Existence and Stability of Nonequilibrium Steady States of Nernst-Planck-Navier-Stokes Systems

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ABSTRACT. We consider the Nernst-Planck-Navier-Stokes system in a bounded domain of $\mathbb{R}^d$, $d = 2, 3$ with general nonequilibrium Dirichlet boundary conditions for the ionic concentrations. We prove the existence of smooth steady state solutions and present a sufficient condition in terms of only the boundary data that guarantees that these solutions have nonzero fluid velocity. We show that all time dependent solutions of the Nernst-Planck-Stokes system in three spatial dimensions, after a finite transient time, become bounded uniformly, independently of their initial size. In addition, we consider one dimensional steady states with steady nonzero currents and show that they are globally nonlinearly stable as solutions in a three dimensional periodic strip, if the currents are sufficiently weak.

1. Introduction

We consider the Nernst-Planck-Navier-Stokes (NPNS) system in a connected, but not necessarily simply connected bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) with smooth boundary. The system models electrodiffusion of ions in a fluid in the presence of an applied electrical potential on the boundary [**22, 24**]. In this paper, we study the case where there are two oppositely charged ionic species with valences ± 1 (e.g. sodium and chloride ions). In this case, the system is given by the Nernst-Planck equations

$$\begin{aligned} \partial_t c_1 + u \cdot \nabla c_1 =& D_1 \mathrm{div}\,(\nabla c_1 + c_1 \nabla \Phi) \\ \partial_t c_2 + u \cdot \nabla c_2 =& D_2 \mathrm{div}\,(\nabla c_2 - c_2 \nabla \Phi) \end{aligned} \tag{1}$$

coupled to the Poisson equation

$$-\epsilon \Delta \Phi = c_1 - c_2 = \rho \tag{2}$$

and to the Navier-Stokes system

$$\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = -K \rho \nabla \Phi, \quad \mathrm{div}\, u = 0. \tag{3}$$

Above c_1 and c_2 are the local ionic concentrations of the cation and anion, respectively, ρ is a rescaled local charge density, u is the fluid velocity, and Φ is a rescaled electrical potential. The constant $K > 0$ is a coupling constant given by the product of Boltzmann's constant k_B and the absolute temperature T_K. The constants D_i are the ionic diffusivities, $\epsilon > 0$ is a rescaled dielectric permittivity of the solvent proportional to the square of the Debye length, and $\nu > 0$ is the kinematic viscosity of the fluid. The dimensional counterparts of Φ and ρ are given by $(k_B T_k/e)\Phi$ and $e\rho$, respectively, where e is elementary charge.

It is well known that for certain *equilibrium boundary conditions,* (see (5) and (7) below) the NPNS system (1)-(3) admits a unique steady solution, with vanishing velocity $u^* = 0$, and with concentrations c_i^* related to Φ^* which uniquely solves a nonlinear Poisson-Boltzmann equation

$$\begin{aligned} -\epsilon \Delta \Phi^* &= c_1^* - c_2^* \\ c_1^* &= Z_1^{-1} e^{-\Phi^*} \\ c_2^* &= Z_2^{-1} e^{\Phi^*} \end{aligned} \tag{4}$$

with $Z_i > 0$ constant for $i = 1, 2$. The equilibria of the Nernst-Planck-Navier-Stokes system are unique minimizers of a total energy that is nonincreasing in time on time dependent solutions [**6**]. For equilibrium boundary conditions it is known that for $d = 2$ the unique steady states are globally stable [**3, 6**] and for

Key words and phrases. electroconvection, ionic electrodiffusion, Poisson-Boltzmann, Nernst-Planck, Navier-Stokes.
MSC Classification: 35Q30, 35Q35, 35Q92.

$d = 3$ locally stable [**8, 28**]. The equilibrium boundary conditions include the cases where c_i obey blocking boundary conditions

$$(\partial_n c_1 + c_1 \partial_n \Phi)_{|\partial\Omega} = (\partial_n c_2 - c_2 \partial_n \Phi)_{|\partial\Omega} = 0 \tag{5}$$

(here, ∂_n is the normal derivative) and Φ obeys Dirichlet, Neumann, or Robin boundary conditions. Also included is the case where c_i obey a mix of blocking and Dirichlet boundary conditions and Φ obeys Dirichlet boundary conditions in such a way that the electrochemical potentials

$$\mu_1 = \log c_1 + \Phi, \quad \mu_2 = \log c_2 - \Phi \tag{6}$$

are each constant on the boundary portions where c_i obey Dirichlet boundary conditions. That is, if c_i satisfy Dirichlet boundary conditions on $S_i \subset \partial\Omega$ (possibly $S_i = \partial\Omega$), then boundary conditions such that

$$\mu_{1|S_1} = \text{constant}, \quad \mu_{2|S_2} = \text{constant} \tag{7}$$

yield an equilibrium boundary condition. In general, arbitrary deviations from such situations can produce instabilities and even chaotic behavior for time dependent solutions to NPNS [**10, 16, 21, 25, 26, 27, 32**]. Furthermore, steady states in nonequilibrium configurations are not known in general, and not known to always be unique [**18, 19**].

The various boundary conditions for c_i and Φ all have physical interpretations, and we refer the reader to [**3, 6, 9, 10, 18, 29**] for relevant discussions. In this paper, we consider only Dirichlet boundary conditions for both c_i and Φ, together with no-slip boundary conditions for u,

$$\begin{aligned} c_{i|\partial\Omega} &= \gamma_i > 0, \quad 1 \le i \le 2 \\ \Phi_{|\partial\Omega} &= W \\ u_{|\partial\Omega} &= 0. \end{aligned} \tag{8}$$

For simplicity, we assume $\gamma_i, W \in C^\infty(\partial\Omega)$, but we do not restrict their size, nor do we require them to be constant. For c_i, the Dirichlet boundary conditions model, for example, ion-selectivity at an ion-selective membrane or some fixed concentration of ions at the boundary layer-bulk interface. Dirichlet boundary conditions for Φ model an applied electric potential on the boundary.

There is a large literature on the well-posedness of the time dependent NPNS system [**3, 6, 8, 9, 12, 17, 20, 28, 29**], as well as the uncoupled Nernst-Planck [**1, 2, 4, 13, 15, 18**] and Navier-Stokes systems [**5, 30**]. Some of the aforementioned studies, in addition to [**7, 14, 19**], study several aspects of the steady state Nernst-Planck equations including existence, uniqueness, stability, and asymptotic behavior.

Thus far, in the context of NPNS, steady states have mostly been studied in the case of equilibrium boundary conditions. In these cases, the corresponding steady states are the unique Nernst-Planck steady states, together with zero fluid flow $u^* \equiv 0$. In this paper in Section 2, Theorem 1, we prove the existence of smooth steady state solutions to the NPNS system (1)-(3) subject to arbitrary (large data) Dirichlet boundary conditions (8). In addition, we derive a sufficient condition, depending solely on the boundary data such that the steady state solution has nonzero fluid flow $u^* \not\equiv 0$ (Theorem 2). Thus the two main results of Section 2 give the existence of steady states for NPNS that are not obtained by existing theory for Nernst-Planck equations, and include in particular the steady solutions with nonzero flow for which instability and chaos have been observed experimentally and numerically.

In Section 3 we consider the time dependent solutions of the Nernst-Planck-Stokes system in 3D and show, using a maximum principle that solutions obey long time bounds that are independent of the size of the initial data. This result is also valid for the NPNS system in 2D and the NPNS system in 3D under the assumption of globally bounded smooth velocities. The maximum principle is for a two-by-two parabolic system with unequal diffusivities. The bound applies for situations far away from equilibrium, when the solutions have nontrivial dynamics, and establishes the existence of an absorbing ball. This is a first step in proving the existence and finite dimensionality of the global attractor, a task that will be pursued elsewhere.

In Section 4, we consider the Nernst-Planck-Stokes (NPS) system in the periodic channel $\Omega = (0, L) \times \mathbb{T}^2$ with piecewise constant (i.e. constant at $x = 0$ and $x = L$, respectively) boundary conditions. In this case, we derive sufficient conditions, depending only on boundary data and parameters, such that NPS admits a one dimensional, globally stable steady state solution that corresponds to a *steady (nonzero) current* solution with the fluid at rest. These are non-Boltzmann states (whose currents identically vanish). The stability condition can be thought of as a smallness condition on the magnitude of the ionic currents or, equivalently, as a small perturbation from equilibrium condition. The main result of Section 4, Theorem 4, is preceded by an analysis of one dimensional steady currents.

1.1. Notation. Unless otherwise stated, we denote by C a positive constant that depends only on the parameters of the system, the domain, and the initial and boundary conditions. The value of C may differ from line to line.

We denote by $L^p = L^p(\Omega)$ the Lebesgue spaces and by $W^{s,p} = W^{s,p}(\Omega)$, $H^s = H^s(\Omega) = W^{s,2}$ the Sobolev spaces. We denote by $\langle \cdot, \cdot \rangle$ the $L^2(\Omega)$ inner product, and we write $dV = dx\, dy\, dz$ for the volume element in three dimensions, and dx in one dimension. We write dS for the surface element.

We denote by $\partial_x, \partial_y, \partial_z, \partial_t$ the partial derivaties with respect to x, y, z, t, respectively, and also use ∂_x to mean $\frac{d}{dx}$ in a one dimensional setting.

We denote $z_1 = 1, z_2 = -1$ for the valences of the ionic species.

2. Steady State Nernst-Planck-Navier-Stokes

We consider the steady state Nernst-Planck-Navier-Stokes sytem

$$u \cdot \nabla c_1 = \mathrm{div}\,(\nabla c_1 + c_1 \nabla \Phi) \tag{9}$$

$$u \cdot \nabla c_2 = \mathrm{div}\,(\nabla c_2 - c_2 \nabla \Phi) \tag{10}$$

$$-\Delta \Phi = c_1 - c_2 = \rho \tag{11}$$

$$u \cdot \nabla u - \Delta u + \nabla p = -\rho \nabla \Phi \tag{12}$$

$$\mathrm{div}\, u = 0 \tag{13}$$

on a smooth, connected, bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ together with boundary conditions

$$c_{i|\partial\Omega} = \gamma_i > 0, \quad 1 \le i \le 2 \tag{14}$$

$$\Phi_{|\partial\Omega} = W \tag{15}$$

$$u_{|\partial\Omega} = 0 \tag{16}$$

with $\gamma_i, W \in C^\infty(\partial\Omega)$ not necessarily constant. In the above system, we have taken $D_i = \epsilon = \nu = K = 1$, as the values of these parameters do not play a significant role in the results of this section. In this section we first prove the existence of a smooth solution to (9)-(13) with boundary conditions (14)-(16). Then, we derive a sufficient condition depending on just γ_i and W and their derivatives that guarantees that any steady state solution (c_i^*, Φ^*, u^*) of (9)-13) with (14)-(16) has nonzero fluid flow i.e. $u^* \not\equiv 0$.

THEOREM 1. *For arbitrary boundary conditions (14)-(16) on a smooth, connected, bounded domain* $\Omega \subset \mathbb{R}$ *(*$d = 2, 3$*), there exists a smooth solution* (c_i, Φ, u) *of the steady state Nernst-Planck-Navier-Stokes system (9)-(13) such that* $c_i \ge 0$.

REMARK 1. *For the steady state Nernst-Planck system, any regular enough solution is necessarily nonnegative (i.e.* $c_i \ge 0$*) if we assume* $\gamma_i \ge 0$*. This follows from the fact that the quantities* $c_1 e^{\Phi}$, $c_2 e^{-\Phi}$ *each satisfy a maximum principle (see Section 4.1). However, in the case of steady state NPNS, where such a maximum principle does not hold, the nonnegativity of* c_i *must be built into the construction.*

REMARK 2. *In general, it is unknown whether steady state solutions of the two or three dimensional NPNS system, with Dirichlet boundary conditions for* c_i*, are unique. Given the available nonuniqueness results for the Navier-Stokes equations in some special cases* [**23, 30, 31**]*, it is reasonable to expect that uniqueness for NPNS does not hold in full generality. Nonetheless, uniqueness holds in some perturbative regimes, see for example Theorem 5, where the argument is given for the Nernst-Planck-Stokes system.*

PROOF. Throughout the proof, we assume $d = 3$. Some steps are streamlined if we assume $d = 2$, but the proof for $d = 3$ nonetheless works for $d = 2$.

The proof consists of two main steps. We first show the existence of a solution to a parameterized approximate NPNS system. Then, we extract a convergent subsequence and show that the limit satisfies the original system.

Step 1. The approximate system. The approximate system is given by:

$$0 = \operatorname{div}\left(\nabla c_1^\delta + \chi_\delta(c_1^\delta)\nabla\Phi^\delta - u^\delta\chi_\delta(c_1^\delta)\right) \tag{17}$$

$$0 = \operatorname{div}\left(\nabla c_2^\delta - \chi_\delta(c_2^\delta)\nabla\Phi^\delta - u^\delta\chi_\delta(c_2^\delta)\right) \tag{18}$$

$$-\Delta\Phi^\delta = \chi_\delta(c_1^\delta) - \chi_\delta(c_2^\delta) = \rho^\delta \tag{19}$$

$$u^\delta\cdot\nabla u^\delta - \Delta u^\delta + \nabla p^\delta = -\rho^\delta\nabla\Phi^\delta \tag{20}$$

$$\operatorname{div} u^\delta = 0 \tag{21}$$

with boundary conditions

$$c_{i\,|\partial\Omega}^\delta = \gamma_i > 0, \quad 1 \le i \le 2 \tag{22}$$

$$\Phi^\delta{}_{|\partial\Omega} = W \tag{23}$$

$$u^\delta{}_{|\partial\Omega} = 0. \tag{24}$$

Here, χ_δ is a smooth cutoff function, which converges pointwise to the following function as $\delta \to 0$,

$$l : y \mapsto \begin{cases} y, & y \ge 0 \\ 0, & y \le 0. \end{cases} \tag{25}$$

We define χ_δ by first fixing a smooth, nondecreasing function $\chi : \mathbb{R} \to \mathbb{R}^+$ such that

$$\chi : y \mapsto \begin{cases} y, & y \ge 1 \\ 0, & y \le 0. \end{cases} \tag{26}$$

Then, we set $\chi_\delta(y) = \delta\chi(\frac{y}{\delta})$. We state below some elementary properties of χ_δ:

1) $\chi_\delta \ge 0$
2) $\chi_\delta(y) = y$ for $y \ge \delta$, $\chi^\delta(y) = 0$ for $y \le 0$
3) χ_δ is nondecreasing
4) $|\chi_\delta'(y)| \le a$ where $a = \sup\chi'$ and so $\chi_\delta(y) \le a|y|$, $|\chi_\delta(x) - \chi_\delta(y)| \le a|x - y|$.

