via a reduction to ODE [39, 23]. Self-similarity is found in compressible fluids and in many other physical systems when dynamics are determined by local interactions. Compressible Euler equations for instance exhibit self-similar explosions [38, 45, 43], implosions [29, 32, 34, 3, 41] and shocks [4, 5].

Incompressible fluids are not local, the outside matters. This makes the link between possible singularities and local self-similar behavior rather tenuous. There have been a number of recent works where self-similar incompressible singularities have been reported, either as rigorous proofs or in computational studies. In all of them there is a remnant of compression due to either the presence of boundaries [12, 13, 14] and [33, 46, 47, 48, 49], or the lack of smoothness of vorticity [24, 11, 26, 27, 9, 16]. The goal of this paper is to analyze constraints on putative self-similar singularities for the incompressible 3D Euler equations with smooth initial data, in the absence of boundaries.

We recall the vorticity formulation of incompressible 3D Euler equations:

$$\partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u, \qquad \nabla \cdot u = 0, \qquad \omega = \nabla \times u. \tag{1.1}$$

The smooth and localized initial data ω_0 for the Cauchy problem associated to (1.1) is specified at time $t = 0$, and the equations are posed on the whole space ($x \in \mathbb{R}^3$). It is well known that singularities of any kind cannot arise in finite time from smooth and localized initial data unless the vorticity becomes infinite, such that its maximum magnitude is not time integrable. This is the well-known Beale-Kato-Majda criterion [2]. We say that (1.1) exhibits a self-similar singularity if there exists a first time $T_* > 0$, a space location $x_* \in \mathbb{R}^3$, a similarity exponent $\gamma > 0$, and a vorticity profile Ω such that

$$\omega(x,t) = \frac{1}{T_* - t}\, \Omega\left(\frac{x - x_*}{(T_* - t)^\gamma}\right) + \text{l.o.t.} \qquad \text{as} \qquad t \to T_*^-. \tag{1.2}$$

The factor $(T_* - t)^{-1}$ appearing in (1.2) is unavoidable, and consistent with [2]. A more precise meaning of a self-similar singularity for (1.1) will be given later in the paper; see Section 2 for an asymptotically self-similar blowup, and Section 3 for a globally self-similar blowup.

If a true self-similar singularity forms in (1.1), the value of the exponent γ is an important and powerful dynamic property, and needs to be understood. Finding a profile Ω is a well known challenge. Because access to self-similar solutions is fraught with numerical and theoretical difficulties, it is useful to establish strict mathematical guardrails and to provide computational benchmarks.

The Euler equations have a great variety of different types of solutions. Blow up of smooth solutions with infinite kinetic energy has been proved [**15, 20, 28, 42**]. In this paper we prove that if the initial data of (1.1) has finite kinetic energy, then a solution behaving as in (1.2) must have $\gamma \geq 2/5$; see Theorem 2.1.

Beyond a general intrinsic interest in lower bounds for γ as a benchmark for ongoing computational and analytical studies, in the second half of the paper we focus on the hypothetical case when γ lies below the parabolic threshold of $1/2$, relevant for incompressible 3D Navier-Stokes. At $\gamma = 1/2$, the advection operator $\partial_t + u \cdot \nabla$ and the dissipative operator $-\Delta$ are in exact balance when acting on vorticities in the form (1.2). Liouville theorems [**36, 44, 40, 8**] rule out certain globally self-similar solutions for the 3D incompressible Navier-Stokes equations with $\gamma = 1/2$. Under reasonable assumptions, $\gamma > 1/2$ cannot be the exponent of an approximate self-similar blow up for 3D incompressible Navier-Stokes equations (see Proposition 3.1). For self-similar scaling with $\gamma < 1/2$, the viscous term yields a vanishingly small force competing with the Euler nonlinearity. If self-similar solutions of Euler equations with $\gamma < 1/2$ are found, they can be used to obtain 3D Navier-Stokes blow up as a perturbation of the 3D Euler one, allowing the inviscid blowup to be used to find a viscous one. This is indeed a rigorously proven fact in the realm of compressible flows: globally self-similar implosion singularities for 3D compressible isentropic Euler equations can be used to provide asymptotically self-similar implosion singularities for 3D compressible barotropic Navier-Stokes equations with constant viscosity coefficients; see [**34, 35**] and [**3, 41**]. In the process of establishing the blow up [**35, 3, 41**], in addition to having $\gamma < 1/2$ the authors prove and use that the inviscid similarity profile has infinite regularity (real-analyticity) in a neighborhood of all sonic points (stagnation points on the self-similar fast-acoustic characteristics).

An analogous approach to establish blow up for incompressible viscous flows hinges on finding self-similar blowup of (1.1) with $\gamma < 1/2$. This is advocated in [**46**] and [**47**].

The paper [**25**] identifies an outgoing property (cf. (3.36)) as being an essential ingredient for the existence of self-similar profiles. This outgoing property quantifies the statement that all self-similar Lagrangian trajectories that originate from nonzero labels must escape to infinity as (self-similar) time diverges. In particular, this condition implies that the self-similar Lagrangian flow has no stagnation points, except for the trivial one at the origin.

In this paper we prove that if an outgoing globally self-similar smooth solution to incompressible 3D Euler equations exists, then we must have $\gamma \geq 1/2$; see Theorem 3.8, Theorem 3.10, and Theorem 4.3. Thus, in this case, the approach proposed to produce incompressible Navier-Stokes singularities via Euler self-similar solutions must fail.

When restricting our analysis to axisymmetric similarity profiles, we prove stronger results. If the velocity has nonzero swirl at a stagnation point of the self-similar Lagrangian flow map, then $\gamma = 1/2$; see Theorem 4.4. Alternatively, irrespective of the swirl value, if the velocity field is C^1 smooth in a neighborhood of all stagnation points of the self-similar Lagrangian flow map, and if all these stagnation points lie on the axis of symmetry, then $\gamma \geq 1/2$; see Theorem 4.6.

Independently of the outgoing property, if a globally self-similar vorticity profile Ω exists, then it cannot be too small, as measured by a certain scaling invariant norm, see Theorem 3.4.

Our proofs are based on basic, well known properties of incompressible Euler and Navier-Stokes equations. We chose to present rigorous results, with explicit minimal assumptions. The fact that $\gamma \geq 2/5$ for finite kinetic energy flows, is such a result. The relevance of the exponent $\gamma = 1/2$ in Euler flows is one of the main messages of this paper. This exponent, which is natural for Navier-Stokes self-similarity, is natural for Euler equations as well. This is due to the invariance of circulation, a quantity with dimension of kinematic viscosity.

2. On asymptotically self-similar blowup with finite kinetic energy

A 3D incompressible Euler singularity cannot be just a vorticity magnitude singularity—the spatial gradient of vorticity must also blow up fast enough. For $\mu \in (0,1)$, let

$$\ell_\mu(t) := \min\left\{ L_0; \left(\frac{[\omega(\cdot,t)]_\mu}{\|u(\cdot,t)\|_{L^2}}\right)^{-\frac{2}{2\mu+5}} \right\}$$

be the length scale formed with the vorticity Hölder seminorm $[\omega(\cdot,t)]_\mu := \sup_{0<|x-y|\le L_0} \frac{|\omega(x,t)-\omega(y,t)|}{|x-y|^\mu}$ and the L^2 norm of velocity, where L_0 is an arbitrary reference length scale. It was shown in [18, Theorem 1, p. 38] that if

$$\int_0^{T_*} \ell_\mu(t)^{-\frac{5}{2}}\,dt < \infty,$$

then no singularities can occur from smooth and localized data at time T_*. This result follows easily from the representation of vorticity and from the conservation of kinetic energy.

In [18] it is also remarked as a consequence that if the vorticity is assumed to have a globally self-similar blowup, namely

$$\omega(x,t) = \frac{1}{T_*-t}\Omega\left(\frac{x}{L(t)}\right),$$

then we must have

$$\int_0^{T_*} L(t)^{-\frac{5}{2}}\,dt = \infty.$$

In particular, if $L(t) = L_0(1-\frac{t}{T_*})^\gamma$ as in (1.2), then $\gamma \ge 2/5$ is necessary for blow up. Here we revisit this result, without assuming the existence of a globally self-similar profile; instead, we assume only local behavior consistent with self-similarity. Furthermore, we link the putative exponent γ to the behavior of the velocity away from the local self-similar behavior. In [7] it was shown that self-similar blow up cannot occur if the vorticity profile decays at infinity fast enough. This is not what we do here, our assumed behavior does not invoke a profile, but if a profile existed, it would have required only that the profile's first derivatives be bounded in a fixed ball. We make assumptions about the nature of the true solution of (1.1), which would be automatically satisfied if a self-similar C^1-smooth vorticity profile existed.

THEOREM 2.1. *Assume the 3D Euler equation (1.1) has initial data $\omega_0 \in C^1(\mathbb{R}^3)$ with finite kinetic energy, i.e. $u_0 = (-\Delta)^{-1}\nabla\times\omega_0 \in L^2(\mathbb{R}^3)$. Assume that the resulting local-in-time smooth solution blows up at some finite time $T_* > 0$. If there exists $\gamma > 0$ such that*

$$\sup_{t\in[0,T_*)} (T_*-t)^{1+\gamma}\|\nabla\omega(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} < \infty, \tag{2.1}$$

then $\gamma \ge 2/5$.

REMARK 2.2. A few comments concerning Theorem 2.1 are in order:

(a) For a self-similar blowup (see (1.2)) with profile Ω, assumption (2.1) is automatic if $\nabla\Omega \in L^\infty$.

(b) The proof of Theorem 2.1 does not use the full assumption (2.1). If the blowup occurs at some point $x_* \in \mathbb{R}^3$, we only use that $\sup_{t\in[0,T_*)}(T_*-t)^{1+\gamma}\|\nabla\omega(\cdot,t)\|_{L^\infty(B(t))} < \infty$, where $B(t)$ is the ball of radius $(T_*-t)^\gamma$ centered at x_*.

(c) If in addition to (2.1) we have that

$$\sup_{t\in[0,T_*)} \|u(\cdot,t)\|_{L^p(\mathbb{R}^3)} < \infty, \tag{2.2}$$

for some $p \ge 2$, then the lower bound on γ becomes

$$\gamma \ge \frac{p}{p+3}.$$

In particular, if $u \in L^\infty(0,T_*;L^3(\mathbb{R}^3))$, then $\gamma \geq 1/2$.