The existence of a solution to (17)-(24) follows as an application of Schaefer's fixed point theorem [**11**], which we state below:

PROPOSITION 1. *Suppose X is a Banach space and $E : X \to X$ is continuous and compact. If the set*

$$\{v \in X \mid v = \lambda E(v) \textit{ for some } 0 \le \lambda \le 1\} \tag{27}$$

is bounded in X, then E has a fixed point.

Step 2. Fixed point reformulation. In order to apply this fixed point theorem, we first reformulate the problem (17)-(24). Letting Γ_i be the unique harmonic function on Ω satisfying $\Gamma_{i|\partial\Omega} = \gamma_i$, and introducing

$$q_i^\delta = c_i^\delta - \Gamma_i, \quad 1 \le i \le 2 \tag{28}$$

we rewrite (17)-(19),

$$-\Delta q_1^\delta = \operatorname{div}\left(-u^\delta\chi_\delta(c_1^\delta) + \chi_\delta(c_1^\delta)\nabla(-\Delta_W)^{-1}\rho^\delta\right) = R_1^\delta(q_1^\delta, q_2^\delta, u^\delta) \tag{29}$$

$$-\Delta q_2^\delta = \operatorname{div}\left(-u^\delta\chi_\delta(c_2^\delta) - \chi_\delta(c_2^\delta)\nabla(-\Delta_W)^{-1}\rho^\delta\right) = R_2^\delta(q_1^\delta, q_2^\delta, u^\delta) \tag{30}$$

where $(-\Delta_W)^{-1}$ maps g to the unique solution f of

$$-\Delta f = g \text{ in } \Omega, \quad f_{|\partial\Omega} = W. \tag{31}$$

Above we view R_i^δ as functions of q_i^δ and u^δ, with c_i^δ and ρ^δ related to q_i^δ via (28) and $\rho^\delta = \chi_\delta(c_1^\delta) - \chi_\delta(c_2^\delta)$. Thus we write (29)-(30) as

$$q_i^\delta = (-\Delta_D)^{-1} R_i^\delta(q_1^\delta, q_2^\delta, u^\delta), \quad 1 \le i \le 2 \tag{32}$$

where $-\Delta_D$ is the Laplace operator on Ω associated with homogeneous Dirichlet boundary conditions. As for the Navier-Stokes subsystem, we first project the equations onto the space of divergence free vector fields using the Leray projection [**5**]

$$\mathbb{P} : (L^2(\Omega))^3 \to H \tag{33}$$

where H is the closure of

$$\mathcal{V} = \{f \in (C_0^\infty(\Omega))^3 \,|\, \operatorname{div} f = 0\} \tag{34}$$

in $(L^2(\Omega))^3$ and is a Hilbert space endowed with the L^2 inner product. Then (20), (21) is given by

$$Au^\delta + B(u^\delta, u^\delta) = -\mathbb{P}(\rho^\delta \nabla \Phi^\delta) \tag{35}$$

where

$$A = \mathbb{P}(-\Delta) : \mathcal{D}(A) = (H^2(\Omega))^3 \cap V \to H \tag{36}$$

$$V = \text{closure of } \mathcal{V} \text{ in } (H_0^1(\Omega))^3 = \{f \in (H_0^1(\Omega))^3 \,|\, \operatorname{div} f = 0\}. \tag{37}$$

As it is well known, the Stokes operator A is invertible and $A^{-1} : H \to \mathcal{D}(A)$ is bounded and self-adjoint on H and compact as a mapping from H into V. The space V is a Hilbert space endowed with the Dirichlet inner product

$$\langle f, g\rangle_V = \int \nabla f : \nabla g \, dV. \tag{38}$$

For $f, g \in V$, $B(f,g) = \mathbb{P}(f \cdot \nabla g)$ and B may be viewed as a continuous, bilinear mapping such that

$$B : (f,g) \in V \times V \mapsto \left(h \in V \mapsto \int_\Omega (f \cdot \nabla g) \cdot h \, dV \right) \in V' \tag{39}$$

where V' is the dual space of V. We note that we may also view A as an invertible mapping $A : V \to V'$. It is with this viewpoint that we write (35) as

$$u^\delta = A^{-1} R_u^\delta(q_1^\delta, q_2^\delta, u^\delta) \tag{40}$$

$$R_u^\delta(q_1^\delta, q_2^\delta, u^\delta) = -(B(u^\delta, u^\delta) + \mathbb{P}(\rho^\delta \nabla (-\Delta_W)^{-1} \rho^\delta)). \tag{41}$$

Thus, setting

$$X = H_0^1(\Omega) \times H_0^1(\Omega) \times V \tag{42}$$

and

$$E = E_1 \times E_2 \times E_u : (f,g,h) \in X \mapsto ((-\Delta_D)^{-1} R_1^\delta(f,g,h), (-\Delta_D)^{-1} R_2^\delta(f,g,h), A^{-1} R_u^\delta(f,g,h)) \tag{43}$$

we seek to show the existence of a weak solution $(\tilde q_1, \tilde q_2, \tilde u) = (\tilde q_1^\delta, \tilde q_2^\delta, \tilde u^\delta) \in X$ to (29), (30), (35) by verifying the hypotheses of Proposition 1 for the operator E and showing that E has a fixed point in X.

Step 3. Continuity and compactness of E. First we prove that E indeed maps X into X and does so continuously and compactly.

LEMMA 1. *The operator $E = E_1 \times E_2 \times E_u : X \to X$ is continuous and compact.*

PROOF. We start with compactness. Since $(-\Delta_D)^{-1}$ (and A^{-1}) maps $L^{\frac32}$ (resp. $(L^{\frac32})^3$) continuously into $W^{2,\frac32} \cap H_0^1$ (resp. $(W^{2,\frac32})^3 \cap V$), by Rellich's theorem, the maps $(-\Delta_D)^{-1} : L^{\frac32} \to H_0^1$ and $A^{-1} : (L^{\frac32})^3 \to V$ are compact. Thus for compactness of E, it suffices to show that

$$\begin{aligned} &(f,g,h) \in X \mapsto R_i^\delta(f,g,h) \in L^{\frac32}, \quad 1 \le i \le 2 \\ &(f,g,h) \in X \mapsto R_u^\delta(f,g,h) \in (L^{\frac32})^3 \end{aligned} \tag{44}$$

are bounded. To this end, we compute

$$\begin{aligned}\|R_1^\delta(f,g,h)\|_{L^{\frac{3}{2}}} \le &\|h\|_{L^6}\|\nabla\chi_\delta(f+\Gamma_1)\|_{L^2}\\ &+\|\nabla\chi_\delta(f+\Gamma)\|_{L^2}\|\nabla(-\Delta_W)^{-1}(\chi_\delta(f+\Gamma_1)-\chi_\delta(g+\Gamma_2))\|_{L^6}\\ &+\|\chi_\delta(f+\Gamma)\|_{L^6}\|(\chi_\delta(f+\Gamma_1)-\chi_\delta(g+\Gamma_2))\|_{L^2}\\ \le &C(\|h\|_V+\|f\|_{H^1}+\|g\|_{H^1}+1)(\|f\|_{H^1}+1).\end{aligned} \tag{45}$$

In the last inequality we used the continuous embeddings $H^1 \hookrightarrow L^6$ and the fact that $|\chi_\delta'| \le a$. Entirely similar estimates give

$$\|R_2^\delta(f,g,h)\|_{L^{\frac{3}{2}}} \le C(\|h\|_V+\|f\|_{H^1}+\|g\|_{H^1}+1)(\|g\|_{H^1}+1). \tag{46}$$

Lastly we estimate R_u^δ,

$$\begin{aligned}\|R_u^\delta(f,g,h)\|_{L^{\frac{3}{2}}} \le &\|B(h,h)\|_{L^{\frac{3}{2}}}\\ &+\|(\chi_\delta(f+\Gamma_1)-\chi_\delta(g+\Gamma_2))\nabla(-\Delta_W)^{-1}(\chi_\delta(f+\Gamma_1)-\chi_\delta(g+\Gamma_2))\|_{L^{\frac{3}{2}}}\\ \le &\|h\|_{L^6}\|h\|_V\\ &+\|(\chi_\delta(f+\Gamma_1)-\chi_\delta(g+\Gamma_2))\|_{L^2}\|\nabla(-\Delta_W)^{-1}(\chi_\delta(f+\Gamma_1)-\chi_\delta(g+\Gamma_2))\|_{L^6}\\ \le &C(1+\|h\|_V^2+\|f\|_{L^2}^2+\|g\|_{L^2}^2).\end{aligned} \tag{47}$$

The bounds (45)-(47) show that the operators from (44) are indeed bounded, and thus E is compact.

Continuity of E follows from the fact that the components of E are sums of compositions of the following continuous operations

$$\begin{aligned} f\in H_1 &\mapsto f+\Gamma_i \in H_1\\ f\in H_1 &\mapsto \chi_\delta(f)\in L^4\\ f\in L^2 &\mapsto \nabla(-\Delta_W)^{-1}f\in (H^1)^3\subset (L^4)^3\\ (f,g)\in L^4\times L^4 &\mapsto fg\in L^2\\ f\in (L^2)^d &\mapsto (-\Delta_D)^{-1}\mathrm{div}\, f\in H_0^1\\ (f,g)\in V\times V &\mapsto B(f,g)\in V'\\ f\in (L^2)^d &\mapsto \mathbb{P}f\in H\\ f\in V' &\mapsto A^{-1}f\in V.\end{aligned} \tag{48}$$

This completes the proof of the lemma. □

Step 4. Uniform a priori H^1 bounds. Now it remains to establish uniform a priori bounds (c.f. (27)). We fix $\lambda\in[0,1]$ and assume that for some $(\tilde q_1,\tilde q_2,\tilde u)\in X$ we have

$$(\tilde q_1,\tilde q_2,\tilde u)=\lambda E(\tilde q_1,\tilde q_2,\tilde u). \tag{49}$$

That is, for all $\psi_1,\psi_2\in H_0^1(\Omega)$ and $\psi_u\in V$ we assume we have

$$\int_\Omega \nabla\tilde q_i\cdot\nabla\psi_i\,dV=\lambda\int_\Omega R_i^\delta(\tilde q_1,\tilde q_2,\tilde u)\psi_i\,dV,\quad 1\le i\le 2 \tag{50}$$

$$\int_\Omega \nabla\tilde u:\nabla\psi_u\,dV=\lambda\int_\Omega R_u^\delta(\tilde q_1,\tilde q_2,\tilde u)\cdot\psi_u\,dV. \tag{51}$$

We make the choice of test functions $\psi_i=\tilde q_i$ and first estimate the resulting integral on the right hand side of (50) for $i=1$, omitting for now the factor λ. Introducing the following primitive of χ_δ

$$Q_\delta(y)=\int_0^y\chi_\delta(s)\,ds \tag{52}$$

we have, integrating by parts,

$$\begin{aligned}\int_\Omega R_1^\delta(\tilde q_1,\tilde q_2,\tilde u)\tilde q_1\,dV =&\int_\Omega \tilde u\cdot\nabla Q_\delta(\tilde c_1)\,dV-\int_\Omega(\tilde u\cdot\nabla\Gamma_1)\chi_\delta(\tilde c_1)\,dV\\ &-\int_\Omega \chi_\delta(\tilde c_1)\nabla(-\Delta_W)^{-1}\tilde\rho\cdot\nabla\tilde q_1\,dV\\ =&I_1^{(1)}+I_2^{(1)}+I_3^{(1)}\end{aligned}\tag{53}$$

where $\tilde q_i=\tilde c_i-\Gamma_i, 1\le i\le 2$ and $\tilde\rho=\chi_\delta(\tilde c_1)-\chi_\delta(\tilde c_2)$. Because $\tilde u$ is divergence-free, it follows after an integration by parts that

$$I_1^{(1)}=0.\tag{54}$$

Next, estimating $I_2^{(1)}$ we have, using the Poincaré inequality twice,

$$|I_2^{(1)}|=\left|\int_\Omega(\tilde u\cdot\nabla\Gamma_1)\chi_\delta(\tilde c_1)\,dV\right|\le C\|\tilde u\|_H\|\tilde c_1\|_{L^2}\le\frac12\|\nabla\tilde q_1\|_{L^2}^2+C\|\tilde u\|_V^2+C.\tag{55}$$

Lastly we estimate $I_3^{(1)}$,

$$\begin{aligned}I_3^{(1)}=&-\int_\Omega\chi_\delta(\tilde c_1)\nabla(-\Delta_W)^{-1}\tilde\rho\cdot\nabla\tilde c_1\,dV+\int_\Omega\chi_\delta(\tilde c_1)\nabla(-\Delta_W)^{-1}\tilde\rho\cdot\nabla\Gamma_1\,dV\\ =&-\int_\Omega\nabla Q_\delta(\tilde c_1)\cdot\nabla(-\Delta_W)^{-1}\tilde\rho\,dV+\int_\Omega\chi_\delta(\tilde c_1)\nabla(-\Delta_W)^{-1}\tilde\rho\cdot\nabla\Gamma_1\,dV\\ =&-\int_\Omega\nabla(Q_\delta(\tilde c_1)-Q_\delta(\Gamma_1))\cdot\nabla(-\Delta_W)^{-1}\tilde\rho\,dV-\int_\Omega\nabla Q_\delta(\Gamma_1)\cdot\nabla(-\Delta_W)^{-1}\tilde\rho\,dV\\ &+\int_\Omega\chi_\delta(\tilde c_1)\nabla(-\Delta_W)^{-1}\tilde\rho\cdot\nabla\Gamma_1\,dV\\ =&-\int_\Omega(Q_\delta(\tilde c_1)-Q_\delta(\Gamma_1))\tilde\rho\,dV-\int_\Omega\nabla Q_\delta(\Gamma_1)\cdot\nabla(-\Delta_W)^{-1}\tilde\rho\,dV\\ &+\int_\Omega\chi_\delta(\tilde c_1)\nabla(-\Delta_W)^{-1}\tilde\rho\cdot\nabla\Gamma_1\,dV.\end{aligned}\tag{56}$$

Analogous computations for $i=2$ in (50) yield on the right hand side

$$\int_\Omega R_2^\delta(\tilde q_1,\tilde q_2,\tilde u)\tilde q_2\,dV=I_1^{(2)}+I_2^{(2)}+I_3^{(2)}\tag{57}$$

where $I_j^{(2)}$, $j=1,2,3$ satisfy

$$\begin{aligned}I_1^{(2)}=&0\\ |I_2^{(2)}|\le&\frac12\|\nabla\tilde q_2\|_{L^2}^2+C\|\tilde u\|_V^2+C\\ I_3^{(2)}=&\int_\Omega(Q_\delta(\tilde c_2)-Q_\delta(\Gamma_2))\tilde\rho\,dV+\int_\Omega\nabla Q_\delta(\Gamma_2)\cdot\nabla(-\Delta_W)^{-1}\tilde\rho\,dV\\ &-\int_\Omega\chi_\delta(\tilde c_2)\nabla(-\Delta_W)^{-1}\tilde\rho\cdot\nabla\Gamma_2\,dV.\end{aligned}\tag{58}$$

Thus, summing (50) in i we obtain

$$\begin{aligned}\frac12\sum_i\|\nabla\tilde q_i\|_{L^2}^2+\lambda\int_\Omega(Q_\delta(\tilde c_1)-Q_\delta(\tilde c_2))\tilde\rho\,dV\le&C\lambda(1+\|\tilde\rho\|_{L^1}+\|\nabla(-\Delta_W)^{-1}\tilde\rho\|_{L^1}\\ &+\|\tilde u\|_V^2+\sum_i\|\tilde c_i\|_{L^2}\|\nabla(-\Delta_W)^{-1}\tilde\rho\|_{L^2}).\end{aligned}\tag{59}$$

Next, using the bounds

$$\begin{aligned} \|\tilde{c}_i\|_{L^2} &\le \|\tilde{q}_i\|_{L^2} + C \le C\|\nabla \tilde{q}_i\|_{L^2} + C, \quad 1 \le i \le 2 \\ \|\tilde{\rho}\|_{L^1} &\le C\|\tilde{\rho}\|_{L^3} \\ \|\nabla(-\Delta_W)^{-1}\tilde{\rho}\|_{L^1} &\le C\|\nabla(-\Delta_W)^{-1}\tilde{\rho}\|_{L^2} \le C\|\tilde{\rho}\|_{L^3} + C \end{aligned} \tag{60}$$

we obtain from (59), using Young's inequalities

$$\frac{1}{4}\sum_i \|\nabla \tilde{q}_i\|_{L^2} + \lambda \int_\Omega (Q_\delta(\tilde{c}_1) - Q_\delta(\tilde{c}_2))\tilde{\rho}\, dV \le C + \theta\lambda\|\tilde{\rho}\|_{L^3}^3 + C\lambda\|\tilde{u}\|_V^2 \tag{61}$$

where θ is a small constant to be chosen later. Next we prove the following bound,

$$\int_\Omega (Q_\delta(\tilde{c}_1) - Q_\delta(\tilde{c}_2))\tilde{\rho}\, dV \ge \frac{1}{4}\|\tilde{\rho}\|_{L^3}^3 - C. \tag{62}$$

Prior to establishing this lower bound, we prove the following lemma, which shows that $Q_\delta \approx \frac{\chi_\delta^2}{2}$.

LEMMA 2. *$|Q_\delta(y) - \frac{\chi_\delta^2}{2}(y)| \le \frac{\delta^2}{2}$ for all $y \in \mathbb{R}$.*

PROOF. For $y \le 0$, we have $Q_\delta(y) = \chi_\delta(y) = 0$ so we may assume $y > 0$. Suppose $y \ge \delta$. Then

$$Q_\delta(y) = \int_0^\delta \chi_\delta(s)\, ds + \int_\delta^y s\, ds \le \delta^2 + \frac{1}{2}(y^2 - \delta^2) = \frac{\delta^2}{2} + \frac{\chi_\delta^2(y)}{2} \tag{63}$$

and similarly

$$Q_\delta(y) = \int_0^\delta \chi_\delta(s)\, ds + \int_\delta^y s\, ds \ge \frac{1}{2}(y^2 - \delta^2) = -\frac{\delta^2}{2} + \frac{\chi_\delta^2(y)}{2}. \tag{64}$$

Thus the lemma holds for $y \ge \delta$. Lastly, suppose $y \in (0, \delta)$. Then, using the monotonicity of χ_δ,

$$Q_\delta(y) = \int_0^y \chi_\delta(s)\, ds \le y\chi_\delta(y) \le \delta\chi_\delta(y) \le \frac{\delta^2}{2} + \frac{\chi_\delta^2(y)}{2}. \tag{65}$$

On the other hand, we have

$$Q_\delta(y) \ge 0 \ge -\frac{\delta^2}{2} + \frac{\chi_\delta^2(y)}{2}. \tag{66}$$

This completes the proof of the lemma. □

Now we proceed with the proof of (62). We split $\Omega = \{\tilde{\rho} \ge 0\} \cup \{\tilde{\rho} < 0\}$. Restricted to $\{\tilde{\rho} \ge 0\}$, we have, using Lemma 2,

$$Q_\delta(\tilde{c}_1) - Q_\delta(\tilde{c}_2) \ge \frac{\chi_\delta^2(\tilde{c}_1)}{2} - \frac{\chi_\delta^2(\tilde{c}_2)}{2} - \delta^2 = \frac{1}{2}(\chi_\delta(\tilde{c}_1) + \chi_\delta(\tilde{c}_2))\tilde{\rho} - \delta^2 \tag{67}$$

and, restricted to $\{\tilde{\rho} < 0\}$, again using the lemma, we have

$$Q_\delta(\tilde{c}_1) - Q_\delta(\tilde{c}_2) \le \frac{\chi_\delta^2(\tilde{c}_1)}{2} - \frac{\chi_\delta^2(\tilde{c}_2)}{2} + \delta^2 = \frac{1}{2}(\chi_\delta(\tilde{c}_1) + \chi_\delta(\tilde{c}_2))\tilde{\rho} + \delta^2. \tag{68}$$

It follows that

$$\begin{aligned} \int_\Omega (Q_\delta(\tilde{c}_1) - Q_\delta(\tilde{c}_2))\tilde{\rho}\, dV &\ge \frac{1}{2}\int_\Omega \tilde{\rho}^2(\chi_\delta(\tilde{c}_1) + \chi_\delta(\tilde{c}_2))\, dV - \delta^2\|\tilde{\rho}\|_{L^1} \\ &\ge \frac{1}{2}\int_\Omega |\tilde{\rho}|^3\, dV - C\|\tilde{\rho}\|_{L^3} \\ &\ge \frac{1}{4}\|\tilde{\rho}\|_{L^3}^3 - C \end{aligned} \tag{69}$$

where in the second inequality we used the fact that, because $\chi_\delta \geq 0$, we have $|\tilde{\rho}| \leq \chi_\delta(\tilde{c}_1) + \chi_\delta(\tilde{c}_2)$. Thus we have established (62). Now we return to (61) and select $\theta = 1/8$. Then using the bound (62), we ultimately have

$$\frac{1}{4}\sum_i \|\nabla \tilde{q}_i\|_{L^2}^2 + \frac{\lambda}{8}\|\tilde{\rho}\|_{L^3}^3 \leq C + \tilde{C}\lambda\|\tilde{u}\|_V^2 \tag{70}$$

with $\tilde{C}$ depending only on data and parameters. Next, we proceed to (51). Choosing $\psi_u = \tilde{u}$, we have on the right hand side, omitting for now the prefactor λ,

$$\int_\Omega R_u^\delta(\tilde{q}_1, \tilde{q}_2, \tilde{u}) \cdot \tilde{u}\, dV = -\int_\Omega B(\tilde{u}, \tilde{u}) \cdot \tilde{u}\, dV - \int_\Omega \mathbb{P}(\tilde{\rho}\nabla(-\Delta_W)^{-1}\tilde{\rho}) \cdot \tilde{u}\, dV. \tag{71}$$

On one hand, using the self-adjointness of the projection $\mathbb{P}$ and the fact that $\tilde{u}$ is divergence-free we have

$$\int_\Omega B(\tilde{u}, \tilde{u}) \cdot \tilde{u}\, dV = \int_\Omega (\tilde{u} \cdot \nabla \tilde{u}) \cdot \tilde{u}\, dV = \frac{1}{2}\int_\Omega \tilde{u} \cdot \nabla |\tilde{u}|^2\, dV = 0. \tag{72}$$

On the other hand, again using the self-adjointness of $\mathbb{P}$, we have

$$\int_\Omega \mathbb{P}(\tilde{\rho}\nabla(-\Delta_W)^{-1}\rho) \cdot \tilde{u}\, dV = \int_\Omega \tilde{\rho}\nabla(-\Delta_W)^{-1}\tilde{\rho} \cdot \tilde{u}\, dV. \tag{73}$$

Thus far, from (51) we have

$$\|\tilde{u}\|_V^2 = -\lambda \int_\Omega \tilde{\rho}\nabla(-\Delta_W)^{-1}\tilde{\rho} \cdot \tilde{u}\, dV. \tag{74}$$

To control the integral on the right hand side, we return to (50), taking this time the test functions $\psi_1 = -\psi_2 = \phi_0 = (-\Delta_D)^{-1}\tilde{\rho}$. Then, on the right hand side, we have, summing in i, integrating by parts, and omitting for now the prefactor λ

$$\begin{aligned}
\int_\Omega [R_1^\delta(\tilde{q}_1, \tilde{q}_2, \tilde{u}) - R_2^\delta(\tilde{q}_1, \tilde{q}_2, \tilde{u})]\phi_0\, dV = &\int_\Omega \tilde{u}\tilde{\rho} \cdot \nabla\phi_0\, dV \\
&- \sum_i \int_\Omega \chi_\delta(\tilde{c}_i)|\nabla\phi_0|^2\, dV - \sum_i \int_\Omega \chi_\delta(\tilde{c}_i)\nabla\phi_W \cdot \nabla\phi_0\, dV \\
\leq &\int_\Omega \tilde{\rho}\nabla(-\Delta_W)^{-1}\tilde{\rho} \cdot \tilde{u}\, dV - \int_\Omega \tilde{\rho}\nabla\phi_W \cdot \tilde{u}\, dV \\
&- \frac{1}{2}\sum_i \int_\Omega \chi_\delta(\tilde{c}_i)|\nabla\phi_0|^2\, dV + \frac{1}{2}\int_\Omega \chi_\delta(\tilde{c}_i)|\nabla\phi_W|^2\, dV \\
\leq &C + \int_\Omega \tilde{\rho}\nabla(-\Delta_W)^{-1}\tilde{\rho} \cdot \tilde{u}\, dV + \frac{1}{2}\|\tilde{u}\|_V^2 + \theta\|\tilde{\rho}\|_{L^3}^3 \\
&- \frac{1}{2}\sum_i \int_\Omega \chi_\delta(\tilde{c}_i)|\nabla\phi_0|^2\, dV + \theta\sum_i \|\nabla\tilde{q}_i\|_{L^2}^2
\end{aligned} \tag{75}$$

where θ is a constant to be chosen. Above, we have denoted $\phi_W = (-\Delta_W)^{-1}\tilde{\rho} - \phi_0$ i.e. ϕ_W is the unique harmonic function on Ω whose values on the boundary are given by W.

On the other hand, we bound the left hand side of (50) in two different ways depending on whether $0 \leq \lambda \leq \Lambda$ or $\Lambda < \lambda \leq 1$, where Λ is determined below (see (80)). We first consider the case $0 \leq \lambda \leq \Lambda$. Then, we bound the left hand side of (50) as follows,

$$\sum_i \left|\int_\Omega \nabla\tilde{q}_i \cdot \nabla\phi_0\, dV\right| \leq C + C_q \sum_i \|\nabla\tilde{q}_i\|_{L^2}^2 \tag{76}$$

where $C_q > 0$ depends only on data and parameters. Collecting the estimates (75), (76), we have from (50),

$$\begin{aligned} 0 \le \frac{\lambda}{2}\sum_i \int_\Omega \chi_\delta(\tilde{c}_i)|\nabla\phi_0|^2\,dV \le& C + \lambda\int_\Omega \tilde{\rho}\nabla(-\Delta_W)^{-1}\tilde{\rho}\cdot\tilde{u}\,dV \\ &+ \frac{1}{2}\|\tilde{u}\|_V^2 + \lambda\theta\|\tilde{\rho}\|_{L^3}^3 + (C_q+\theta)\sum_i \|\nabla\tilde{q}_i\|_{L^2}^2. \end{aligned} \tag{77}$$

Above, we keep track of the prefactor λ only where needed and bound $\lambda \le 1$ if this suffices. Then, adding (77) to (74), we obtain

$$\frac{1}{2}\|\tilde{u}\|_V^2 \le C + \lambda\theta\|\tilde{\rho}\|_{L^3}^3 + (C_q+\theta)\sum_i \|\nabla\tilde{q}_i\|_{L^2}^2. \tag{78}$$