PROOF OF THEOREM 2.1. We recall from [18] (see also [21, 19]) that the magnitude of vorticity satisfies

$$\partial_t|\omega| + u\cdot\nabla|\omega| = \alpha|\omega|\,, \tag{2.3}$$

where the stretching factor α is defined as

$$\alpha(x,t) := \frac{3}{4\pi}\text{p.v.}\int_{\mathbb{R}^3}(\hat{y}\cdot\xi(x,t))\left(\hat{y}\cdot(\omega(x+y,t)\times\xi(x,t))\right)\frac{dy}{|y|^3}, \tag{2.4}$$

where $\hat{y} = y/|y| \in \mathbb{S}^2$ and $\xi(x,t) = \omega(x,t)/|\omega(x,t)| \in \mathbb{S}^2$.

We consider a smooth cutoff function $\chi\colon \mathbb{R}_+ \to \mathbb{R}$, with $\chi \equiv 1$ on $[0,1]$, and $\chi \equiv 0$ on $[2,\infty)$. Let $R(t) > 0$, to be determined later. We decompose $\alpha = \alpha_{\rm in} + \alpha_{\rm out}$, where

$$\begin{aligned}\alpha_{\rm in}(x,t) &:= \frac{3}{4\pi}\text{p.v.}\int_{\mathbb{R}^3}(\hat{y}\cdot\xi(x,t))\left(\hat{y}\cdot(\omega(x+y,t)\times\xi(x,t))\right)\chi\Big(\frac{|y|}{R(t)}\Big)\frac{dy}{|y|^3}\\ &= \frac{3}{4\pi}\int_{\mathbb{R}^3}(\hat{y}\cdot\xi(x,t))\left(\hat{y}\cdot((\omega(x+y,t)-\omega(x,t))\times\xi(x,t))\right)\chi\Big(\frac{|y|}{R(t)}\Big)\frac{dy}{|y|^3},\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\alpha_{\rm out}(x,t) &:= \frac{3}{4\pi}\text{p.v.}\int_{\mathbb{R}^3}(\hat{y}\cdot\xi(x,t))\left(\hat{y}\cdot(\omega(x+y,t)\times\xi(x,t))\right)\left(1-\chi\Big(\frac{|y|}{R(t)}\Big)\right)\frac{dy}{|y|^3}\\ &= \frac{3}{4\pi}\int_{\mathbb{R}^3}u(x+y,t)\cdot\left((5(\hat{y}\cdot\xi)^2-1)\hat{y}-2(\hat{y}\cdot\xi)\xi\right)\left(1-\chi\Big(\frac{|y|}{R(t)}\Big)\right)\frac{dy}{|y|^4}\\ &\quad - \frac{3}{4\pi R(t)}\int_{\mathbb{R}^3}(\hat{y}\cdot\xi)u(x+y,t)\cdot(\hat{y}\times(\xi\times\hat{y}))\,\chi'\Big(\frac{|y|}{R(t)}\Big)\frac{dy}{|y|^3}.\end{aligned} \tag{2.6}$$

In the inner integral we used that $\omega(x,t)\times\xi(x,t) = 0$, while in the outer integral we have integrated by parts with respect to curl_y. We make two claims:

- There exists a constant $C_{\rm in} > 0$ depending only on χ, such that
$$|\alpha_{\rm in}(x,t)| \leq C_{\rm in}R(t)\|\nabla\omega(\cdot,t)\|_{L^\infty(B_{2R(t)}(x))}. \tag{2.7}$$
This estimate follows directly from (2.5) upon noting that $|\hat{y}| = 1 = |\xi(x,t)|$, and by bounding $|\omega(x+y,t)-\omega(x,t)| \leq |y|\|\nabla\omega(\cdot,t)\|_{L^\infty(B_{2R(t)})}$.
- For any $p \geq 1$, there exists a constant $C_{\rm out} > 0$ depending only on p and χ, such that
$$|\alpha_{\rm out}(x,t)| \leq C_{\rm out}R(t)^{-1-\frac{3}{p}}\|u(\cdot,t)\|_{L^p(\mathbb{R}^3)}. \tag{2.8}$$
This estimate follows directly from (2.6), using Hölder's inequality and $|\hat{y}| = 1 = |\xi(x,t)|$.

The theorem now immediately follows from (2.7)–(2.8), by letting $p = 2$. Indeed, classical solutions of the 3D Euler equations conserve their kinetic energy, $\|u(\cdot,t)\|_{L^2(\mathbb{R}^3)} = \|u_0\|_{L^2(\mathbb{R}^3)}$, and hence we have an a-priori bound for $\mathsf{RHS}_{(2.8)}$. Moreover, assumption (2.1) implies an upper bound for $\mathsf{RHS}_{(2.7)}$. Optimizing in $R(t)$, and using (2.1) we obtain the pointwise bound

$$|\alpha(x,t)| \leq C(T_*-t)^{-\frac{5(1+\gamma)}{7}}\left(\sup_{t\in[0,T_*)}(T_*-t)^{1+\gamma}\|\nabla\omega(\cdot,t)\|_{L^\infty(B_{2R(t)}(x))}\right)^{\frac{5}{7}}\|u_0\|_{L^2(\mathbb{R}^3)}^{\frac{2}{7}},$$

where $C = C(C_{\rm in}, C_{\rm out}) > 0$. Thus, if $5(1+\gamma)/7 < 1$—which is equivalent to $\gamma < 2/5$—then $\int_0^{T_*}\|\alpha(\cdot,t)\|_{L^\infty}dt < \infty$, and so by (2.3) and the Beale-Kato-Majda criterion, no blowup can occur at time T_*. Therefore, a singularity necessitates $\gamma \geq 2/5$.

If in addition to (2.1) we also know that (2.2) holds, using (2.8) we may analogously prove

$$|\alpha(x,t)| \le C(T_* - t)^{-\frac{(p+3)(1+\gamma)}{2p+3}} \left(\sup_{t\in[0,T_*)} (T_* - t)^{1+\gamma} \|\nabla\omega(\cdot,t)\|_{L^\infty(B_{2R(t)}(x))} \right)^{\frac{p+3}{2p+3}} \sup_{t\in[0,T_*)} \|u(\cdot,t)\|_{L^p(\mathbb{R}^3)}^{\frac{p}{2p+3}},$$

where $C = C(C_{\rm in}, C_{\rm out}, p) > 0$. Thus, if $(p+3)(1+\gamma)/(2p+3) < 1$—which is equivalent to $\gamma < p/(p+3)$—then $\int_0^{T_*} \|\alpha(\cdot,t)\|_{L^\infty} dt < \infty$, and no blowup can occur at time T_*. Therefore, a singularity necessitates $\gamma \ge p/(p+3)$, as claimed in item (c) of Remark 2.2. □

3. On globally self-similar blowup

In order to obtain better lower bounds on the similarity exponent $\gamma \ge 2/5$, we analyze the hypothetical case in which the 3D incompressible Euler equations (1.1), written in velocity form as

$$\partial_t u + u\cdot\nabla u + \nabla p = 0, \qquad \nabla\cdot u = 0, \tag{3.1}$$

admit a globally self-similar singularity at a time $T_* > 0$, at a single point $x_* \in \mathbb{R}^3$.

3.1. The self-similar ansatz. Due to Galilean symmetry, we assume without loss of generality that the singularity occurs at the space location $x_* = 0$, and that the similarity profile for the velocity vanishes at this point. Due to time-rescaling symmetry, we take without loss of generality $T_* = 1$. That is, we investigate the globally self-similar ansatz:

$$\begin{aligned} u(x,t) &= (1-t)^{\gamma-1} U(y), \qquad p(x,t) = (1-t)^{2(\gamma-1)} P(y), \\ \omega(x,t) &= \frac{1}{1-t}\Omega(y), \qquad y = \frac{x}{(1-t)^\gamma}. \end{aligned} \tag{3.2}$$

This ansatz is less restrictive than it seems. A more general form, $\omega(x,t) = A(t)\widetilde{\Omega}\left(\frac{x-x(t)}{\ell(t)}, \tau(t)\right)$ leads to the form above. Indeed, writing the Euler equations in vorticity form and using the fact that the velocity and vorticity are related by the linear relation $\omega = \nabla\times u$, we see that the nonlinear term $u\cdot\nabla\omega - \omega\cdot\nabla u$ scales with amplitude A^2 under the general ansatz above. We thus arrive at the ODE $\dot{A} = A^2$ where $\dot{A}$ is a time derivative [1]. This leads inevitably to $A(t) = (T-t)^{-1}$. The rest follows dividing the Euler equation by A^2 and taking the simplest situation, when the ensuing coefficients are constant. Then the constant coefficient in front of $y\cdot\nabla\Omega$ leads to $\ell(t) = \ell_0\left(1 - \frac{t}{T}\right)^\gamma$. The term due to $x(t)$ is integrated, because A and ℓ are known. It follows that $x(t) = x_* - \frac{c}{\gamma}\ell(t)$ where c is a constant, and this is dealt with by translating $\widetilde{\Omega}$.

3.1.1. *The stationary self-similar PDE.* The similarity profiles U, P, Ω appearing in (3.2) satisfy the (stationary) self-similar Euler equation in velocity form

$$(1-\gamma)U + \gamma(y\cdot\nabla)U + (U\cdot\nabla)U + \nabla P = 0, \qquad \nabla\cdot U = 0, \tag{3.3}$$

or equivalently, the (stationary) self-similar Euler equation in vorticity form

$$\Omega + \gamma(y\cdot\nabla)\Omega + (U\cdot\nabla)\Omega = (\Omega\cdot\nabla)U, \qquad \Omega = \nabla\times U, \qquad \nabla\cdot U = 0. \tag{3.4}$$

As mentioned earlier, due to Galilean symmetry, we assume throughout this section that the self-similar velocity profile U vanishes at the origin. We also only consider profiles U which are sub-linear at infinity.[2] We summarize these properties as

$$U(0) = 0, \qquad \lim_{|y|\to\infty} |y|^{-1}|U(y)| = 0\,. \tag{3.5}$$

[1]This time derivative can be with respect to an arbitrary internal clock, that is: $\frac{d}{dt}$ can be replaced by $\frac{1}{f(t)}\frac{d}{dt}$ where $f(t)$ is an arbitrary non-vanishing function.