Choosing $\theta = 1$ and multiplying this last inequality by $\frac{1}{8(C_q+1)}$ and adding it to (70), we obtain

$$\frac{1}{8}\sum_i \|\nabla\tilde{q}_i\|_{L^2}^2 + \frac{1}{32(C_q+1)}\|\tilde{u}\|_V^2 \le \tilde{R} \tag{79}$$

for some $\tilde{R}$ depending on data and parameters, but not on λ or δ, provided (c.f. (70))

$$\lambda \le \Lambda = \frac{1}{32\tilde{C}(C_q+1)}. \tag{80}$$

Now we consider the case $\Lambda < \lambda \le 1$. In this case, we estimate the left hand side of (50) as follows,

$$\begin{aligned} \sum_i \left|\int_\Omega \nabla\tilde{q}_i\cdot\nabla\phi_0\,dV\right| \le& C + \theta\sum_i \|\nabla\tilde{q}_i\|_{L^2}^2 + \Lambda\theta\|\tilde{\rho}\|_{L^3}^3 \\ \le& C + \theta\sum_i \|\nabla\tilde{q}_i\|_{L^2}^2 + \lambda\theta\|\tilde{\rho}\|_{L^3}^3. \end{aligned} \tag{81}$$

Then, combining (75), (81), we have

$$\begin{aligned} 0 \le \frac{\lambda}{2}\sum_i \int_\Omega \chi_\delta(\tilde{c}_i)|\nabla\phi_0|^2\,dV \le& C + \lambda\int_\Omega \tilde{\rho}\nabla(-\Delta_W)^{-1}\tilde{\rho}\cdot\tilde{u}\,dV \\ &+ \frac{1}{2}\|\tilde{u}\|_V^2 + 2\lambda\theta\|\tilde{\rho}\|_{L^3}^3 + 2\theta\sum_i \|\nabla\tilde{q}_i\|_{L^2}^2. \end{aligned} \tag{82}$$

Then adding (82) to (74), we obtain

$$\frac{1}{2}\|\tilde{u}\|_V^2 \le C + 2\lambda\theta\|\tilde{\rho}\|_{L^3}^3 + 2\theta\sum_i \|\nabla\tilde{q}_i\|_{L^2}^2. \tag{83}$$

Now we multiply (83) by $2(1+\tilde{C})$ (c.f. (70)) and choose θ small enough so that

$$4(1+\tilde{C})\theta \le \frac{1}{8}. \tag{84}$$

Adding the resulting inequality to (70), we obtain

$$\frac{1}{8}\sum_i \|\nabla\tilde{q}_i\|_{L^2}^2 + \|\tilde{u}\|_V^2 \le \overline{R} \tag{85}$$

for some $\overline{R}$ depending on data and parameters, but not on λ or δ.

Step 5. Existence of solutions to approximate system. The two estimates (79) and (85) verify the hypotheses of Proposition 1, and thus it follows that there exists a weak solution $(q_1^\delta, q_2^\delta, u^\delta) \in X$ to (29), (30), (35) satisfying

$$\sum_i \|q_i^\delta\|_{H^1} + \|u^\delta\|_V \le R \tag{86}$$

for some $R > 0$ independent of δ.

Step 6. Smoothness of approximate solutions. We establish that in fact $(q_1^\delta, q_2^\delta, u^\delta)$ is smooth and satisfies uniform H^2 estimates.

LEMMA 3. *If $(q_1^\delta, q_2^\delta, u^\delta) \in X$ is a weak solution to (29), (30), (35), then $(q_1^\delta, q_2^\delta, u^\delta)$ is smooth, that is,*

$$(q_1^\delta, q_2^\delta, u^\delta) \in X^k = H^k(\Omega) \times H^k(\Omega) \times (H^k(\Omega))^3 \quad \textit{for all } k > 0. \tag{87}$$

Furthermore, there exists $C_R > 0$ independent of δ so that

$$\|q_1^\delta\|_{H^2} + \|q_2^\delta\|_{H^2} + \|u^\delta\|_{H^2} \le C_R. \tag{88}$$

PROOF. First we verify (88). From the estimates (45), (46), (47) (taking $(f,g,h) = (q_1^\delta, q_2^\delta, u^\delta)$) and the uniform bound (86) it follows that

$$\begin{aligned}
\sum_i \|q_i^\delta\|_{W^{2,\frac{3}{2}}} + \|u^\delta\|_{W^{2,\frac{3}{2}}} \le& C\left(\sum_i \|\Delta q_i^\delta\|_{L^{\frac{3}{2}}} + \|Au^\delta\|_{L^{\frac{3}{2}}}\right) \\
=& C\left(\sum_i \|R_i^\delta(q_1^\delta, q_2^\delta, u^\delta)\|_{L^{\frac{3}{2}}} + \|R_u^\delta(q_1^\delta, q_2^\delta, u^\delta)\|_{L^{\frac{3}{2}}}\right) \\
\le& C\left(1 + \sum_i \|q_i^\delta\|_{H^1}^2 + \|u^\delta\|_V^2\right) \\
\le& \bar{C}_R
\end{aligned} \tag{89}$$

where $\bar{C}_R$ is independent of δ. Then, due to the embedding $W^{2,\frac{3}{2}} \hookrightarrow W^{1,3}$, it follows that

$$\sum_i \|q_i^\delta\|_{W^{1,3}} + \|u^\delta\|_{W^{1,3}} \le \tilde{C}_R \tag{90}$$

for $\tilde{C}_R$ independent of δ. Now we estimate R_i^δ, R_u^δ in L^2,

$$\begin{aligned}
\|R_i^\delta(q_1^\delta, q_2^\delta, u^\delta)\|_{L^2} \le& C(\|u^\delta\|_{L^6}\|\nabla c_i^\delta\|_{L^3} + \|\nabla c_i^\delta\|_{L^3}\|\nabla(-\Delta_W)^{-1}\rho^\delta\|_{L^6} + \|c_i^\delta\|_{L^3}\|\rho^\delta\|_{L^6}), \quad 1 \le i \le 2 \\
\|R_u^\delta(q_1^\delta, q_2^\delta, u^\delta)\|_{L^2} \le& C(\|u^\delta\|_{L^6}\|\nabla u\|_{L^3} + \|\rho^\delta\|_{L^3}\|\nabla(-\Delta_W)^{-1}\rho^\delta\|_{L^6})
\end{aligned} \tag{91}$$

and since all the terms on the right hand sides are bounded independently of δ due to (90), we find that Δq_i^δ, Au^δ are bounded in L^2 independently of δ, and (88) follows.

Higher regularity follows by induction. Indeed, suppose

$$(q_1^\delta, q_2^\delta, u^\delta) \in X^k \text{ for some integer } k \ge 3. \tag{92}$$

We show that $(q_1^\delta, q_2^\delta, u^\delta) \in X^{k+1}$ follows. By elliptic regularity, it suffices to show that $R_i^\delta(q_1^\delta, q_2^\delta, u^\delta)$ and $R_u^\delta(q_1^\delta, q_2^\delta, u^\delta)$ are in H^{k-1}, and so we compute

$$\begin{aligned}
\|R_i^\delta(q_1^\delta, q_2^\delta, u^\delta)\|_{H^{k-1}} \le& C(\|u^\delta \cdot \nabla c_i^\delta\|_{H^{k-1}} + \|\nabla c_i^\delta \cdot \nabla(-\Delta_W)^{-1}\rho^\delta\|_{H^{k-1}} + \|c_i^\delta \rho^\delta\|_{H^{k-1}}) \\
\le& C(\|u^\delta\|_{H^{k-1}}\|\nabla c_i^\delta\|_{H^{k-1}} + \|\nabla c_i^\delta\|_{H^{k-1}}\|\nabla(-\Delta_W)^{-1}\rho^\delta\|_{H^{k-1}} \\
&+ \|c_i^\delta\|_{H^{k-1}}\|\rho^\delta\|_{H^{k-1}}) \\
\le& C(\|u^\delta\|_{H^{k-1}}\|c_i^\delta\|_{H^k} + \|c_i^\delta\|_{H^k}(1 + \|\rho^\delta\|_{H^{k-2}}) + \|c_i^\delta\|_{H^{k-1}}\|\rho^\delta\|_{H^{k-1}}) \\
<& \infty \quad \text{by (92)}
\end{aligned} \tag{93}$$

where in the second inequality, we used the fact that H^s is an algebra for $s > \frac{3}{2}$; that is,

$$\|fg\|_{H^s} \le C\|f\|_{H^s}\|g\|_{H^s}.$$

Similarly, we estimate

$$\begin{aligned}\|R_u^\delta(q_1^\delta,q_2^\delta,u^\delta)\|_{H^{k-1}} \le& C(\|u^\delta\cdot\nabla u^\delta\|_{H^{k-1}}+\|\rho^\delta\nabla(-\Delta_W)^{-1}\rho^\delta\|_{H^{k-1}})\\ \le& C\|u^\delta\|_{H^{k-1}}\|u^\delta\|_{H^k}+\|\rho^\delta\|_{H^{k-1}}(1+\|\rho^\delta\|_{H^{k-2}})\\ <&\infty \quad \text{by (92)}\end{aligned} \tag{94}$$

where in the first inequality, we used the fact that $\mathbb{P}$ is continuous as a mapping $\mathbb{P}: H^{k-1}\to H^{k-1}$ [**30**].

The proof of the smoothness of $(q_1^\delta,q_2^\delta,u^\delta)$ is thus complete once we verify the base case of $k=3$. It suffices to show that $R_i^\delta(q_1^\delta,q_2^\delta,u^\delta)$ and $R_u^\delta(q_1^\delta,q_2^\delta,u^\delta)$ are in H^1:

$$\begin{aligned}\|R_i^\delta(q_1^\delta,q_2^\delta,u^\delta)\|_{H^1} \le& C(\|u^\delta\cdot\nabla c_i^\delta\|_{H^1}+\|\mathrm{div}\,(\chi_\delta(c_i^\delta)\nabla(-\Delta_W)^{-1}\rho^\delta)\|_{H^1})\\ \le& C(\|u^\delta\cdot\nabla c_i^\delta\|_{L^2}+\|\nabla(u^\delta\cdot\nabla c_i^\delta)\|_{L^2}+\|\chi_\delta(c_i^\delta)\nabla(-\Delta_W)^{-1}\rho^\delta\|_{H^2})\\ \le& C(\|u^\delta\|_{L^\infty}\|\nabla c_i^\delta\|_{L^2}+\|\nabla u^\delta\|_{L^4}\|\nabla c_i^\delta\|_{L^4}+\|u^\delta\|_{L^\infty}\|\nabla\nabla c_i^\delta\|_{L^2}\\ &+\|\chi_\delta(c_i^\delta)\|_{H^2}\|\nabla(-\Delta_W)^{-1}\rho^\delta\|_{H^2})\\ <&\infty,\quad 1\le i\le 2 \quad \text{by (88)}\\ \|R_u^\delta(q_1^\delta,q_2^\delta,u^\delta)\|_{H^1} \le& C\|u^\delta\cdot\nabla u^\delta\|_{H^1}+\|\rho^\delta\nabla(-\Delta_W)^{-1}\rho^\delta\|_{H^1})\\ \le& C(\|u^\delta\cdot\nabla u^\delta\|_{L^2}+\|\nabla(u^\delta\cdot\nabla u^\delta)\|_{L^2}\\ &+\|\rho^\delta\nabla(-\Delta_W)^{-1}\rho^\delta\|_{L^2}+\|\nabla(\rho^\delta\nabla(-\Delta_W)^{-1}\rho^\delta\|_{L^2})\\ \le& C(\|u^\delta\|_{L^\infty}\|u^\delta\|_V+\|\nabla u^\delta\|_{L^4}^2+\|u^\delta\|_{L^\infty}\|\nabla\nabla u^\delta\|_{L^2}\\ &+\|\rho^\delta\|_{L^4}\|\nabla(-\Delta_W)^{-1}\rho^\delta\|_{L^4}+\|\nabla\rho^\delta\|_{L^2}\|\nabla(-\Delta_W)^{-1}\rho^\delta\|_{L^\infty}\\ &+\|\rho^\delta\|_{L^4}\|\nabla\nabla(-\Delta_W)^{-1}\rho^\delta\|_{L^4})\\ <&\infty \quad \text{by (88)}\end{aligned} \tag{95}$$

where again, we used the fact that $\mathbb{P}: H^1\to H^1$ is continuous. Thus the proof of the lemma is complete. $\square$

Now we finish off the proof of Theorem 1

Step 7. Nonnegativity of concetrations and passing to the limit.. First, we establish the nonnegativity of c_i^δ. We recall that c_i^δ satisfies

$$-\Delta c_i^\delta=-(u^\delta\cdot\nabla c_i^\delta)\chi_\delta'(c_i^\delta)+z_i(\nabla c_i^\delta\cdot\nabla\Phi^\delta)\chi_\delta'(c_i^\delta)-\chi_\delta(c_i^\delta)\rho^\delta,\quad 1\le i\le 2 \tag{96}$$

where $z_1=1=-z_2$. Suppose c_i^δ attains a negative value in Ω, and suppose that at $x_0\in\Omega$ we have $c_i^\delta(x_0)=\inf_\Omega c_i^\delta<0$. Then consider the largest ball B centered at x_0 so that $c^\delta_{i\,|B}\le 0$. Since $c^\delta_{i\,|\partial\Omega}=\gamma_i>0$, we necessarily have $\bar{B}\subset\Omega$ and for some $y\in\partial B$ we have $c_i^\delta(y)=0$. Furthermore, c_i^δ satisfies

$$-\Delta c_i^\delta=-(u^\delta\cdot\nabla c_i^\delta)\chi_\delta'(c_i^\delta)+z_i(\nabla c_i^\delta\cdot\nabla\Phi^\delta)\chi_\delta'(c_i^\delta). \tag{97}$$

in B. However, since $c^\delta_{i\,|B}$ attains its global minimum in B, the strong maximum principle implies that $c^\delta_{i\,|B}\equiv\inf_\Omega c_i^\delta<0$; however this contradicts the fact that $c_i^\delta(y)=0$ for $y\in\partial B$. Therefore $c_i^\delta\ge 0$.

Now, due to (88), there is a sequence $\delta_j\to 0$ as $j\to\infty$ and $(c_1,c_2,u)\in X^2$ so that $(c_1^{\delta_j},c_2^{\delta_j},u^{\delta_j})\to(c_1,c_2,u)$ strongly in X^1, pointwise almost everywhere, and weakly in X^2 as $j\to\infty$. And, we take

$$\Phi=(-\Delta_W)^{-1}\rho. \tag{98}$$

Since c_i is the pointwise almost everywhere limit of nonnegative functions, we have $c_i\ge 0$ almost everywhere, and after redefining c_i on a set of measure zero, we assume henceforth that $c_i\ge 0$ everywhere.

Now we verify that (c_1,c_2,u) together with (98) is a weak solution of (9)-(16). Since the trace operator is continuous from $H^1(\Omega)$ into $H^{\frac12}(\partial\Omega)$ and (c_1,c_2,u) is the strong X^1 limit of $(c_1^{\delta_j},c_2^{\delta_j},u^{\delta_j})$, we have that the boundary conditions (14)-(16) are satisfied in the sense of traces. Next, we verify that (c_1, c_2, u) satisfies (9)-(13) in the weak sense: for any $(\psi_1, \psi_2, \psi_u) \in X$, we have

$$
\begin{aligned}
\int_\Omega \nabla c_i \cdot \nabla \psi_i \, dV &= -\int_\Omega (-uc_i + c_i \nabla(-\Delta_W)^{-1}\rho) \cdot \nabla \psi_i \, dV, \quad 1 \le i \le 2 \\
\int_\Omega \nabla u : \nabla \psi_u \, dV &= -\int_\Omega (B(u,u) + \mathbb{P}(\rho \nabla(-\Delta_W)^{-1}\rho)) \cdot \psi_u \, dV.
\end{aligned}
\tag{99}
$$

Prior to establishing these equalities, we show that

$$
\|\chi_{\delta_j}(c_i^{\delta_j}) - c_i\|_{L^6} \to 0 \quad \text{as } j \to \infty, \quad 1 \le i \le 2. \tag{100}
$$

Indeed, we have

$$
\int_\Omega |\chi_{\delta_j}(c_i^{\delta_j}) - c_i|^6 \, dV \le C \left(\int_\Omega |\chi_{\delta_j}(c_i^{\delta_j}) - \chi_{\delta_j}(c_i)|^6 \, dV + \int_\Omega |\chi_{\delta_j}(c_i) - c_i|^6 \, dV \right). \tag{101}
$$

The first integral on the right hand side converges to 0 because

$$
\|\chi_{\delta_j}(c_i^{\delta_j}) - \chi_{\delta_j}(c_i)\|_{L^6} \le a \|c_i^{\delta_j} - c_i\|_{L^6} \le C \|c_i^{\delta_j} - c_i\|_{H^1} \to 0. \tag{102}
$$

The second integral also converges to 0 due to the dominated convergence theorem and the fact that for each $x \in \Omega$, we have $\chi_{\delta_j}(c_i(x)) = c_i(x)$ for all j sufficiently large since $\chi_\delta(y) = y$ for all $\delta \le y$ if $y > 0$ and for all δ if $y = 0$. Thus, we can now compute

$$
\left| \int_\Omega \nabla c_i \cdot \nabla \psi_i \, dV - \int_\Omega \nabla c_i^{\delta_j} \cdot \nabla \psi_i \, dV \right| \le \|\psi_i\|_{H^1} \|c_i - c_i^{\delta_j}\|_{H^1} \to 0, \quad 1 \le i \le 2
$$

$$
\begin{aligned}
\left| \int_\Omega u c_i \cdot \nabla \psi_i \, dV - \int_\Omega u^{\delta_j} \chi_{\delta_j}(c_i^{\delta_j}) \cdot \nabla \psi_i \, dV \right| \le & \|\psi_i\|_{H^1} (\|u\|_{L^3} \|c_i - \chi_{\delta_j}(c_i^{\delta_j})\|_{L^6} \\
& + \|\chi_{\delta_j}(c_i^{\delta_j})\|_{L^3} \|u - u^{\delta_j}\|_{L^6}) \to 0, \quad 1 \le i \le 2
\end{aligned}
$$

$$
\begin{aligned}
&\left| \int_\Omega c_i \nabla(-\Delta_W)^{-1}\rho \cdot \nabla \psi_i \, dV - \int_\Omega \chi_{\delta_j}(c_i^{\delta_j}) \nabla(-\Delta_W)^{-1}\rho^{\delta_j} \cdot \nabla \psi_i \, dV \right| \\
\le & \|\psi_i\|_{H^1} (\|\nabla(-\Delta_W)^{-1}\rho\|_{L^3} \|c_i - \chi_{\delta_j}(c_i^{\delta_j})\|_{L^6} \\
& + \|\nabla(-\Delta_D)^{-1}(\rho - \rho^{\delta_j})\|_{L^6} \|\chi_{\delta_j}(c_i^{\delta_j})\|_{L^3}) \to 0, \quad 1 \le i \le 2
\end{aligned}
$$

$$
\left| \int_\Omega \nabla u : \nabla \psi_u \, dV - \int_\Omega \nabla u^{\delta_j} : \nabla \psi_u \, dV \right| \le \|\psi_u\|_V \|u - u^{\delta_j}\|_V \to 0
$$

$$
\left| \int_\Omega B(u,u) \cdot \psi_u \, dV - \int_\Omega B(u^{\delta_j}, u^{\delta_j}) \cdot \psi_u \right| \le \|\psi_u\|_{L^6} (\|u - u^{\delta_j}\|_{L^3} \|u\|_V + \|u^{\delta_j}\|_{L^3} \|u - u^{\delta_j}\|_V) \to 0
$$

$$
\begin{aligned}
&\left| \int_\Omega \rho \nabla(-\Delta_W)^{-1}\rho \cdot \psi_u \, dV - \int_\Omega \rho^{\delta_j} \nabla(-\Delta_W)^{-1}\rho^{\delta_j} \cdot \psi_u \right| \\
\le & \|\psi\|_{L^6} (\|\rho - \rho^{\delta_j}\|_{L^2} \|\nabla(-\Delta_W)^{-1}\rho\|_{L^3} + \|\rho^{\delta_j}\|_{L^2} \|\nabla(-\Delta_D)^{-1}(\rho - \rho^{\delta_j})\|_{L^3}) \to 0.
\end{aligned}
$$

The above computations, together with the fact that $(c_1^\delta, c_2^\delta, u^\delta)$ satisfy (17)-(21), imply (99).

Finally, the smoothness of (c_1, c_2, u) follows from the same bootstrapping scheme as in the proof of Lemma 3. The proof of Theorem 1 is now complete. □

The equilibria of the Nernst-Planck-Navier-Stokes system are unique minimizers of a total energy that is nonincreasing in time on solutions [**6**] and they arise when certain equilibrium boundary conditions (5), (7) are supplied. The potential then obeys Poisson-Boltzmann equations (4) which provide the unique steady state solution (c_1^*, c_2^*) of the Nernst-Planck equations (1) with zero fluid velocity $u^* \equiv 0$. However, in many cases of physical interest, the boundary conditions are not suitable for equilibrium, and an electrical potential gradient generates (experimentally [**25**] or numerically [**10, 21**]) nontrivial fluid flow which is responsible for the instability. Thus, it is relevant to derive rigorous conditions under which the steady state whose existence is guaranteed by Theorem 1 has nonzero fluid velocity $u^* \not\equiv 0$. We derive below one such condition.

THEOREM 2. *Suppose (c_1^*, c_2^*, u^*) is a solution to (9)-(13) with boundary conditions (14)-(16). Suppose in addition that the boundary conditions satisfy*

$$\int_{\partial\Omega} (\gamma_1 - \gamma_2)(n_i\partial_j - n_j\partial_i)W\, dS \neq 0 \quad \textit{or} \quad \int_{\partial\Omega} W(n_i\partial_j - n_j\partial_i)(\gamma_1 - \gamma_2)\, dS \neq 0. \tag{103}$$

for some $i, j \in \{x, y, z\}, i \neq j$, where n_i are the components of the unit normal vector along $\partial\Omega$. Then $u^* \not\equiv 0$.

REMARK 3. *We note that if $i, j \in \{x, y, z\}$ with $i \neq j$, then $n_i\partial_j - n_j\partial_i$ is a vector field tangent to $\partial\Omega$, so that the integrals in (103) are well defined. Indeed, the characteristic directions of $n_i\partial_j - n_j\partial_i$ are $n_ie_j - n_je_i$ (with e_k the canonical basis of $\mathbb{R}^d$) and $n \cdot (n_ie_j - n_je_i) = 0$ shows that these are tangent to $\partial\Omega$. Thus, the condition (103) can be checked with just knowledge of the values of c_i and Φ on $\partial\Omega$.*

PROOF. If $(c_1^*, c_2^*, u^* \equiv 0)$ is a solution to (9)-(13), then $\rho^*\nabla\Phi$ must be a gradient force i.e.

$$\rho^*\nabla\Phi^* = \nabla F \tag{104}$$

for some smooth F. Thus a sufficient condition for $u^* \not\equiv 0$ is

$$\nabla \times (\rho^*\nabla\Phi^*) \not\equiv 0. \tag{105}$$

In turn, a sufficient condition for (105) is

$$\int_{\Omega} \nabla\rho^* \times \nabla\Phi^*\, dV \neq 0. \tag{106}$$

Integrating the above integral by parts, moving the derivative off ρ^*, we obtain the following equivalent condition

$$\int_{\partial\Omega} \rho^*(n_i\partial_j - n_j\partial_i)\Phi\, dS \neq 0 \quad \text{for some } i, j \in \{x, y, z\},\, i \neq j. \tag{107}$$

And by the remark following the statement of Theorem 2, this is equivalent to the condition

$$\int_{\partial\Omega} (\gamma_1 - \gamma_2)(n_i\partial_j - n_j\partial_i)W\, dS \neq 0 \quad \text{for some } i, j \in \{x, y, z\},\, i \neq j. \tag{108}$$

Similarly, by moving the derivative off Φ^* in (106), we obtain the equivalent condition

$$\int_{\partial\Omega} W(n_i\partial_j - n_j\partial_i)(\gamma_1 - \gamma_2)\, dS \neq 0 \quad \text{for some } i, j \in \{x, y, z\},\, i \neq j. \tag{109}$$

This completes the proof. □

3. Maximum principle and long time behavior of solutions

In this section we investigate the long time behavior of the system

$$\partial_t c_1 + u\cdot\nabla c_1 = D_1 \mathrm{div}\,(\nabla c_1 + c_1\nabla\Phi) \tag{110}$$
$$\partial_t c_2 + u\cdot\nabla c_2 = D_2 \mathrm{div}\,(\nabla c_2 - c_2\nabla\Phi) \tag{111}$$
$$-\epsilon\Delta\Phi = \rho = c_1 - c_2 \tag{112}$$
$$\partial_t u - \nu\Delta u + \nabla p = -K\rho\nabla\Phi \tag{113}$$
$$\mathrm{div}\, u = 0. \tag{114}$$

The global existence and uniqueness of smooth solutions of this system with Dirichlet boundary conditions is proved in [**9**]. Here we prove a maximum/minimum principle for the ionic concentrations c_i, which in particular gives us time independent L^∞ bounds (see also [**7**]). In addition, we show that the Dirichlet boundary data of c_i are *attracting* in the sense that $\max\{\sup_\Omega c_1, \sup_\Omega c_2\}$ and $\min\{\sup_\Omega c_1, \sup_\Omega c_2\}$ converge monotonically to the extremal values of the boundary values in the limit of $t\to\infty$.

The restriction to the Stokes subsystem is due to lack of information on global regularity for Navier-Stokes solutions in 3D, thus limiting the analysis of long time behavior. The results below do extend to 2D NPNS and apply to 3D NPNS under the assumption of regularity of velocity. The modifications to the proofs required in these cases are straightforward but will not be pursued here.

We consider a general smooth, bounded, connected domain $\Omega\subset\mathbb{R}^3$ with boundary conditions

$$\begin{aligned} c_{i|\partial\Omega} &= \gamma_i > 0, \quad 1\le i\le 2\\ \Phi_{|\partial\Omega} &= W\\ u_{|\partial\Omega} &= 0\end{aligned} \tag{115}$$

where γ_i and W are smooth and not necessarily constant.

THEOREM 3. *Suppose* $(c_1\ge 0, c_2\ge 0, u)$ *is the unique, global smooth solution to (110)-(114) on* Ω *with smooth initial conditions* $(c_1(0)\ge 0, c_2(0)\ge 0, u(0))$ *(with div* $u(0) = 0$*) and boundary conditions (115). Then,*

(I) For $i = 1,2$ *and all* $t\ge 0$

$$\min\{\inf_\Omega c_1(0), \inf_\Omega c_2(0), \underline{\gamma}\} \le c_i(t,x) \le \max\{\sup_\Omega c_1(0), \sup_\Omega c_2(0), \overline{\gamma}\} \tag{116}$$

where $\underline{\gamma} = \min_i \inf_{\partial\Omega}\gamma_i$ *and* $\overline{\gamma} = \max_i \sup_{\partial\Omega}\gamma_i$*. In particular*

$$\overline{M}(t) = \max_i \sup_\Omega c_i(t,x), \quad \underline{M}(t) = \min_i \inf_\Omega c_i(t,x) \tag{117}$$

are nonincreasing and nondecreasing on $(0,\infty)$*, respectively.*

(II) For all $\delta > 0$*, there exists* T *depending on* δ*,* Ω *and initial and boundary conditions such that for all* $t\ge T$ *we have*

$$\underline{\gamma} - \delta \le \underline{M}(t) \le \overline{M}(t) \le \overline{\gamma} + \delta.$$

The theorem is a consequence of the following proposition.

PROPOSITION 2. *Suppose* $v_i : [0,\infty)\times\bar\Omega\to\mathbb{R}$*,* $i = 1,2$ *is a nonnegative, smooth solution to*

$$\begin{aligned}\partial_t v_1 &= d_1\Delta v_1 + b_1\cdot\nabla v_1 - p_1(v_1 - v_2)\\ \partial_t v_2 &= d_2\Delta v_2 + b_2\cdot\nabla v_2 + p_2(v_1 - v_2)\end{aligned} \tag{118}$$

with time independent, smooth Dirichlet boundary conditions

$$v_{i|\partial\Omega} = g_i > 0, \quad 1\le i\le 2 \tag{119}$$

where $d_i > 0$ *are constants,* $b_i = b_i(t,x)$ *are smooth vector fields, and* $p_i = p_i(t,x)\ge 0$*. Then*

(I') For both i and all $t \geq 0$

$$\min\{\inf_\Omega v_1(0), \inf_\Omega v_2(0), \underline{g}\} \leq v_i(t,x) \leq \max\{\sup_\Omega v_1(0), \sup_\Omega v_2(0), \overline{g}\} \tag{120}$$

where $\underline{g} = \min_i \inf_{\partial\Omega} g_i$ and $\overline{g} = \max_i \sup_{\partial\Omega} g_i$. In particular

$$\overline{V}(t) = \max_i \sup_\Omega v_i(t,x), \quad \underline{V}(t) = \min_i \inf_\Omega v_i(t,x) \tag{121}$$

are nonincreasing and nondecreasing on $(0,\infty)$, respectively.