[2]These are the profiles for which we may (typically) employ a localization or cutoff procedure at large values of y, to ensure that the velocity field u in original (x,t) coordinates has finite kinetic energy.

3.1.2. *Space rescaling and normalization.* Note that if U and Ω are a solution of (3.4) on $\mathbb{R}^3$, then $U_\lambda(y) = \lambda U(\frac{y}{\lambda})$ and $\Omega_\lambda(y) = \Omega(\frac{y}{\lambda})$, are also a solution of (3.4) on $\mathbb{R}^3$. Under this rescaling, we have[3]

$$\|\nabla \Omega_\lambda\|_{L^\infty(\mathbb{R}^3)} = \lambda^{-1} \|\nabla \Omega\|_{L^\infty(\mathbb{R}^3)}, \qquad \text{and} \qquad \|\Omega_\lambda\|_{L^p(\mathbb{R}^3)} = \lambda^{\frac{3}{p}} \|\Omega\|_{L^p(\mathbb{R}^3)}. \tag{3.6}$$

In order to fix units, throughout this section we consider profiles which are normalized via (3.6) to satisfy

$$\|\nabla \Omega\|_{L^\infty(\mathbb{R}^3)} = 1. \tag{3.7}$$

3.1.3. *Behavior of the profiles at space infinity.* While the assumption of sublinearity of U at infinity (cf. (3.5)) is necessary for any reasonable self-similar profile, the self-similar PDEs (3.3) and (3.4) impose more stringent assumptions. Indeed, (3.5) dictates that the term $\gamma y \cdot \nabla$ is stronger than the term $U \cdot \nabla$ for large values of $|y|$. Thus, the behavior of U and Ω as $|y| \to \infty$ must respect the kernel of the operators $(1-\gamma) + \gamma y \cdot \nabla$ (for U) and $1 + \gamma y \cdot \nabla$ (for Ω). We deduce that the leading order behavior of the self-similar profiles in the far field is given by

$$|U| \sim |y|^{\frac{\gamma-1}{\gamma}}, \qquad \text{and} \qquad |\Omega| \sim |y|^{-\frac{1}{\gamma}}, \qquad \text{as} \qquad |y| \to \infty.$$

It is convenient to quantify the above asymptotic descriptions. Recalling that $U(0) = 0$, we assume that there exists a constant $\mathsf{C}_\flat > 0$ such that

$$|U(y)| \leq \mathsf{C}_\flat |y| \langle y \rangle^{-\frac{1}{\gamma}}, \qquad \text{and} \qquad |\Omega(y)| + |\nabla U(y)| \leq \mathsf{C}_\flat \langle y \rangle^{-\frac{1}{\gamma}}, \qquad \text{for all} \qquad y \in \mathbb{R}^3. \tag{3.8}$$

3.2. A remark for the viscous case. We consider here hypothetical blow up for the 3D incompressible Navier-Stokes equation (set $\mathsf{RHS}_{(3.1)} = \Delta u$ instead of 0), where the vorticity $\omega(x,t)$ is approximately self-similar.

Proposition 3.1. *Let $\omega(x,t)$ be a solution of the 3D incompressible Navier-Stokes equation which can be written as*

$$\omega(x,t) = \omega_{\rm in}(x,t) + \omega_{\rm out}(x,t),$$

where $\omega_{\rm out}$ is regular, in the sense that

$$\int_0^1 \|\omega_{\rm out}(\cdot,t)\|^4_{L^2(\mathbb{R}^3)} dt \leq \Gamma < \infty, \tag{3.9}$$

and where $\omega_{\rm in}$ is supported in $B_{\ell(t)}(x(t))$ with $\ell(t) \sim (1-t)^\gamma$, and satisfies

$$\|\omega_{\rm in}(\cdot,t)\|_{L^2(B_{\ell(t)}(x(t)))} \leq C(1-t)^{-1+\frac{3\gamma}{2}} \tag{3.10}$$

for some $C < \infty$. Then, if the solution blows up at time $t = 1$, it follows that $\gamma \leq \frac{1}{2}$.

Remark 3.2. The assumption (3.10) is satisfied if

$$\omega_{\rm in}(x,t) \sim \frac{1}{1-t} \Omega\left(\frac{x - x(t)}{(1-t)^\gamma}\right) \qquad \text{in} \qquad B_{\ell(t)}(x(t)), \qquad \text{as} \qquad t \to 1^-,$$

with $|\Omega(y)| \leq C$ for $|y| \leq 1$, and $\ell(t) \sim (1-t)^\gamma$. The assumption (3.9) is satisfied for instance if $\omega_{\rm out}(x,t) = 0$ for $|x - x(t)| \leq \ell(t)$,

$$|\omega_{\rm out}(x,t)| \leq C_1 |x - x(t)|^{-\frac{1}{\gamma}}, \qquad \text{for} \qquad \ell(t) \leq |x - x(t)| \leq 1,$$

or, more generally, if

$$\int_0^1 \|\omega_{\rm out}(t)\|^4_{L^2(B_1(x(t)))} dt \leq C_2 < \infty, \tag{3.11}$$

and if, for instance

$$\sup_{|x - x(t)| \geq 1} |\omega_{\rm out}(x,t)| \leq C_3.$$

[3] As we shall see later (cf. (3.8)), Ω is only expected to lie in $L^p(\mathbb{R}^3)$ when $p > 3\gamma$.

Indeed, because $\omega_{\rm out}(x,t) = \omega(x,t)$ for all $|x - x(t)| \geq 1$ we have $|\omega_{\rm out}(x,t)|^2 \leq C_3|\omega(x,t)|$ for $|x-x(t)| \geq 1$. Then $\|\omega_{\rm out}(t)\|^2_{L^2(\mathbb{R}^3\setminus B(x(t),1))} \leq C_3\|\omega(t)\|_{L^1(\mathbb{R}^3)}$. As it is well known, $\sup_t \|\omega(t)\|_{L^1(\mathbb{R}^3)}$ is bounded in terms of the initial data [17], so the assumptions imply that $\omega_{\rm out} \in L^4(dt; L^2(\mathbb{R}^3))$.

Thus, the assumption (3.9) is satisfied if ω is a smooth vorticity matched to an inner self-similar blow up ansatz with asymptotic behavior at infinity of the type (3.8). The main point is that both the inner blow up profile and the matching vorticity magnitude majorant $z(x,t) = |x - x(t)|^{-\frac{1}{\gamma}}\mathbf{1}_{\{\ell(t)\leq|x-x(t)|\leq 1\}}$ have the property that they belong to $L^4(dt; L^2(\mathbb{R}^3))$ if $\gamma > \frac{1}{2}$, which is a class of non-blow up.

Proof of Proposition 3.1. We add the contributions from the inner (3.10) and outer (3.9) pieces and deduce

$$\int_0^1 \|\omega(\cdot,t)\|^4_{L^2(\mathbb{R}^3)} dt \leq 16\left(\Gamma + \frac{C^4}{6\gamma - 3}\right).$$

If $\gamma > \frac{1}{2}$, the right hand side is finite, and there is no blow up because of a well-known regularity criterion [22]. □

3.3. Self-similar vortex stretching. We revisit aspects of the proof of Theorem 2.1, under the additional assumption that a globally self-similar profile exists (as specified in Section 3.1).

Denote the vorticity direction vector as

$$\Xi(y) = \frac{\Omega(y)}{|\Omega(y)|},$$

and define the self-similar vortex stretching factor by

$$\mathsf{A}(y) := \Xi^i(y)\partial_i U^j(y)\Xi^j(y).$$

Taking the dot product of (3.4) with $\Omega|\Omega|^{p-2}$ we deduce the identity

$$|\Omega|^p + \tfrac{1}{p}(\gamma y + U)\cdot\nabla(|\Omega|^p) = \mathsf{A}|\Omega|^p. \tag{3.12}$$

In view of (3.8) we expect $|\Omega|$ to be integrable at infinity only for $p > 3\gamma$, which is why we stated (3.12) with general p, as opposed to the standard $p = 2$ formulation.

By the very definition of A, the fact that $|\Xi(y)| = 1$, and using assumption (3.8) we have that $|\mathsf{A}(y)| \leq \mathsf{C}_\flat\langle y\rangle^{-\frac{1}{\gamma}}$ for all $y \in \mathbb{R}^3$. We aim to obtain a direct bound for the vortex stretching term; from (2.4), we have that

$$\mathsf{A}(y) = \frac{3}{4\pi}\text{p.v.}\int_{\mathbb{R}^3} (\hat{z}\cdot\Xi(y))\,[\hat{z}\cdot(\Omega(y+z)\times\Xi(y))]\,\frac{dz}{|z|^3}, \tag{3.13}$$

where $\hat{z} = z/|z|$. Let $\chi = \chi(|z|)$ be a smooth radially decreasing cutoff function, with $\chi \equiv 1$ in B_1 and $\chi \equiv 0$ in $B_2^{\complement}$. Let $L > 0$ be a length scale to be optimized later. Then, from (3.13), and (3.8), we obtain that for $p > 3\gamma$:

$$\begin{aligned}&\left|\mathsf{A}(y) - \frac{3}{4\pi}\int_{\mathbb{R}^3} (\hat{z}\cdot\Xi(y))\,[\hat{z}\cdot(\Omega(y+z)\times\Xi(y))]\,\chi\left(\tfrac{|z|}{L}\right)\frac{dz}{|z|^3}\right| \\ &\qquad \leq \frac{3^{1/p}(p-1)^{(p-1)/p}}{(4\pi)^{1/p}}\|\Omega\|_{L^p(\mathbb{R}^3)}L^{-\frac{3}{p}}.\end{aligned} \tag{3.14}$$

On the other hand, since $\nabla\Omega \in L^\infty(\mathbb{R}^3)$, using the cancellation property $\Omega(y)\times\Xi(y) = 0$ and Taylor's theorem we obtain the estimate

$$\frac{3}{4\pi}\left|\int_{\mathbb{R}^3} (\hat{z}\cdot\Xi(y))\,\big[\hat{z}\cdot\big(\big(\Omega(y+z) - \Omega(y)\big)\times\Xi(y)\big)\big]\,\chi\left(\tfrac{|z|}{L}\right)\frac{dz}{|z|^3}\right| \leq 6\|\nabla\Omega\|_{L^\infty(\mathbb{R}^3)}L. \tag{3.15}$$

Optimizing the choice of L in (3.14) and (3.15), we deduce the pointwise bound

$$|\mathsf{A}(y)| \leq \mathsf{C}_p \|\nabla\Omega\|_{L^\infty(\mathbb{R}^3)}^{\frac{3}{p+3}} \|\Omega\|_{L^p(\mathbb{R}^3)}^{\frac{p}{p+3}}, \qquad \mathsf{C}_p := 2 \cdot 6^{\frac{3}{p+3}} \left(\tfrac{3}{4\pi}\right)^{\frac{1}{p+3}} (p-1)^{\frac{p-1}{p+3}}, \tag{3.16}$$

for all $y \in \mathbb{R}^3$. We note that the quantity present in (3.16), namely $\|\nabla\Omega\|_{L^\infty(\mathbb{R}^3)}^{\frac{3}{p+3}} \|\Omega\|_{L^p(\mathbb{R}^3)}^{\frac{p}{p+3}}$, is scaling invariant under the natural scaling from (3.6).