(II') Suppose, in addition to the preceding hypotheses, that b_i is uniformly bounded in time. Then for all $\delta > 0$, there exists $0 < T^* = T^*(\delta, d_i, \sup_t \|b_i(t)\|_{L^\infty}, g_i, v_i(t=0), \Omega)$ such that for all $t \geq T^*$ we have

$$\underline{g} - \delta \leq \underline{V}(t) \leq \overline{V}(t) \leq \overline{g} + \delta.$$

PROOF. We prove just the lower bound in (I') as the upper bound can be established analogously. If either $\inf_\Omega v_1(t=0) = 0$ or $\inf_\Omega v_2(t=0) = 0$ then the lower bound holds trivially as we are assuming that $v_i \geq 0$. So we assume $v_1(t=0), v_2(t=0) > 0$.

We define

$$\underline{\underline{V}}(t) = \min_{0 \leq s \leq t} \underline{V}(s).$$

We show that $\underline{V}$ and $\underline{\underline{V}}$ are both locally Lipschitz (i.e. Lipschitz continuous on every interval $[0,T]$). Indeed, assigning to each $t \geq 0$ a point $x_i(t) \in \bar{\Omega}$ such that $v_i(t, x_i(t)) = \underline{V}_i(t) = \inf_\Omega v_i(t,x)$, we have for $s < t \leq T$

$$\frac{v_i(t,x_i(t)) - v_i(s,x_i(t))}{t-s} \leq \frac{\underline{V}_i(t) - \underline{V}_i(s)}{t-s} \leq \frac{v_i(t,x_i(s)) - v_i(s,x_i(s))}{t-s} \tag{122}$$

and so

$$\left| \frac{\underline{V}_i(t) - \underline{V}_i(s)}{t-s} \right| \leq \sup_{[0,T]\times\bar{\Omega}} |\partial_t v_i(t,x)| = \underline{L}_i^T, \quad 1 \leq i \leq 2 \tag{123}$$

implying that $\underline{V}_i(t)$ is locally Lipschitz. Next, assigning to each $t \geq 0$ an $i(t) \in \{1,2\}$ such that $\underline{V}_{i(t)}(t) = \underline{V}(t)$ we have for $s < t \leq T$

$$\frac{\underline{V}_{i(t)}(t) - \underline{V}_{i(t)}(s)}{t-s} \leq \frac{\underline{V}(t) - \underline{V}(s)}{t-s} \leq \frac{\underline{V}_{i(s)}(t) - \underline{V}_{i(s)}(s)}{t-s} \tag{124}$$

and thus

$$\left| \frac{\underline{V}(t) - \underline{V}(s)}{t-s} \right| \leq \max_i \underline{L}_i^T = \underline{L}^T. \tag{125}$$

Thus $\underline{V}_i$ is locally Lipschitz. Lastly consider $\underline{\underline{V}}$. Fixing $s < t \leq T$, we assume without loss of generality that $\underline{\underline{V}}(t) \neq \underline{\underline{V}}(s)$. In particular since $\underline{\underline{V}}$ is nonincreasing, this implies $\underline{\underline{V}}(t) < \underline{\underline{V}}(s)$. Then consider

$$t^* = \inf\{t' \in [s,t] | \, \underline{V}(t') = \underline{\underline{V}}(t)\}. \tag{126}$$

Since $\underline{V}(s) \geq \underline{\underline{V}}(s) > \underline{\underline{V}}(t)$, we necessarily have $t^* > s$. Thus,

$$\left| \frac{\underline{\underline{V}}(t) - \underline{\underline{V}}(s)}{t-s} \right| = \frac{\underline{\underline{V}}(s) - \underline{\underline{V}}(t)}{t-s} \leq \frac{\underline{V}(s) - \underline{V}(t^*)}{t-s} \leq \frac{\underline{V}(s) - \underline{V}(t^*)}{t^*-s} \leq \underline{L}^T \tag{127}$$

and local Lipschitz continuity of $\underline{\underline{V}}$ follows.

Due to the Lipschitz continuity, we have in particular that $\underline{V}_i, \underline{V}, \underline{\underline{V}}$ are differentiable almost everywhere and the set $A_T = \{t \in (0,T) | \, \underline{V}_1'(t), \underline{V}_2'(t), \underline{V}'(t), \underline{\underline{V}}'(t) \text{ exist}\}$ has full measure, $|A_T| = T$. To complete the proof of the lower bound in (I'), we prove the following lemma.

LEMMA 4. *For all $t \in A_T$ we have*

(i) For each i and for all $x \in \bar{\Omega}$ such that $v_i(t,x) = \underline{V}_i(t)$, we have $\underline{V}_i'(t) = \partial_t v_i(t,x)$

(ii) $\underline{V}'(t) = \underline{V}_i'(t)$ for all i such that $\underline{V}(t) = \underline{V}_i(t)$

(iii) if $\underline{\underline{V}}'(t) < 0$, then $\underline{\underline{V}}(t) = \underline{V}(t)$ and $\underline{\underline{V}}'(t) = \underline{V}'(t)$.

PROOF. We fix $t \in A_T$. To see (i), we fix i and $x \in \bar{\Omega}$ such that $v_i(t,x) = \underline{V}_i(t)$ and compute for $0 < s < t < T$

$$\frac{\underline{V}_i(t) - \underline{V}_i(s)}{t-s} \geq \frac{v_i(t,x) - v_i(s,x)}{t-s} \tag{128}$$

and taking the limit $s \to t^-$ we see that $\underline{V}_i'(t) \geq \partial_t v_i(t,x)$. Similarly for $0 < t < s < T$ we have

$$\frac{\underline{V}_i(s) - \underline{V}_i(t)}{s-t} \leq \frac{v_i(s,x) - v_i(t,x)}{s-t} \tag{129}$$

and we obtain $\underline{V}_i'(t) \leq \partial_t v_i(t,x)$ upon taking the limit $s \to t^+$. An analogous argument gives us (ii).

Now we show (iii). Assume $\underline{\underline{V}}'(t) < 0$, then for all $s < t$ we have $\underline{\underline{V}}(s) > \underline{\underline{V}}(t)$ for otherwise, since $\underline{\underline{V}}$ is nonincreasing we have $\underline{\underline{V}}(s) = \underline{\underline{V}}(t)$ for some $s < t$. But then, again since $\underline{\underline{V}}$ is nonincreasing we have $\underline{\underline{V}}(r) = \underline{\underline{V}}(t)$ for all $r \in [s,t]$. However, it then follows that the left sided derivative of $\underline{\underline{V}}$ at t is zero, which contradicts our assumption $\underline{\underline{V}}'(t) < 0$. By the same argument we see that for all $s > t$ we have $\underline{\underline{V}}(s) < \underline{\underline{V}}(t)$.

It now follows that $\underline{\underline{V}}(t) = \underline{V}(t)$. Indeed, otherwise, there exists $s < t$ such that $\underline{V}(s) = \underline{\underline{V}}(t)$. But by the argument in the previous paragraph, we have $\underline{V}(s) \geq \underline{\underline{V}}(s) > \underline{\underline{V}}(t)$, which gives us a contradiction.

Now to prove that $\underline{\underline{V}}'(t) = \underline{V}'(t)$, we compute for $s < t$, using $\underline{\underline{V}}(t) = \underline{V}(t)$

$$\frac{\underline{\underline{V}}(t) - \underline{\underline{V}}(s)}{t-s} \geq \frac{\underline{V}(t) - \underline{V}(s)}{t-s} \tag{130}$$

which gives us $\underline{\underline{V}}'(t) \geq \underline{V}'(t)$ upon taking the limit $s \to t^-$. Similarly, considering $s > t$ we obtain the opposite inequality, thus completing the proof of (iii). □

Now we complete the proof of the lower bound in (I'). Suppose for the sake of contradiction that there exists $T > 0$ such that $\underline{\underline{V}}(T) < \min\{\inf_\Omega v_1(0), \inf_\Omega v_2(0), \underline{g}\}$. Then since $\underline{\underline{V}}$ is locally Lipschitz and hence satisfies the fundamental theorem of calculus, there exists $t \in A^T$ such that $\underline{\underline{V}}'(t) < 0$ and $\underline{\underline{V}}(t) < \min\{\inf_\Omega v_1(0), \inf_\Omega v_2(0), \underline{g}\}$. Then it follows from Lemma 4 that for some i and some $x \in \bar{\Omega}$, we have $v_i(t,x) = \underline{\underline{V}}(t)$ and $\partial_t v_i(t,x) < 0$. Also, since $\underline{\underline{V}}(t) < \underline{g}$ we have that $x \in \Omega$. We assume without loss of generality that $i = 1$. Then evaluating (110) at (t,x) and using the fact that $v_2(t,x) \geq \underline{\underline{V}}(t)$ we find that

$$\begin{aligned} 0 > \partial_t v_1 =& d_1 \Delta v_1 + b_1 \cdot \nabla v_1 - p_1(v_1 - v_2) \\ \geq& -p_1(\underline{\underline{V}}(t) - \underline{\underline{V}}(t)) \\ =& 0 \end{aligned} \tag{131}$$

which gives us a contradiction. Therefore $\underline{V}(t) \geq \underline{\underline{V}}(t) \geq \min\{\inf_\Omega v_1(0), \inf_\Omega v_2(0), \underline{g}\}$ for all t. The monotonicity of $\underline{V}$ follows from the same argument by replacing the initial time 0 by some arbitrary time $s > 0$. Then we obtain for all $t > s$

$$\underline{V}(t) \geq \underline{\underline{V}}(t) \geq \min\{\inf_\Omega v_1(s), \inf_\Omega v_2(s)\} = \underline{V}(s). \tag{132}$$

Above we removed the $\underline{\gamma}$ from the minimum, as this is redundant for $s > 0$.

Now we prove the lower bound statement of (II'). The upper bound statement is proved similarly.

We assume without loss of generality that $0 \notin \bar{\Omega}$ so that there exists $\underline{\alpha}, \overline{\alpha} > 0$ such that $\underline{\alpha} \leq |x| \leq \overline{\alpha}$ for all $x \in \bar{\Omega}$. Then we define

$$w_i = v_i - \epsilon |x|^\lambda, \quad 1 \leq i \leq 2 \tag{133}$$

where $\lambda > 0$ is chosen large enough so that for each i we have

$$\frac{d_i}{2}(\lambda + 1) \geq \sup_{t \geq 0,\, x \in \Omega} |b_i(t,x) \cdot x|. \tag{134}$$

and $\epsilon > 0$ is chosen small enough that

$$\epsilon \overline{\alpha}^\lambda \leq \delta. \tag{135}$$

The functions w_i satisfy the equations

$$\partial_t w_i = d_i \Delta w_i + b_i \cdot \nabla w_i + \epsilon d_i \lambda(\lambda+1)|x|^{\lambda-2} + \epsilon\lambda|x|^{\lambda-2} b_i \cdot x - z_i p_i(w_1 - w_2), \quad 1 \le i \le 2 \tag{136}$$

where $z_1 = 1 = -z_2$.

By the same proof as in Lemma 4 and the discussion leading up to it, we know that the functions

$$\underline{W}_i(t) = \inf_{\Omega} w_i(t,x), 1 \le i \le 2, \quad \underline{W}(t) = \min_i \underline{W}_i(t) \tag{137}$$

are locally Lipschitz and thus differentiable almost everywhere. In addition, for each $t > 0$ where $\underline{W}'(t)$, $\underline{W}_1'(t)$, $\underline{W}_2'(t)$ all exist, we have that for each i and $x \in \bar{\Omega}$ such that $w_i(x,t) = \underline{W}(t)$, the time derivatives coincide, $\partial_t w_i(x,t) = \underline{W}'(t)$.

Now, if initially, at time $t = 0$, we have $\underline{g} - \epsilon \sup_{\partial\Omega} |x|^{\lambda} \le \underline{W}(0)$, then since $\underline{W} \le \underline{V}$ we have, using (135),

$$\underline{g} - \delta \le \underline{g} - \epsilon \overline{\alpha}^{\lambda} \le \underline{V}(0). \tag{138}$$

Then, since $\underline{V}$ is monotone nondecreasing, the lower bound in (II') follows.

Now suppose $\underline{W}(0) < \underline{g} - \epsilon \sup_{\partial\Omega} |x|^{\lambda}$. Then in particular, we have

$$\underline{W}(0) < \min_i \inf_{\partial\Omega} w_i = \min_i \inf_{\partial\Omega} (g_i - \epsilon|x|^{\lambda}). \tag{139}$$

Thus by continuity, we have that

$$\underline{W}(t) < \min_i \inf_{\partial\Omega} (g_i - \epsilon|x|^{\lambda}) \tag{140}$$

holds on some interval $[0, T^*)$ where $T^* \in (0, \infty]$ can be chosen to be maximal so that if T^* is finite, we have $\underline{W}(T^*) = \min_i \inf_{\partial\Omega}(g_i - \epsilon|x|^{\lambda})$.

We claim that indeed $T^* < \infty$. The lower bound of (II') follows from this claim since

$$\begin{aligned}\underline{W}(T^*) = \min_i \inf_{\partial\Omega}(g_i - \epsilon|x|^{\lambda}) \Rightarrow \min_i \inf_{\Omega}(v_i(T^*) - \epsilon|x|^{\lambda}) \ge \underline{g} - \epsilon \sup_{\partial\Omega} |x|^{\lambda} \ge \underline{g} - \delta \\ \Rightarrow \underline{V}(T^*) \ge \underline{g} - \delta\end{aligned} \tag{141}$$

and the lower bound continues to hold for all $t \ge T^*$ due to the monotonicity of $\underline{V}$.