We wish to pair the pointwise upper bound for A in (3.16) with a lower bound. In order to achieve this, we evaluate (3.12) at a global maximum of $|\Omega|$, to directly obtain:

Proposition 3.3. *If y_* is a (self-similar) space location at which $|\Omega| \not\equiv 0$ attains its global maximum, then $\mathsf{A}(y_*) = 1$.*

By combining the bound (3.16) with Proposition 3.3, we have thus established:

Theorem 3.4. *Let (U, Ω) be any solution of (3.4) which satisfies the bound (3.8) for some $\gamma > 0$. Then, the vorticity profile cannot be too small, in the sense Ω must obey the scaling invariant lower bound*

$$\|\nabla\Omega\|_{L^\infty(\mathbb{R}^3)}^{\frac{3}{p+3}} \|\Omega\|_{L^p(\mathbb{R}^3)}^{\frac{p}{p+3}} \geq \mathsf{C}_p^{-1}, \tag{3.17}$$

for any $p > 3\gamma$, where the constant $\mathsf{C}_p > 0$ is as defined in (3.16).

Proof of Theorem 3.4. Since (3.16) must hold at $y_* = \operatorname{argmax}_{y\in\mathbb{R}^3}|\Omega|$, the bound (3.17) follows from $\mathsf{A}(y_*) = 1$, cf. Proposition 3.3. ☐

Remark 3.5. Under the normalization (3.7), and using the definition of C_p, we obtain from (3.17) the requirement

$$\|\Omega\|_{L^p(\mathbb{R}^3)} \geq \tfrac{1}{2} \left(\tfrac{\pi}{1296}\right)^{\frac{1}{p}} (p-1)^{-1+\frac{1}{p}}, \tag{3.18}$$

whenever $p > 3\gamma$.

3.4. Self-similar Lagrangian particle trajectories. It is convenient to denote the total transport velocity appearing in (3.3) and (3.4) as

$$V(y) := \gamma y + U(y). \tag{3.19}$$

The self-similar Lagrangian trajectories are defined as solutions of the autonomous ODE

$$\frac{d}{d\tau} Y(a,\tau) = V(Y(a,\tau)) \tag{3.20}$$

with $Y(a,0) = a$. Let $X = X(a,t)$ denote the Lagrangian particle trajectories associated to the velocity field $u = u(x,t)$; that is, $\frac{d}{dt}X(a,t) = u(X(a,t),t)$ for all $t \in (0,1)$, with initial datum $X(a,0) = a$. The self-similar Lagrangian particle trajectories are related to X via

$$Y(a,\tau) = \frac{X(a,t)}{(1-t)^\gamma}, \qquad \text{with} \qquad \tau = -\log(1-t). \tag{3.21}$$

We note that for $a = 0$ we have $Y(0,\tau) = 0$ for all $\tau \in \mathbb{R}$; the origin is a fixed point of the dynamical system induced by (3.20).

3.4.1. *The self-similar Cauchy formula and the Weber formula.* The Cauchy vorticity (vector) transport formula $\omega(X(a,t),t) = (\nabla_a X)(a,t)\omega_0(a)$ transformed into self-similar coordinates becomes

$$\Omega(Y(a,\tau)) = e^{-(\gamma+1)\tau} (\nabla_a Y)(a,\tau)\Omega(a), \tag{3.22}$$

for all $a \in \mathbb{R}^3$ and all $\tau \in \mathbb{R}$. The difference between (3.22) and the classical Cauchy formula is that $\det(\nabla_a Y)(a,\tau) = e^{3\gamma\tau}$ (instead of $= 1$), because the vector field $\gamma y + U(y)$ is not divergence free; we have $\nabla \cdot (\gamma y + U(y)) = 3\gamma$. The classical Weber formula for Euler equations is

$$u^j(X(a,t),t)\partial_i X^j(a,t) - u^i(a,0) = \partial_i \pi(a,t) \tag{3.23}$$

and is obtained by checking that $\partial_t((\nabla X(a,t))^T u(X(a,t),t))$ is a gradient. The self-similar Weber formula is

$$e^{(1-2\gamma)\tau} U^j(Y(a,\tau))\partial_i Y^j(a,\tau) - U^i(a) = \partial_i q(a,\tau). \tag{3.24}$$

for an appropriate scalar function q. The proof of the Weber formula (3.24) follows either from changing variables in (3.23), or directly in similarity variables from the observation stemming from (3.3) that

$$\partial_\tau((\nabla Y)^T U(Y)) + (1-2\gamma)(\nabla Y)^T U(Y) + \nabla\left(P(Y) - \frac{|U(Y)|^2}{2}\right) = 0$$

holds at each a. Thus,

$$e^{(1-2\gamma)\tau}((\nabla Y)^T U(Y))(a,\tau) - U(a) = -\nabla\int_0^\tau e^{(1-2\gamma)s}\left(P(Y) - \frac{|U(Y)|^2}{2}\right)ds$$

is a gradient.

3.4.2. *The self-similar Kelvin circulation theorem.* Let $C(0) \subset \mathbb{R}^3$ be a simple closed curve parametrized as $C(0) = \{a(\lambda) \colon \lambda \in \mathbb{T}\}$. The circulation of U on a loop $C(0)$ is the integral of the 1-form $U(y)dy$ on that loop, which is computed as

$$\Gamma_{\rm ss}(0) = \oint_{C(0)} U dy = \int_{\mathbb{T}} U^j(a(\lambda))\dot{a}^j(\lambda)d\lambda, \tag{3.25}$$

where we have denoted $\dot{a}^j = \frac{d}{d\lambda}a^j$. We note that V (recall (3.19)) and U have the same circulation on any loop because the 1-forms $V(y)dy$ and $U(y)dy$ differ by an exact 1-form, $d\frac{\gamma|y|^2}{2}$.

The self-similar Weber formula (3.24) says that the pull back of the 1-form $e^{(1-2\gamma)\tau}U(y)dy$ under the diffeomorphism $Y(a,\tau)$ is the sum of the form $U(a)da$ and an exact 1-form, dq. The circulation of U on $C(\tau)$ where $C(\tau) = Y(C(0),\tau)$ is the push forward (image) of the loop $C(0)$ is

$$\Gamma_{\rm ss}(\tau) = \oint_{C(\tau)} U(y)dy. \tag{3.26}$$

The self-similar Weber formula (3.24) implies

$$e^{(1-2\gamma)\tau}\oint_{C(\tau)} U(y)dy = \oint_{C(0)} U(y)dy. \tag{3.27}$$

Indeed, multiplying (3.24) by $\dot{a}^i(\lambda)$ and integrating $d\lambda$ on $\mathbb{T}$ we have

$$\begin{aligned} &e^{(1-2\gamma)\tau}\Gamma_{\rm ss}(\tau) = e^{(1-2\gamma)\tau}\oint_{C(\tau)} U(y)dy \\ &= e^{(1-2\gamma)\tau}\int_{\mathbb{T}} U^j(Y(a(\lambda),\tau))\,(\partial_i Y^j)(a(\lambda),\tau)\dot{a}^i(\lambda)d\lambda \\ &= \Gamma_{\rm ss}(0) \end{aligned} \tag{3.28}$$

In original variables, the Weber formula says that the pull back of the 1-form $u(x,t)dx$ under the Lagrangian flow $X(a,t)$ is the sum of the time independent 1-form $u(a,0)da$ and an exact 1-form. As a consequence, Kelvin's circulation theorem states that the circulation $\Gamma(t)$ around the material loop $C(t) = \{X(a(\lambda),t) \colon \lambda \in \mathbb{T}\}$ is conserved. That is, we have $\frac{d}{dt}\Gamma(t) = 0$, where

$$\Gamma(t) = \oint_{C(t)} u(x,t)dx = \int_{\mathbb{T}} u^j(X(a(\lambda),t),t)(\partial_i X^j)(a(\lambda),t)\dot{a}^i(\lambda)d\lambda.$$

Changing to self-similar variables using (3.21) and time $\tau = -\log(1-t)$ we note that

$$\Gamma(t) = e^{(1-2\gamma)\tau}\Gamma_{\rm ss}(\tau).$$

REMARK 3.6. Because $e^{(1-2\gamma)\tau}\Gamma_{\rm ss}(\tau)$ is constant in time, if the integral $\int_{\mathbb{T}}(\ldots)d\lambda$ appearing in (3.28) is nonzero, and remains bounded as $\tau \to \infty$, then we must have $\gamma = 1/2$ in order to avoid a vanishing/blowing up exponential factor. Thus, the self-similar Kelvin circulation theorem identifies the similarity exponent $\gamma = 1/2$ as being distinguished for 3D Euler, without knowledge of any viscous regularization. It also indicates that fixed points of (3.20), and by extension points on the stable manifold of these fixed points, play an important role in the analysis. A result in this direction is offered in Theorem 4.4 below.

3.4.3. *The self-similar Bernoulli function.* Note that $\nabla \times V = \nabla \times U = \Omega$ since $\nabla \times y = 0$. In (3.3) we rewrite the first three terms as

$$(1-\gamma)U + (\gamma y + U)\cdot\nabla U = (1-\gamma)(V-\gamma y) - \gamma V + V\cdot\nabla V$$
$$= -\gamma(1-\gamma)\nabla\frac{|y|^2}{2} + (1-2\gamma)V + \Omega\times V + \nabla\frac{|V|^2}{2}.$$

After rearranging, we obtain that (3.3) is equivalent to

$$(1-2\gamma)\,V + \Omega\times V + \nabla\mathcal{H} = 0, \tag{3.29}$$

where the *self-similar Bernoulli function*[4] defined by

$$\mathcal{H}(y) := \frac{1}{2}|\gamma y + U(y)|^2 + P(y) + \frac{\gamma(\gamma-1)}{2}|y|^2. \tag{3.30}$$

Dotting (3.29) with V yields the transport identity

$$V\cdot\nabla\mathcal{H} = (2\gamma-1)|V|^2, \tag{3.31}$$

which, restated along self-similar Lagrangian trajectories (3.20) is

$$\frac{d}{d\tau}\mathcal{H}(Y(a,\tau)) = (2\gamma-1)|V(Y(a,\tau))|^2. \tag{3.32}$$

Two immediate consequences follow. First, for $\gamma < 1/2$ the Bernoulli function is *strictly decreasing* along any non-stationary trajectory:

$$\gamma < \tfrac{1}{2} \qquad \Longrightarrow \qquad \tfrac{d}{d\tau}\mathcal{H}(Y(a,\tau)) \le 0, \tag{3.33}$$

with equality if and only if $V(Y(a,\tau)) = 0$. Second, since $|V(y)| \sim \gamma|y|$ for $|y| \gg 1$ (cf. (3.8)) and for $\gamma < 1$ we have $P(y) = \mathsf{o}(1)$ as $|y| \to \infty$,[5] the far-field behavior of the Bernoulli function is

$$\mathcal{H}(y) = \frac{\gamma(2\gamma-1)}{2}|y|^2 + \mathsf{o}(|y|^2) \qquad \text{as } |y|\to\infty. \tag{3.34}$$

In particular, for $\gamma < 1/2$ we have $\mathcal{H}(y) \to -\infty$ as $|y| \to \infty$.

3.5. An outgoing property? We note that all trajectories with sufficiently large labels escape to infinity. Indeed, the bound (3.8) implies that for all $y \in \mathbb{R}^3$ such that

$$|y| \ge R_\flat := \max\left\{1, \left(\frac{2\mathsf{C}_\flat}{\gamma}\right)^\gamma\right\}, \tag{3.35}$$

we have $|U(y)\cdot y| \le \frac{\gamma}{2}|y|^2$, and therefore $(\gamma y + U(y))\cdot y \ge \frac{\gamma}{2}|y|^2$. From (3.20) we then immediately obtain that if $|a| \ge R_\flat$, then $|Y(a,\tau)|$ is a strictly increasing function of τ, and we have the bounds $|a|e^{\frac{\gamma}{2}\tau} \le |Y(a,\tau)| \le |a|e^{\frac{3\gamma}{2}\tau}$, and $e^{\frac{\gamma}{2}\tau} \le |\nabla_a Y(a,\tau)| \le e^{\frac{3\gamma}{2}\tau}$, for all $\tau \ge 0$.

The exponential escape to infinity of self-similar Lagrangian flow maps (as exhibited above) is well-known to play a fundamental role, both in proving the existence and the stability of self-similar profiles. In the context of the compressible Euler equations, see [**5, 34, 3, 10**]. In our incompressible 3D Euler setting we only know that trajectories $Y(a,\tau)$ for which $|a|$ is sufficiently large escape to infinity, and that the trajectory emanating from $a = 0$ is frozen at the origin.

[4] See [**36**] for use of the analogous Bernoulli function in the case of 3D incompressible Navier-Stokes.

[5] A bound for the pressure may be obtained as in [**36**]. Taking the divergence of (3.3) we have that $P = P_h + \mathcal{R}_i\mathcal{R}_j(U^iU^j)$, where P_h is harmonic and $\mathcal{R}$ is the vector of Riesz-transforms. For $\gamma < 1$, the second term in this expression decays to 0 as $|y| \to \infty$. From (3.3) and (3.8) we also have that $|\nabla P|$ (and hence also $|\nabla P_h|$) decays to 0 as $|y| \to \infty$ when $\gamma < 1$, showing that P_h is a constant. Since the pressure is defined only up to a constant, $P_h \equiv 0$.

A priori, we cannot rule out the existence of points $y \in \mathbb{R}^3 \setminus \{0\}$ for which $V(y) = \gamma y + U(y) = 0$. In the compressible setting, such points exist—*sonic points*—and they are well known to cause tremendous headaches, such as severe instabilities, even within the class of radially symmetric solutions [**29, 34, 3**]. For incompressible 3D Euler equations, a *global outgoing property*, quantified as a lower bound of the type

$$V(y) \cdot y = (\gamma y + U(y)) \cdot y \geq c_* |y|^2, \qquad \text{for all } y \in \mathbb{R}^3, \qquad \text{for some } c_* > 0, \tag{3.36}$$

has been recently identified by Elgindi [**25**] as one of the two fundamental assumptions needed to prove the existence of solutions to (3.4); see [**25**, Hypothesis 6.7]. Assumption (3.36) implies that the only zero of V occurs at $y = 0$ (since $c_* > 0$). In this section we want to allow for a generalized outgoing property, which allows for the degenerate case $c_* = 0$, allows the function V to vanish away from the origin, and only makes assumptions that are local, near the nodal set of V.

Definition 3.7. *We say that the field $V(y) = \gamma y + U(y)$ satisfies the* local outgoing property *if the nodal set*

$$\mathcal{N}_V := \{y_* \in \mathbb{R}^3 \colon V(y_*) = 0\}$$

is finite,[6] *and if there exists $c_* \geq 0$ and $\varepsilon_* > 0$ such that for all $y_* \in \mathcal{N}_V$ we have*

$$V(y) \cdot (y - y_*) \geq c_* |y - y_*|^2, \qquad \text{when} \qquad |y - y_*| \leq \varepsilon_*. \tag{3.37}$$

A few remarks concerning Definition 3.7 are in order:

- Note that $\mathcal{N}_V \neq \emptyset$ because $0 \in \mathcal{N}_V$ always.
- The global outgoing property (3.36) implies the local outgoing property of Definition 3.7. Indeed, upon evaluating (3.36) at any $y_* \in \mathcal{N}_V$ we deduce that $0 = V(y_*) \cdot y_* \geq c_* |y_*|^2$; this shows $y_* = 0$ since $c_* > 0$. Thus $\mathcal{N}_V = \{0\}$. Condition (3.37) thus holds for any $\varepsilon_* > 0$.
- When $\mathcal{N}_V = \{0\}$, (3.37) generalizes (3.36) not just in the local-in-y nature of the inequality; when $c_* = 0$ it also allows for $V(y) \cdot y = \gamma |y|^2 + y \cdot U(y)$ to vanish as $|y|^{2n}$ for $n \geq 2$ as $|y| \to 0$.

Our next result points out a significant consequence of assumption (3.37).

Theorem 3.8. *Let U be a C^2 smooth globally self-similar velocity profile for 3D incompressible Euler. If there exists $y_* \in \mathcal{N}_V$ such that $\Omega(y_*) \neq 0$, and if the local outgoing property (3.37) holds* in an open neighborhood *of $y = y_*$ for some $c_* \geq 0$, then*

$$\gamma \geq \frac{1}{2} + c_*.$$

Proof of Theorem 3.8. We rewrite the self-similar vorticity equation (3.4) as

$$\Omega + V \cdot \nabla \Omega = \mathbb{S} \Omega, \tag{3.38}$$

where $\mathbb{S} = \frac{1}{2}((\nabla U) + (\nabla U)^\intercal)$ is the rate of strain matrix. If $y_* \in \mathbb{R}^3$ is a zero of V, restricting (3.38) to $y = y_*$, we deduce that the matrix

$$\mathbb{S}_{y_*} := \mathbb{S}(y_*)$$

has 1 as an eigenvalue, with associated eigenvector $\Omega(y_*) \neq 0$. Now $\mathbb{S}_{y_*}$ is a real, symmetric, traceless matrix ($\nabla \cdot U = 0$) and we denote its eigenvalues (which are real) as $1, \lambda_2, \lambda_3$, with $\lambda_2 + \lambda_3 = -1$.

Since $V(y_*) = 0$, for $|y - y_*| \ll 1$ we have

$$\begin{aligned} V(y) \cdot (y - y_*) &= (y - y_*) \otimes (y - y_*) \colon (\nabla V)(y_*) + \mathcal{O}(|y - y_*|^3) \\ &= (y - y_*) \otimes (y - y_*) \colon (\gamma \mathrm{Id} + \mathbb{S}_{y_*}) + \mathcal{O}(|y - y_*|^3). \end{aligned}$$

Thus, if the outgoing condition (3.37) holds in an open neighborhood of y_*, then for all $|y - y_*| \ll 1$ we have $\gamma + \hat{z} \cdot \mathbb{S}_{y_*} \hat{z} + \mathcal{O}(|y - y_*|) \geq c_*$, where $\hat{z} = \frac{y - y_*}{|y - y_*|}$. Passing $y \to y_*$, we deduce that the

[6] As discussed below (3.35), the bound (3.8) implies $\mathcal{N}_V \subset B_{R_\flat}(0)$. Hence, if we assume that $\mathcal{N}_V$ has *no accumulation points*, then this set must automatically be finite.

smallest eigenvalue of the matrix $\mathbb{S}_{y_*}$, satisfies $\lambda_{\min}(\mathbb{S}_{y_*}) \geq c_* - \gamma$. Thus, the sum of the two-smallest eigenvalues of $\mathbb{S}_{y_*}$ is $\geq 2(c_* - \gamma)$, and due to the zero trace property, the largest eigenvalue of the matrix $\mathbb{S}_{y_*}$ satisfies $\lambda_{\max}(\mathbb{S}_{y_*}) \leq 2(\gamma - c_*)$. We have thus shown that the eigenvalues of $\mathbb{S}_{y_*}$ satisfy

$$[\lambda_{\min}(\mathbb{S}_{y_*}), \lambda_{\max}(\mathbb{S}_{y_*})] \subseteq [c_* - \gamma, 2(\gamma - c_*)].$$

In particular, because the eigenvalue 1 lies in this interval, we deduce $1 \leq 2(\gamma - c_*)$. The claimed lower bound $\gamma \geq c_* + 1/2$ now follows. □

The following result is a consequence of Theorem 3.8, and its proof.