It remains to prove the claim that $T^* < \infty$. Indeed, let us fix a time t such that (140) holds. Then at time t, the value $\underline{W}(t)$ is attained by w_i, for some i, at some interior point $x \in \Omega$. This point is a global minimum of w_i at time t. We assume without loss of generality that $i = 1$. Thus evaluating (136) at (t,x) and using (134) and the fact that $w_2(t,x) \ge w_1(t,x)$ we have

$$\begin{aligned}\partial_t w_1 =& d_1 \Delta w_1 + b_1 \cdot \nabla w_1 + \epsilon d_1 \lambda(\lambda+1)|x|^{\lambda-2} + \epsilon\lambda|x|^{\lambda-2} b_1 \cdot x - p_1(w_1 - w_2) \\ \ge& \epsilon d_1 \lambda(\lambda+1)|x|^{\lambda-2} - \epsilon\lambda|x|^{\lambda-2} \sup_{y\in\Omega} |b_1(t,y) \cdot y| \\ \ge& \frac{1}{2}\epsilon d_1 \lambda(\lambda+1)|x|^{\lambda-2} \\ \ge& \frac{1}{2}\epsilon d_1 \lambda(\lambda+1)\underline{\alpha}^{\lambda-2}.\end{aligned} \tag{142}$$

If $t \in A_{T^*} = \{t \in (0,T^*) | \underline{W}'(t), \underline{W}_1'(t), \underline{W}_2'(t) \text{ exist}\}$, then by the same argument as in Lemma 4, we have $\underline{W}'(t) = \partial_t w_1(t,x)$ and thus

$$\underline{W}'(t) \ge \frac{1}{2}\epsilon d_1 \lambda(\lambda+1)\underline{\alpha}^{\lambda-2}. \tag{143}$$

In general, the relation

$$\underline{W}'(t) \ge \min_i \frac{1}{2}\epsilon d_i \lambda(\lambda+1)\underline{\alpha}^{\lambda-2} = \tilde{\beta}. \tag{144}$$

holds for every time $t \in A_{T^*}$. Since $\underline{W}, \underline{W}_1, \underline{W}_2$ are each locally Lipschitz (and hence differentiable almost everwhere and satisfy the fundamental theorem of calculus), if (140) holds on $[0,\infty)$ (i.e. if $T^* = \infty$), then we obtain

$$\infty > \liminf_{t\to\infty} \underline{W}(t) = \underline{W}(0) + \liminf_{t\to\infty} \int_0^t \underline{W}'(s)\,ds \geq \underline{W}(0) + \liminf_{t\to\infty}(t\tilde{\beta}) = \infty \tag{145}$$

which gives us a contradiction. Thus $T^* < \infty$, and in fact since, by (I'),

$$\begin{aligned} \max\{\sup_\Omega v_1(0), \sup_\Omega v_2(0), \overline{g}\} - \epsilon\underline{\alpha}^\lambda \geq \underline{V}(T^*) - \epsilon\underline{\alpha}^\lambda \geq \underline{W}(T^*) =& \underline{W}(0) + \int_0^{T^*} \underline{W}'(s)\,ds \\ \geq& \underline{W}(0) + T^*\tilde{\beta}, \end{aligned} \tag{146}$$

we have that T^* is bounded above by a constant depending ultimately on $\delta, d_i, \sup_t \|b_i(t)\|_{L^\infty}$, the intitial and boundary conditions, and the domain,

$$T^* \leq \frac{1}{\tilde{\beta}}(\max\{\sup_\Omega v_1(0), \sup_\Omega v_2(0), \overline{g}\} - \underline{W}(0) - \epsilon\underline{\alpha}^\lambda). \tag{147}$$

This completes the proof of the lower bound of (II') and thus of the proposition. □

REMARK 4. *We note that the preceding proof, which relies on a maximum principle argument, does not depend on the spatial dimension d. Thus, Proposition 2 holds for smooth, bounded, connected domains in any spatial dimension.*

We now prove Theorem 3 using Proposition 2.

PROOF. We take $v_i = c_i$, $d_i = D_i$, $g_i = \gamma_i$, $p_i = c_i/\epsilon$, and

$$b_i = -u + D_i z_i \nabla\Phi, \quad 1 \leq i \leq 2 \tag{148}$$

in the proposition, and thus (I) follows. In order to show (II), it suffices to verify that b_i is uniformly bounded in time for each i. By (I) and (112), we have that $\sup_t \|\nabla\Phi(t)\|_{L^\infty} < \infty$. Thus it only remains to establish a uniform bound on $\|u\|_{L^\infty}$. To this end, we prove below that $\|Au\|_{L^2}$ is uniformly bounded in time, from which the desired result follows due to the embedding $H^2 \hookrightarrow L^\infty$.

Step 1. Uniform $L^\infty_t H^1_x$ bound on u. Applying the Leray projection to (113), we obtain

$$\partial_t u + \nu Au = -K\mathbb{P}(\rho\nabla\Phi). \tag{149}$$

Multiplying (149) by Au and integrating by parts, we obtain, using (I),

$$\frac{1}{2}\frac{d}{dt}\|u\|_V^2 + \frac{\nu}{2}\|Au\|_H^2 \leq C'\|\mathbb{P}(\rho\nabla\Phi)\|_H^2 \leq C, \tag{150}$$

Then using the Stokes regularity estimate

$$\|u\|_V \leq C\|Au\|_H \tag{151}$$

we have

$$\frac{d}{dt}\|u|_V^2 \leq -C\|u|_V^2 + C \tag{152}$$

from which it follows that

$$\sup_t \|u(t)\|_V < \infty. \tag{153}$$

Step 2. Local uniform $L^2_t H^1_x$ bounds on c_i. Multiplying (110) by c_1 and integrating by parts we obtain

$$\frac{1}{2}\frac{d}{dt}\|c_1\|_{L^2}^2 - D_1\int_\Omega c_1\Delta c_1\,dV = D_1\int_\Omega c_1\nabla c_1\cdot\nabla\Phi - \frac{c_1^2\rho}{\epsilon}\,dV \tag{154}$$

and writing $\Delta c_1 = \Delta(c_1 - \Gamma_1)$ where Γ_1 is the unique harmonic function on Ω satisfying $\Gamma_{1|\partial\Omega} = \gamma_1$, we obtain after integrations by parts, Young's inequalities, and the uniform bounds on c_i,

$$\frac{1}{2}\frac{d}{dt}\|c_1\|_{L^2}^2 + \frac{D_1}{2}\|\nabla c_1\|_{L^2}^2 \leq C. \tag{155}$$

Then, integrating in time and again using the uniform bound on c_1, we obtain for all $t \geq 0$ and $\tau > 0$,

$$\int_t^{t+\tau} \|\nabla c_1(s)\|_{L^2}^2 \, ds \leq C(1+\tau) \tag{156}$$

where C is independent of t and τ. Similar estimates for $i = 2$ give us

$$\int_t^{t+\tau} \|\nabla c_i(s)\|_{L^2}^2 \, ds \leq \bar{C}(1+\tau), \quad 1 \leq i \leq 2 \tag{157}$$

with $\bar{C}$ independent of t and τ.

Step 3. Uniform $L_t^\infty H_x^1$ bounds on c_i Multiplying (110) by $-\Delta c_1$, integrating by parts, and using uniform bounds on c_i, we obtain

$$\frac{d}{dt}\|\nabla c_1\|_{L^2}^2 + \|\Delta c_1\|_{L^2}^2 \leq C + C\|\nabla c_1\|_{L^2}^2. \tag{158}$$

Now fix any $t > 1$. By (157), there exists $t_0 \in (\lfloor t \rfloor - 1, \lfloor t \rfloor)$ such that $\|\nabla c_1(t_0)\|_{L^2}^2 \leq 2\bar{C}$ (here $\lfloor t \rfloor$ denotes the largest integer not exceeding t). Then from (158) and (157), we have

$$\begin{aligned} \|\nabla c_1(t)\|_{L^2}^2 \leq& \|\nabla c_1(t_0)\|_{L^2}^2 + C(t-t_0) + C\int_{t_0}^t \|\nabla c_1(s)\|_{L^2}^2 \, ds \\ \leq& 2\bar{C} + 2C + 3C\bar{C} \end{aligned} \tag{159}$$

where the final term does not depend on t. After similar estimates for ∇c_2, we obtain

$$\sup_t \|\nabla c_i(t)\|_{L^2} < \infty, \quad 1 \leq i \leq 2. \tag{160}$$

Step 4. Local uniform $L_t^2 H_x^2$ bounds on c_i Integrating (158) and using (160), we obtain

$$\int_t^{t+\tau} \|\Delta c_1(s)\|_{L^2}^2 \, ds \leq C(1+\tau) \tag{161}$$

for C independent of t and τ. The same method yields the corresponding estimate for $i = 2$, and thus we have for C independent of t and τ

$$\int_t^{t+\tau} \|\Delta c_i(s)\|_{L^2}^2 \, ds \leq C(1+\tau), \quad 1 \leq i \leq 2. \tag{162}$$

Step 5. Local uniform $L_t^2 L_x^2$ bounds on $\partial_t c_i$. Multiplying (110) by $\partial_t c_1$ and integrating by parts, we obtain, using the uniform bounds on u, c_i and ∇c_i,

$$\begin{aligned} \frac{D_1}{2}\frac{d}{dt}\|\nabla c_1\|_{L^2}^2 + \frac{1}{2}\|\partial_t c_1\|_{L^2}^2 &\leq C(\|u\|_V^2\|\nabla c_1\|_{L^3}^2 + \|\nabla c_1\|_{L^2}^2\|\nabla\Phi\|_{L^\infty}^2 + \|c_1\rho\|_{L^2}^2) \\ &\leq C(1 + \|\Delta c_1\|_{L^2}^2) \end{aligned} \tag{163}$$

and integrating in time and using (160), (162), we obtain

$$\int_t^{t+\tau} \|\partial_s c_1(s)\|_{L^2}^2 \, ds \leq C(1+\tau). \tag{164}$$

Similar estimates for $i = 2$ give us

$$\int_t^{t+\tau} \|\partial_s c_i(s)\|_{L^2}^2 \, ds \leq \tilde{C}(1+\tau), \quad 1 \leq i \leq 2 \tag{165}$$

for $\tilde{C}$ independent t and τ.

Step 6. Local uniform $L_t^2 L_x^2$ bounds on $\partial_t u$. Multiplying (149) by $\partial_t u$ and integrating by parts, we have

$$\frac{\nu}{2}\frac{d}{dt}\|u\|_V^2 + \frac{1}{2}\|\partial_t u\|_H^2 \leq \|\rho\nabla\Phi\|_{L^2}^2 \leq C \tag{166}$$

and thus integrating in time, it follows from (153) that

$$\int_t^{t+\tau} \|\partial_s u(s)\|_H^2 \, ds \le C'(1+\tau) \tag{167}$$

where C' is independent of t and τ.

Step 7. Uniform $L_t^\infty L_x^2$ bounds on $\partial_t c_i$ and $\partial_t u$. Differentiating (110) in time, multiplying by $\partial_t c_1$ and integrating by parts, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_t c_1\|_{L^2}^2 + D_1\|\nabla\partial_t c_1\|_{L^2}^2 = &-\int_\Omega (u\cdot\nabla\partial_t c_1)\partial_t c_1 \, dV - \int_\Omega (\partial_t u\cdot\nabla c_1)\partial_t c_1 \, dV \\ &- D_1\int_\Omega \partial_t c_1 \nabla\Phi\cdot\nabla\partial_t c_1 \, dV - D_1\int_\Omega c_1\nabla\partial_t\Phi\cdot\nabla\partial_t c_1 \, dV.\end{aligned} \tag{168}$$

The first integral on the right hand side vanishes because $\operatorname{div} u = 0$. We integrate the second integral by parts once more, and using Young's inequalities and uniform bounds on c_i we obtain

$$\frac{1}{2}\frac{d}{dt}\|\partial_t c_1\|_{L^2}^2 + \frac{D_1}{2}\|\nabla\partial_t c_1\|_{L^2}^2 \le C(\|\partial_t u\|_H^2 + \|\partial_t c_1\|_{L^2}^2 + \|\partial_t c_2\|_{L^2}^2). \tag{169}$$

Similarly for $i = 2$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|\partial_t c_2\|_{L^2}^2 + \frac{D_2}{2}\|\nabla\partial_t c_2\|_{L^2}^2 \le C(\|\partial_t u\|_H^2 + \|\partial_t c_1\|_{L^2}^2 + \|\partial_t c_2\|_{L^2}^2). \tag{170}$$

Next differentiating (149) by time, multiplying by $\partial_t u$ and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\partial_t u\|_H^2 + \frac{\nu}{2}\|\partial_t u\|_V^2 \le C(\|\partial_t\rho\|_{L^2}^2\|\nabla\Phi\|_{L^\infty}^2 + \|\rho\|_{L^\infty}^2\|\nabla\partial_t\Phi\|_{L^2}^2) \le C(\|\partial_t c_1\|_{L^2}^2 + \|\partial_t c_2\|_{L^2}^2). \tag{171}$$

Now adding (169)-(171), we obtain

$$\frac{d}{dt}(\|\partial_t c_1\|_{L^2}^2 + \|\partial_t c_2\|_{L^2}^2 + \|\partial_t u\|_H^2) \le C(\|\partial_t c_1\|_{L^2}^2 + \|\partial_t c_2\|_{L^2}^2 + \|\partial_t u\|_H^2). \tag{172}$$

Finally, using the same method as in Step 3, we use (165), (167) and (172) to obtain

$$\sup_t(\|\partial_t c_1(t)\|_{L^2} + \|\partial_t c_2(t)\|_{L^2} + \|\partial_t u(t)\|_H) < \infty. \tag{173}$$