Proposition 3.9. *Let U be a C^2 smooth similarity profile solving (3.3). Let $y_* \in \mathbb{R}^3$ be a zero of $V(y) = \gamma y + U(y)$. If U satisfies the outgoing property (3.37) locally near $y = y_*$ for some $c_* \geq 0$, and if $\gamma < 1/2 + c_*$, then we must have $\Omega(y_*) = 0$. Additionally, for any $n \in \mathbb{N}$ such that Ω is C^{n+1} smooth in an open neighborhood of y_*, we have that $\nabla^n \Omega(y_*) = 0$. That is, when $\gamma < 1/2 + c_*$, a smooth vorticity profile must vanish to infinite order at $y = y_*$.*

Proof of Proposition 3.9. We prove that $\nabla^m \Omega(y_*) = 0$ for all integers $0 \leq m \leq n$ by induction on m. The fact that $\Omega(y_*) = 0$ follows from Theorem 3.8; this is the base case $m = 0$.

As before, denote $\mathbb{S}_{y_*} = \mathbb{S}(y_*)$, which is real and symmetric. As in the proof of Theorem 3.8, the real eigenvalues of $\mathbb{S}_{y_*} = \mathbb{S}(y_*)$ satisfy

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \in [c_* - \gamma, 2(\gamma - c_*)]. \tag{3.39}$$

Without loss of generality, we take coordinates so that $\mathbb{S}_{y_*}$ is a diagonal matrix. In view of $\Omega(y_*) = 0$, we have that $\nabla U(y_*) = \mathbb{S}_{y_*}$.

Fix an integer $1 \leq m \leq n$, and assume by induction that $\partial^\beta \Omega(y_*) = 0$ for all multi-indices $\beta \in \mathbb{N}_0^3$ with $|\beta| \leq m - 1$. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3$ be a multi-index of length $|\alpha| = m$. We apply ∂^α to (3.38), and evaluate the resulting expression at $y = y_*$; using the Leibniz formula, the inductive assumption on the vanishing of derivatives of Ω, and the fact that $\gamma y_* + U(y_*) = 0$, for each $i \in \{1, 2, 3\}$ we obtain

$$(\partial^\alpha \Omega)^i(y_*) + \alpha_k(\gamma \delta_{jk} + \partial_k U^j(y_*))(\partial^{\alpha - e_k + e_j} \Omega)^i(y_*) = \mathbb{S}_{y_*}^{ij} (\partial^\alpha \Omega)^j(y_*).$$

The summation convention over repeated indices (j and k) is used in the above identity. Since $|\alpha| = m$, $\partial_k U^j(y_*) = \mathbb{S}_{y_*}^{kj}$, and we work in coordinates in which $\mathbb{S}_{y_*} = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$, the above expression simplifies to

$$(\partial^\alpha \Omega)^i(y_*) \left(1 + \gamma m - \lambda_i + \sum_{k=1}^{3} \alpha_k \lambda_k \right) = 0,$$

for each $i \in \{1, 2, 3\}$. Using (3.39) we may further bound

$$1 + m\gamma - \lambda_i + \sum_{j=1}^{3} \alpha_j \lambda_j \geq 1 + m\gamma - \lambda_3 + m\lambda_1 \geq 1 + mc_* + 2c_* - 2\gamma.$$

If $2\gamma < 1 + 2c_*$, then $1 + mc_* + 2c_* - 2\gamma > 0$ and it follows that $(\partial^\alpha \Omega)^i(y_*) = 0$. □

We combine the previous result with a global outgoing property, as stated in (3.37) (or [**25**, Hypothesis 6.7] for the non-degenerate case) and prove the following result:

Theorem 3.10. *Let U be a nontrivial C^2 smooth globally self-similar velocity profile for 3D incompressible Euler equations. Assume that the local outgoing property of Definition 3.7 holds.*

Furthermore, assume that Ω is real-analytic at all $y_* \in \mathcal{N}_V$.[7] *Then,*

$$\gamma \geq \frac{1}{2}.$$

We note that while the conclusion of Theorem 3.10 is similar to that of Theorem 3.8, the assumptions are different. Here we do not assume $\Omega(y_*) \neq 0$, and instead work with the complementary case when $\Omega(y_*) = 0$; the extra assumption is infinite regularity at the nodal points y_* of V (it is reasonable to expect that for $|y - y_*| \ll 1$, (3.4) could be solved via a power series expansion).

PROOF OF THEOREM 3.10. Assume by contradiction that $\gamma < 1/2$. For some integer $N \geq 0$, let $\mathcal{N}_V = \{y_*^{(\ell)}\}_{\ell=0}^N \in \mathbb{R}^3$. Our convention is that $y_*^{(0)} = 0$. Since $\gamma < 1/2 + c_*$, by Proposition 3.9 we have $\nabla^n \Omega(y_*^{(\ell)}) = 0$ for all $n \in \mathbb{N}_0$. The analyticity of Ω near $y_*^{(\ell)}$ shows the existence of a radius $R_* > 0$ such that $\Omega(y) = 0$ for all $|y - y_*^{(\ell)}| < R_*$, for all $0 \leq \ell \leq N$.

Next, we claim that for all $y_0 \in \mathbb{R}^3$ there exists a finite time $\tau_0 = \tau_0(y_0) \leq 0$ such that the backwards-in-time flow map satisfies

$$Y(y_0, \tau_0) \in \mathcal{D} := \bigcup_{\ell=0}^{N} B_{R_*}\big(y_*^{(\ell)}\big). \tag{3.40}$$

Due to the discussion in the first paragraph of Section 3.5, we only need to check condition (3.40) for y_0 belonging to the compact set $\mathcal{K} := \mathcal{D}^{\complement} \cap \overline{B_{R_\flat}(0)}$, where $R_\flat$ is as defined in (3.35).[8] Since all the zeros of $V(y) = \gamma y + U(y)$ lie in the interior of $\mathcal{D}$, since V is continuous and $\mathcal{K}$ is compact, there exists a $c_\flat > 0$ such that

$$|V(y)| \geq c_\flat, \qquad \text{for all} \qquad y \in \mathcal{K} = \mathcal{D}^{\complement} \cap \overline{B_{R_\flat}(0)}.$$

Moreover, recall that if $y_0 \in \overline{B_{R_\flat}(0)}$, then for all $\tau < 0$ we have that $Y(y_0, \tau) \in B_{R_\flat}(0)$. Combining these two facts shows that for any $y_0 \in \mathcal{K}$ and for all $\tau < 0$ such that $Y(y_0, \tau) \notin \mathcal{D}$, we have $|V(Y(y_0, \tau))| \geq c_\flat$. At this stage we recall from (3.32) that the Bernoulli function $\mathcal{H}$ satisfies

$$\frac{d}{d\tau}\mathcal{H}(Y(y_0, \tau)) = (2\gamma - 1)|V(Y(y_0, \tau))|^2 \leq (2\gamma - 1)c_\flat^2$$

for all $\tau < 0$ such that $Y(y_0, \tau) \notin \mathcal{D}$; here we have used again that $\gamma < 1/2$. Integrating this inequality on $[\tau, 0]$ implies

$$\mathcal{H}(Y(y_0, \tau)) \geq \mathcal{H}(y_0) - \tau(1 - 2\gamma)c_\flat^2$$

as long as $Y(y_0, \tau) \in \mathcal{K}$. Since the Bernoulli function $\mathcal{H}$ is continuous and $\mathcal{K}$ is compact, we deduce that τ cannot be taken to be arbitrarily negative; there must exist $\tau_0 \leq 0$ such that $Y(y_0, \tau_0) \in \mathcal{D}$. Since we have previously shown that $\Omega \equiv 0$ on $\mathcal{D}$, it follows that for any $y_0 \in \mathbb{R}^3$ there exists $\tau_0 \leq 0$ such that $\Omega(Y(y_0, \tau_0)) = 0$.

To conclude the proof, we appeal to the self-similar Cauchy formula (3.22), which states that

$$e^{(\gamma+1)\tau}\Omega(Y(y_0, \tau)) = (\nabla_a Y)(y_0, \tau)\Omega(y_0)$$

for all $y_0 \in \mathbb{R}^3$ and all $\tau \in \mathbb{R}$. Evaluating the above expression at $\tau = \tau_0$, using that $\Omega(Y(y_0, \tau_0)) = 0$, and noting that $\det(\nabla_a Y)(y_0, \tau_0) = e^{3\gamma\tau_0} \neq 0$ (so that this matrix is invertible), we deduce $\Omega(y_0) = 0$. Since y_0 was arbitrary we deduce $\Omega \equiv 0$, a contradiction. Thus $\gamma \geq 1/2$. ☐

[7] The high regularity of the profile in a neighborhood of fixed points of the self-similar Lagrangian flow played a crucial role in the proofs of finite time blowup for the compressible Navier-Stokes equations via self-similar imploding solutions of compressible Euler; see the arguments presented in [**35**, **3**, **41**]. These papers strongly suggest that lifting a self-similar singularity for 3D incompressible Euler to 3D incompressible Navier-Stokes, would also require this infinite local regularity at Lagrangian fixed points.

[8] This is because when $|y_0| > R_\flat$, we have that $|Y(y_0, \tau)| \leq |y_0|e^{\gamma\tau/2}$, as long as $|Y(y_0, \tau)| \geq R_\flat$. Thus, there exists a sufficiently negative τ such that $|Y(y_0, \tau)| = R_\flat$.

4. Axisymmetric global self-similarity

For smooth initial conditions (for instance C^1 vorticity), whether the 3D Euler equations (1.1) exhibit axisymmetric singularities remains an open problem. As such, it is natural to revisit the discussion of hypothetical globally self-similar solutions from Section 3, and make the additional assumption that the self-similar profile $U(y)$ is axisymmetric around the $\vec{e}_3$ axis.

The focus on axisymmetric profiles is also motivated by ongoing computational studies (see e.g. [**37**, **33**, **30**, **31**, **46**] and references therein), in which solutions U of the full (stationary) self-similar Euler equation (3.3) are sought within this symmetry class, because this dramatically reduces the computational space domain: from $y \in \mathbb{R}^3$ to $(r, z) \in \mathbb{R}_+ \times \mathbb{R}$.

We adopt self-similar cylindrical coordinates (r, z, θ), where $r = \sqrt{y_1^2 + y_2^2}$ and $z = y_3$. The self-similar velocity profile U can be written in this frame as

$$U(y) = U_r(r,z)\vec{e}_r + U_\theta(r,z)\vec{e}_\theta + U_z(r,z)\vec{e}_z.$$

The pressure is $P = P(r, z)$. With this notation, (3.3) becomes

$$\begin{aligned}
(1-\gamma)U_r + (\gamma r + U_r)\partial_r U_r + (\gamma z + U_z)\partial_z U_r + \partial_r P &= \frac{1}{r}U_\theta^2,\\
(1-\gamma)U_z + (\gamma r + U_r)\partial_r U_z + (\gamma z + U_z)\partial_z U_z + \partial_z P &= 0,\\
(1-\gamma)U_\theta + (\gamma r + U_r)\partial_r U_\theta + (\gamma z + U_z)\partial_z U_\theta &= -\frac{1}{r}U_r U_\theta,\\
\partial_r U_r + \frac{1}{r}U_r + \partial_z U_z &= 0.
\end{aligned} \tag{4.1}$$

Remark 4.1. Because we consider the self-similar collapse of initially smooth solutions, the swirl component of velocity cannot vanish identically: $U_\theta \not\equiv 0$.

The vorticity vector $\Omega = \nabla \times U$ may be written in components as

$$\Omega(y) = \Omega_r(r,z)\vec{e}_r + \Omega_\theta(r,z)\vec{e}_\theta + \Omega_z(r,z)\vec{e}_z,$$

where

$$\Omega_r = -\partial_z U_\theta, \qquad \Omega_z = \partial_r U_\theta + \tfrac{1}{r}U_\theta, \qquad \Omega_\theta = \partial_z U_r - \partial_r U_z. \tag{4.2}$$

With this notation, (3.4) becomes

$$\begin{aligned}
\Omega_r + (\gamma r + U_r)\partial_r \Omega_r + (\gamma z + U_z)\partial_z \Omega_r &= \Omega_r \partial_r U_r + \Omega_z \partial_z U_r,\\
\Omega_z + (\gamma r + U_r)\partial_r \Omega_z + (\gamma z + U_z)\partial_z \Omega_z &= \Omega_r \partial_r U_z + \Omega_z \partial_z U_z,\\
\Omega_\theta + (\gamma r + U_r)\partial_r \Omega_\theta + (\gamma z + U_z)\partial_z \Omega_\theta &= \frac{1}{r}\big(U_r\Omega_\theta - 2U_\theta\Omega_r\big).
\end{aligned} \tag{4.3}$$

Remark 4.2. Recall that if $\Omega(y)$ is assumed to be continuous, then on the axis of symmetry $\{r = 0\}$ we have $\Omega_r(0, z) = \Omega_\theta(0, z) = 0$. The same reasoning applies for a continuous velocity: $U_r(0, z) = U_\theta(0, z) = 0$. This leaves open the possibility that $\Omega_z(0, z)$ or $U_z(0, z)$ do not vanish identically on the axis of symmetry.

4.1. Lagrangian trajectories in the meridional (r, z)-plane. The ODE system (3.20) for the three-dimensional flow map $Y(a, \tau)$, decomposes into three components:

$$\begin{aligned}
\frac{d}{d\tau}R(r,z,\tau) &= \gamma R(r,z,\tau) + U_r(R(r,z,\tau), Z(r,z,\tau)),\\
\frac{d}{d\tau}Z(r,z,\tau) &= \gamma Z(r,z,\tau) + U_z(R(r,z,\tau), Z(r,z,\tau)),
\end{aligned} \tag{4.4}$$

and

$$\frac{d}{d\tau}\Theta(r,z,\tau) = \frac{1}{R(r,z,\tau)}U_\theta(R(r,z,\tau), Z(r,z,\tau)), \tag{4.5}$$

with initial conditions $R(r,z,0)=r$, $Z(r,z,0)=z$, and $\Theta(r_0,z_0,0)=0$. The dynamics of R and Z are decoupled from that of Θ, reducing the 3D Lagrangian evolution to a 2D autonomous system (4.4) in the meridional plane (R,Z). The full 3D trajectory is this 2D path spun around the $\vec{e}_3$-axis according to (4.5).

4.2. Consequences of an axisymmetric outgoing property. We revisit the discussion from Section 3.5, in the axisymmetric setting.

In axisymmetry, we have that

$$V=\gamma y+U=\underbrace{(\gamma r+U_r)}_{=:V_r}\vec{e}_r+U_\theta\vec{e}_\theta+\underbrace{(\gamma z+U_z)}_{=:V_z}\vec{e}_z.$$

Thus, (r_*,z_*) is a zero of V if and only if $\gamma r_*+U_r(r_*,z_*)=U_\theta(r_*,z_*)=\gamma z_*+U_z(r_*,z_*)=0$. For such a zero we note however that back in Cartesian $\mathbb{R}^3$, for any point y_* on the circle $(r_*,\theta,z_*)_{\theta\in[0,2\pi]}$ we have that $V(y_*)=0$ (by axisymmetry). Thus, if we insist on the finiteness of the nodal set $\mathcal{N}_V\subset\mathbb{R}^3$, as required by Definition 3.7, it follows that $r_*=0$; the aforementioned circle contracts onto a point. Hence the finiteness of $\mathcal{N}_V$ implies that all nodal values of V must lie on the axis of symmetry, and hence with Remark 4.2,

$$\mathcal{N}_V=\{(0,0,z_j)\colon V_z(0,z_j)=\gamma z_j+U_z(0,z_j)=0,\ -M\le j\le N\}$$

for some integers $M,N\ge 0$. For convenience we let $z_0=0$, which always yields a nodal value.

In the vicinity of zeroes $(0,z_*)$ of V, inequality (3.37) from Definition 3.7 becomes

$$rV_r(r,z)+(z-z_*)V_z(r,z)\ge c_*r^2+c_*(z-z_*)^2, \tag{4.6}$$

for all $(r,z)\in[0,\infty)\times\mathbb{R}$ with $r^2+(z-z_*)^2\le\varepsilon_*^2$.

We record the results of Theorems 3.8 and 3.10 in the axisymmetric setting. Sections 4.3 and 4.4 contain the new results: therein we analyze the case when the outgoing property of Definition 3.7 fails.

THEOREM 4.3. *Let $U\not\equiv 0$ be a C^2 smooth axisymmetric similarity profile solving (4.1). Then:*

(i) Let $(0,z_*)$ be a zero of V, meaning that $\gamma z_*+U_z(0,z_*)=0$. If $\Omega_z(0,z_*)\neq 0$ and if the outgoing property (4.6) holds locally near $(0,z_*)$ for some $c_*\ge 0$, then $\gamma\ge 1/2+c_*$.

(ii) If the nodal set $\mathcal{N}_V\subset\mathbb{R}^3$ is finite, the outgoing property (4.6) holds locally in the vicinity of all nodal values $(0,z_*)\in\mathbb{R}_+\times\mathbb{R}$ for some $c_*\ge 0$, and if Ω is real-analytic at each of these nodal values, then $\gamma\ge 1/2$.

4.3. Failure of the outgoing property: fixed points with nonzero swirl velocity. Here we explore constraints imposed by the existence of a fixed point $(r,z)\in\mathbb{R}_+\times\mathbb{R}$ for the dynamics (4.4) in the meridional plane; i.e., points such that

$$\gamma r+U_r(r,z)=0,\qquad \text{and}\qquad \gamma z+U_z(r,z)=0. \tag{4.7}$$

In this case, we automatically deduce $R(r,z,\tau)=r$, $Z(r,z,\tau)=z$, for all $\tau\in\mathbb{R}$, and $\Theta(r,z,\tau)=\frac{\tau}{r}U_\theta(r,z)\mod 2\pi$. If we additionally know that $U_\theta(r,z)\neq 0$,[9] we have thus obtained a periodic orbit for the full dynamics of the Lagrangian flow map $Y(\cdot,\tau)$ with nonzero circulation. This leads to the following result:

THEOREM 4.4. *Assume that U is a C^2 smooth, axisymmetric, globally self-similar velocity profile for 3D incompressible Euler equations. Assume there exists a fixed point (r,z) for the dynamics in the meridional plane, i.e. (4.7) holds. If $U_\theta(r,z)\neq 0$, then the similarity exponent satisfies $\gamma=\frac{1}{2}$.*

REMARK 4.5. If (r,z) is a solution of (4.7) with $U_\theta(r,z)=0$, then also $\Omega_\theta(r,z)=0$. This follows from the swirl component of the vorticity equation (4.3).

[9]The condition $U_\theta(r,z)\neq 0$ implies $r>0$, due to Remark 4.2.

Proof of Theorem 4.4. We consider the loop $C(0)$ parameterized by (r, z, θ), with $\theta \in [0, 2\pi)$. Due to (4.7), this loop is an invariant set of the flow: $C(\tau) = C(0)$ because the Lagrangian flow (4.4)–(4.5) merely advects points along the loop. Combining (3.28) and (4.7), after a short computation we deduce that

$$\Gamma_{\rm ss}(0) = e^{(1-2\gamma)\tau}\Gamma_{\rm ss}(\tau) = e^{(1-2\gamma)\tau}\int_{\mathbb{T}} U_\theta(r,z) r d\lambda = e^{(1-2\gamma)\tau}\cdot 2\pi r U_\theta(r,z).$$

Since we have assumed $U_\theta(r,z) \neq 0$ we must have that $r > 0$, and we deduce that $\gamma = 1/2$ is the only exponent which conserves the circulation along this loop. □

4.4. Failure of the outgoing property: all fixed points on the axis of symmetry. From Theorem 4.3 we know that if a sufficiently smooth axisymmetric similarity profile $U \not\equiv 0$ solving (4.1) were to *exist for some* $\gamma < 1/2$, then the *local outgoing property of Definition 3.7 must fail*. As discussed in Section 4.2 this failure can be either due to the fact that the nodal set $\mathcal{N}_V \subset \mathbb{R}^3$ is infinite (such a situation was discussed in Section 4.3), or because the local outgoing inequality (4.6) fails. In this section we consider the case that $\mathcal{N}_V \subset \mathbb{R}^3$ is finite, but the local outgoing inequality (4.6) does not hold. In particular, this implies that *all fixed points of the ODE system* (4.4) (the solutions of (4.7)) at which the swirl velocity vanishes,[10] *lie on the axis of symmetry* $\{r = 0\}$.[11] Our main result is:

Theorem 4.6. *Let U be a nontrivial C^2 smooth axisymmetric globally self-similar velocity profile for 3D incompressible Euler equations, solving* (4.1)*, satisfying* (3.5) *and* (3.8)*. Assume that all fixed points of the meridional Lagrangian flow* (4.4) *are isolated, and lie on the axis of symmetry $\{r = 0\}$. Then $\gamma \geq 1/2$.*

Notice that as opposed to Theorem 4.3, here we did not assume the validity of (4.6), nor did we assume the infinite smoothness of Ω at the nodal values.

Before turning to the proof of Theorem 4.6, we record a key topological constraint for the meridional Lagrangian trajectories (4.4).

Lemma 4.7. *Let $D \subset \{r > 0, z \in \mathbb{R}\}$ be a nonempty bounded open set, with $\int_D r\,dr\,dz > 0$. Then D cannot be invariant in time under the flow of* (4.4).

Proof of Lemma 4.7. The 3D incompressibility $\nabla \cdot U = 0$, gives $\nabla\cdot(\gamma y + U(y)) = 3\gamma$. Equivalently, $\partial_r(r(\gamma r + U_r)) + \partial_z(r(\gamma z + U_z)) = 3\gamma r$. By the Reynolds transport theorem, for any region $D(\tau) = \{(R(r,z,\tau), Z(r,z,\tau)) \colon (r,z) \in D\}$ advected by the meridional Lagrangian flow of (4.4), we compute

$$\frac{d}{d\tau}\int_{D(\tau)} r\,dr\,dz = \int_{D(\tau)} \big(\partial_r(r(\gamma r + U_r)) + \partial_z(r(\gamma z + U_z))\big)\,dr\,dz = 3\gamma\int_{D(\tau)} r\,dr\,dz.$$

If D were invariant, meaning that $D(\tau) = D$ for all τ, then the left side of the above vanishes, while the right side equals $3\gamma\int_D r\,dr\,dz > 0$, a contradiction. □

Corollary 4.8. *Assume that all fixed points of the meridional flow* (4.4) *lie on the axis of symmetry $\{r = 0\}$. Then there are no heteroclinic and no homoclinic orbits through the open domain $\{r > 0, z \in \mathbb{R}\}$, which (asymptotically as $\tau \to \pm\infty$) connect two such fixed points.*

Proof of Corollary 4.8. Let $(0, z_-)$ and $(0, z_+)$ be two distinct fixed points on the axis of symmetry: $\gamma z_\pm + U_z(0, z_\pm) = 0$. By contradiction, suppose that there exists a heteroclinic trajectory $\Gamma := \{(R(\tau), Z(\tau)) : \tau \in \mathbb{R}\}$ lying entirely in $\{r > 0, z \in \mathbb{R}\}$, with $(R(\tau), Z(\tau)) \to (0, z_\pm)$

[10] If the swirl velocity does not vanish at a solution of (4.7), Theorem 4.4 already gives $\gamma = 1/2$.

[11] For 3D incompressible Navier-Stokes, any putative finite time axisymmetric singularity must occur on the axis of symmetry [6]. Hence, if we are to lift axisymmetric self-similar blowups from incompressible 3D Euler to incompressible 3D Navier-Stokes, it is natural to search for Euler singularities which lie on $\vec{e}_z$.

as $\tau \to \pm\infty$. The curve Γ together with the axis segment $\{(0,z) : z \in [\min(z_-,z_+),\max(z_-,z_+)]\}$ forms a simple closed curve in the closed half-plane $\{r \geq 0, z \in \mathbb{R}\}$, which bounds a nonempty open region $D \subset \{r > 0, z \in \mathbb{R}\}$ with $\int_D r\,dr\,dz > 0$. The boundary of D consists of the heteroclinic orbit (a trajectory) and a piece of the axis $\{r = 0\}$, which is invariant under the meridional Lagrangian flow (4.4) since $U_r(0,z) \equiv 0$. By uniqueness of ODE solutions, trajectories starting in the interior of D cannot exit through the boundary, so D is invariant under the flow. This contradicts Lemma 4.7.

The proof of nonexistence of homoclinic orbits is the same, except that we do not need to consider the axis segment to determine the domain D. □

Proof of Theorem 4.6. Assume by contradiction that $\gamma < 1/2$. The fixed points of (4.4) on the axis of symmetry $\{r = 0\}$ are precisely the zeros of the function $V_z(0,z) = \gamma z + U_z(0,z)$. Because $U(0) = 0$ (cf. (3.5)), the origin is always a zero: $V_z(0,0) = 0$. Furthermore, the sublinearity assumption (3.5) implies that $V_z(0,z) \sim \gamma z$ as $|z| \to \infty$, so $V_z(0,\cdot)$ has at most finitely many zeros.

Fix an arbitrary point (r_0,z_0) with $r_0 > 0$, and consider its backward trajectory for the associated flow (4.4) in the meridional plane

$$\Gamma_0 = \{(R(\tau),Z(\tau))\colon \tau \leq 0\} = \{(R(r_0,z_0,\tau),Z(r_0,z_0,\tau))\colon \tau \leq 0\},$$

as $\tau \to -\infty$. We claim that Γ_0 is contained in a bounded set. Because $\mathcal{H}$ is non-increasing along forward trajectories (cf. (3.33)), backward in time $\mathcal{H}$ is non-decreasing. The far-field asymptotics (3.34) imply that the superlevel set $\mathcal{S}_c := \{y \in \mathbb{R}^3 : \mathcal{H}(y) \geq c\}$ is bounded for every $c \in \mathbb{R}$, since $\mathcal{H}(y) \to -\infty$ as $|y| \to \infty$ when $\gamma < 1/2$. Setting $c = \mathcal{H}(r_0,z_0)$, the backward trajectory satisfies $\mathcal{H}(R(\tau),Z(\tau)) \geq c$ for all $\tau \leq 0$, and is thus contained in $\Gamma_0 \subset \mathcal{S}_c$.

Next, we claim that the α-limit set of Γ_0 is a single axis fixed point. The backward trajectory Γ_0 is bounded and lies in the two-dimensional meridional half-plane. By the Poincaré–Bendixson theorem, the α-limit set A is a nonempty, compact, connected, flow-invariant subset of the half-plane $\{r \geq 0, z \in \mathbb{R}\}$. As periodic orbits are excluded (on a periodic orbit $(\gamma r + U_r, \gamma z + U_z)$ is nonzero, so $\mathcal{H}$ would strictly decrease over one period, contradicting periodicity), A must either be a single fixed point, or consist of multiple fixed points connected by heteroclinic/homoclinic orbits. Since all fixed points lie on the axis of symmetry $\{r = 0\}$, any heteroclinic/homoclinic orbit in A connecting axis fixed points would pass through $\{r > 0\}$, violating Corollary 4.8. Therefore, $A = \{(0,z_*)\}$ for some axis fixed point z_*, and the full trajectory converges: $(R(\tau),Z(\tau)) \to (0,z_*)$ as $\tau \to -\infty$.

Next, we note that the U_θ equation in (4.1) gives

$$(\gamma r + U_r)\partial_r(rU_\theta) + (\gamma z + U_z)\partial_z(rU_\theta) + (1-2\gamma)(rU_\theta) = 0.$$

Integrating this equation along the backwards trajectory $(R(\tau),Z(\tau))_{\tau\leq 0}$ results in

$$e^{(1-2\gamma)\tau}R(\tau)U_\theta(R(\tau),Z(\tau)) = r_0 U_\theta(r_0,z_0).$$

Since $\gamma < 1/2$, we have $e^{(1-2\gamma)\tau} \to 0$ as $\tau \to -\infty$. Previously we have shown $R(\tau) \to 0$ and $U_\theta(R(\tau),Z(\tau)) \to U_\theta(0,z_*) = 0$ as $\tau \to -\infty$. Since $r_0 > 0$, we deduce $U_\theta(r_0,z_0) = 0$; this fact is a manifestation of the conservation of circulation (3.27). With Remark 4.2, it follows that $U_\theta \equiv 0$, and hence also $\Omega_r \equiv \Omega_z \equiv 0$ on $[0,\infty) \times \mathbb{R}$.

To conclude, we use the Ω_θ equation in (4.3) and the fact that $U_\theta = 0$ to deduce

$$(\gamma r + U_r)\partial_r\big(\tfrac{1}{r}\Omega_\theta\big) + (\gamma z + U_z)\partial_z\big(\tfrac{1}{r}\Omega_\theta\big) + (1+\gamma)\big(\tfrac{1}{r}\Omega_\theta\big) = 0.$$

Integrating this equation along the backwards trajectory $(R(\tau),Z(\tau))_{\tau\leq 0}$ results in

$$e^{(1+\gamma)\tau}\tfrac{1}{R(\tau)}\Omega_\theta(R(\tau),Z(\tau)) = \tfrac{1}{r_0}\Omega_\theta(r_0,z_0).$$

We have $e^{(1+\gamma)\tau} \to 0$ as $\tau \to -\infty$. Moreover, as $\tau \to -\infty$ we have $R(\tau) \to 0$, and hence $\frac{1}{R(\tau)}\Omega_\theta(R(\tau),Z(\tau)) \to (\partial_r\Omega_\theta)(0,z_*)$, which is finite by assumption (Ω is C^1). Thus, for any $r_0 > 0$

we deduce $\Omega_\theta(r_0, z_0) = 0$. With Remark 4.2, it follows that $\Omega_\theta \equiv 0$ on $[0,\infty) \times \mathbb{R}$. This contradicts the assumption that the solution Ω was nonzero; hence, we must have $\gamma \geq 1/2$. □

Acknowledgements

The work of P.C. was partially supported by NSF grant DMS-2106528 and by a Simons Collaboration Grant 601960. The work of M.I. was partially supported by NSF grant DMS-2204614. The work of V.V. was partially supported by the Collaborative NSF grant DMS-2307681 and a Simons Investigator Award.

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