Step 8. Uniform $L_t^\infty H_x^2$ bounds on u. From (149), we have

$$\|Au\|_H \le C(\|\partial_t u\|_H + \|\rho\nabla\Phi\|_{L^2}) \tag{174}$$

and it follows from the preceding estimates that

$$\sup_t \|Au(t)\|_H < \infty. \tag{175}$$

With this bound, the proof of the uniform boundededness of $b_i = -u + D_i z_i \nabla\Phi$ is complete, and thus (II) of the theorem follows from (II') of Proposition 2. □

REMARK 5. *In the proof of Theorem 3 only the following properties of the velocity u are used. For part (I), we only make use of the fact that $x \mapsto u(t,x)$ is a smooth vector field for each t, without need of quantitative information, and for part (II), we additionally make use of the fact that $\|u(t)\|_{L^\infty}$ is uniformly bounded in time. These conditions must be satisfied in order to apply Proposition 2. Then, it follows from Remark 4 that Theorem 3 holds also for the two dimensional NPNS system. Indeed, in two dimensions, the nonlinear term $u\cdot\nabla u$ does not change the properties that u remains smooth and $\|u(t)\|_{L^\infty}$ remains uniformly bounded for all time. The proof of these statements follow from similar steps as in Steps 1-8 in the preceding proof of Theorem 3, taking into account the nonlinear term $u\cdot\nabla u$ and using Gagliardo-Nirenberg interpolation inequalities that hold specifically in two dimensions. Further details are omitted. The preceding comments also apply to the three dimensional NPNS system, but due to the lack of control of u, the smoothness of the vector field and the boundedness of its maximum modulus must be assumed, as they are not known to hold a priori .*

4. Global Stability of Weak Steady Currents

In this section we consider the long time behavior of solutions to the time dependent Nernst-Planck-Stokes (NPS) system 110)-(114). In this section we take the domain to be the three dimensional periodic strip $\Omega = (0,L) \times \mathbb{T} \times \mathbb{T}$ where $\mathbb{T}$ has period 1, and boundary conditions

$$c_i(t,0,y,z) = \alpha_i, \quad c_i(t,L,y,z) = \beta_i, \quad 1 \le i \le 2 \tag{176}$$

$$\Phi(t,0,y,z) = -V, \quad \Phi(t,L,y,z) = 0 \tag{177}$$

$$u(t,0,y,z) = u(t,L,y,z) = 0. \tag{178}$$

Here, we take $\alpha_i, \beta_i, V > 0$ to be *constants*.

In the first subsection of this section, we analyze one dimensional solutions to NPS with boundary conditions (176)-(178) and establish uniform bounds. In the second subsection, we show that *weak current* one dimensional solutions are globally stable. This latter result yields as a corollary the uniqueness of steady state solutions in the setting of small perturbations from equilibrium.

4.1. One Dimensional Steady States. We consider the one dimensional steady state Nernst-Planck system for $x \in (0,L)$

$$0 = \partial_x(\partial_x c_1^* + c_1^* \partial_x \Phi^*) \tag{179}$$

$$0 = \partial_x(\partial_x c_2^* - c_2^* \partial_x \Phi^*) \tag{180}$$

$$-\epsilon \partial_{xx}\Phi^* = \rho^* = c_1^* - c_2^* \tag{181}$$

with boundary conditions corresponding to (176), (177)

$$c_i^*(0) = \alpha_i > 0, \quad c_i^*(L) = \beta_i > 0, \quad 1 \le i \le 2 \tag{182}$$

$$\Phi^*(0) = -V < 0, \quad \Phi^*(L) = 0. \tag{183}$$

As we will see in Section 4.2, one dimensional Nernst-Planck steady states, with zero fluid flow $u^* \equiv 0$, are also steady state solutions to the full three dimensional NPNS system in our current setting of a three dimensional periodic strip with boundary conditions (176)-(178). This is the motivation for the study of these one dimensional solutions.

While the computations of the previous section could be significantly simplified for this one dimensional, no fluid setting, nonetheless the existence of a smooth solution to (179)-(183), with $c_i^* \ge 0$, follows from a streamlined version of the proof of Theorem 1 (see also [**18**]).

In this subsection, we establish uniform bounds on c_i^* and Φ^* that depend exclusively on boundary data. To this end, we recall the electrochemical potentials

$$\mu_i^* = \log c_i^* + z_i \Phi^*, \quad 1 \le i \le 2 \tag{184}$$

and the related variables (a.k.a. Slotboom variables in the semiconductor literature)

$$\eta_i^* = \exp \mu_i^* = c_i^* e^{z_i \Phi^*}, \quad 1 \le i \le 2 \tag{185}$$

where $z_1 = 1 = -z_2$. We refer the reader to [**19**] for a more complete study on the one dimensional steady state Nernst-Planck system.

PROPOSITION 3. *Suppose (c_1^*, c_2^*, Φ^*) is a smooth solution to (179)-(183). Then, the solution satisfies the following uniform bounds:*

(I) $\min\{\alpha_i e^{-z_i V}, \beta_i\} = \lambda_i \le \eta_i^* \le \Lambda_i = \max\{\alpha_i e^{-z_i V}, \beta_i\}$

(II) $\min\{-V, \log(\lambda_1/\Lambda_2)^{\frac{1}{2}}\} = -v \le \Phi^* \le \mathcal{V} = \max\{0, \log(\Lambda_1/\lambda_2)^{\frac{1}{2}}\}$

(III) $\min\{\alpha_1, \alpha_2, \beta_1, \beta_2\} = \underline{\gamma} \le c_i^* \le \overline{\gamma} = \max\{\alpha_1, \alpha_2, \beta_1, \beta_2\}$ *for* $1 \le i \le 2$.

PROOF. Writing (179), (180) in terms of η_i^* we have

$$0 = \partial_x(e^{-z_i\Phi^*}\partial_x \eta_i^*) = e^{-z_i\Phi^*}(-z_i \partial_x \Phi^* \partial_x \eta_i^* + \partial_{xx}\eta_i^*), \quad 1 \le i \le 2. \tag{186}$$

And thus (I) follows from the weak maximum principle and the fact that

$$\eta_i^*(0) = \alpha_i e^{-z_i V}, \quad \eta_i^*(L) = \beta_i \quad 1 \le i \le 2. \tag{187}$$

To show (II), we rewrite (181) as

$$-\epsilon \partial_{xx}\Phi^* = \eta_1^* e^{-\Phi^*} - \eta_2^* e^{\Phi^*}. \tag{188}$$

So if Φ^* attains its global maximum at an interior point $x_0 \in (0, L)$, then

$$\begin{aligned} 0 \le -\epsilon\partial_{xx}\Phi^*(x_0) &= \eta_1^*(x_0)e^{-\Phi^*(x_0)} - \eta_2^*(x_0)e^{\Phi^*(x_0)} \\ \Rightarrow \Phi^*(x_0) &\le \log\left(\frac{\eta_1^*(x_0)}{\eta_2^*(x_0)}\right)^{\frac{1}{2}} \le \log(\Lambda_1/\lambda_2)^{\frac{1}{2}} \end{aligned} \tag{189}$$

and the upper bound in (II) follows. Similarly, if Φ^* attains its global minimum at an interior point $x_0 \in (0, L)$, then

$$\begin{aligned} 0 &\ge \eta_1^*(x_0)e^{-\Phi^*(x_0)} - \eta_2^*(x_0)e^{\Phi^*(x_0)} \\ \Rightarrow -\Phi^*(x_0) &\le \log\left(\frac{\eta_2^*(x_0)}{\eta_1^*(x_0)}\right)^{\frac{1}{2}} \le \log(\Lambda_2/\lambda_1)^{\frac{1}{2}} \end{aligned} \tag{190}$$

and the lower bound in (II) follows.

Lastly, prior to proving (III), we note that by combining (I) and (II) and using the definition of η_i^*, it is possible to obtain upper and lower bounds on c_i^* that depend on boundary data for c_i and Φ^*. Here, instead we establish the bounds in (III), which in particular does not depend on boundary data for Φ^*. We prove only the upper bound as the lower bound can be shown analogously. To do so, we introduce the rescaling $X = x/\epsilon^{\frac{1}{2}}$ so that we can rewrite (179), (180) as

$$-\partial_{XX}c_1^* = \partial_X c_1^* \partial_X \Phi^* - c_1^*(c_1^* - c_2^*) \tag{191}$$

$$-\partial_{XX}c_2^* = -\partial_X c_2^* \partial_X \Phi^* + c_2^*(c_1^* - c_2^*). \tag{192}$$

Suppose that $\max\{c_1^*, c_2^*\}$ attains a global maximal value, $c > \overline{\gamma}$, at an interior point $X_0 \in (0, L/\epsilon^{\frac{1}{2}})$. Assume without loss of generality that this maximum is attained by c_1^*. Then we have

$$\begin{aligned} 0 \le -\partial_{XX}c_1^*(X_0) &= \partial_X c_1^*(X_0)\partial_X\Phi^*(x_0) - c_1^*(X_0)(c_1^*(X_0) - c_2^*(X_0)) \\ &= -c(c - c_2^*(X_0)). \end{aligned} \tag{193}$$

Then since by assumption we have $c \ge c_2^*(X_0)$, we necessarily have that $c = c_2^*(X_0)$, for otherwise, the right hand side of (193) becomes strictly negative. Furthermore, the inequality in (193) is an equality, and we conclude that $\partial_{XX}c_1^*(X_0) = 0$. And since we have shown that c_2^* also attains its global maximum at X_0, by evaluating (192) at X_0 we conclude that $\partial_X c_2^*(X_0) = \partial_{XX}c_2^*(X_0) = 0$.

It follows by induction that for $i = 1, 2$ we have

$$\partial_X^k c_i^*(X_0) = 0 \quad \text{for all } k \ge 1. \tag{194}$$

Indeed, assume, for the sake of induction, that this is true for all $1 \le k' \le k$ where $k \ge 2$. Then differentiating (191),

$$-\partial_X^{k+1}c_1^* = \sum_{j=0}^{k-1}\binom{k-1}{j}\left(\partial_X^{j+1}c_1^*\partial_X^{k-j}\Phi^* - \partial_X^j c_1^*\partial_X^{k-1-j}(c_1^* - c_2^*)\right) \tag{195}$$

and evaluating (195) at X_0 and using the induction hypothesis together with the fact that $c_1^*(X_0) = c_2^*(X_0)$, we conclude $\partial_X^{k+1}c_1^*(X_0) = 0$ as desired. Similarly by differentiating (192) we obtain $\partial_X^{k+1}c_2^*(X_0) = 0$.

Now *if* c_1^*, c_2^* are real analytic, then (194) implies that in fact $c_1^* \equiv c_2^* \equiv c$, but we assumed that $c > \overline{\gamma}$, so this contradiction implies the upper bound $\max\{c_1^*, c_2^*\} \le \overline{\gamma}$.

So to complete the proof of (III) it suffices to establish the real analyticity of c_1^*, c_2^*. To this end, we prove that there exists $C > 0$ such that for all integers $k \geq 0$

$$\begin{aligned} A^{-1} &= \sup_X |\partial_X \Phi^*| \leq \frac{1}{4} C \\ A^k &= \sup_X |\partial_X^k c_1^*| + \sup_X |\partial_X^k c_2^*| \leq \frac{1}{4}(k+1)! C^{k+2}. \end{aligned} \tag{196}$$

We choose C so that (196) holds for A^k, $k = -1, 0, 1$. Now we prove the implication

$$(196) \text{ holds for all } k' \text{ less than or equal to } k \geq 1 \Rightarrow (196) \text{ holds for } k+1. \tag{197}$$

To show this, we see from (195) that

$$\begin{aligned} |\partial_X^{k+1} c_i^*| \leq & \sum_{j=0}^{k-1} \binom{k-1}{j} (A^{j+1} A^{k-j-2} + A^j A^{k-1-j}) \\ \leq & \frac{1}{16} \sum_{j=0}^{k-1} \frac{(k-1)!}{j!(k-1-j)!} ((j+2)!(k-j-1)! C^{k+3} + (j+1)!(k-j)! C^{k+3}) \\ = & \frac{C^{k+3}}{16} (k-1)! \sum_{j=0}^{k-1} ((j+2)(j+1) + (j+1)(k-j)) \\ \leq & \frac{C^{k+3}}{16} (k-1)! k((k+1)k + k^2) \\ \leq & \frac{1}{8} (k+2)! C^{k+3} \end{aligned} \tag{198}$$

and summing in i we obtain

$$A^{k+1} \leq \frac{1}{4}(k+2)! C^{k+3} \tag{199}$$

as desired. Thus c_i^* is real analytic and the proof of the upper bound in (III) and of the proposition is complete. ☐

REMARK 6. *Proposition 3 (I), (II) directly extend to higher dimensional settings, for arbitrary bounded, connected domains with sufficiently smooth boundary, when no fluid is involved (see also* **[18]** *for further generalizations). (III) also extends to higher dimensions, where now,* $\underline{\gamma} = \min_i \inf_{\partial\Omega} c_i, \overline{\gamma} = \max_i \sup_{\partial\Omega} c_i$. *The proof requires appealing to the nonstationary Nernst-Planck system. Indeed, we note that by Theorem 3, for all* $\delta > 0$, *the set* $B_\delta = \{f \in L^\infty \,|\, \underline{\gamma} - \delta \leq f \leq \overline{\gamma} + \delta\}$ *is an absorbing ball for each* c_i *in the sense that, for any initial conditions, solutions to NPNS with boundary conditions 8 satisfy* $c_i(t) \in B_\delta$ *for both* i, *for all large enough times* t. *In particular, any stationary solution to NPNS with boundary conditions* (8) *must lie in* B_δ, *and since this is true for any* $\delta > 0$, *stationary solutions in fact lie in* B_0. *The same reasoning (i.e. using the corresponding version of Theorem 3 for the Nernst-Planck system) and the same conclusion hold for stationary solutions of Nernst-Planck equations.*

4.2. Global Stability. In this last subsection, we study the problem of stability of one dimensional steady currents on the domain $\Omega = (0, L) \times \mathbb{T}^2$.

DEFINITION 1. *We say that* $(c_1^*, c_2^*, u^* \equiv 0)$ *is a* ***(one dimensional) steady current*** *solution to (110)-(114) with boundary conditions (176)-(178) on the three dimensional periodic strip* $\Omega = (0, L) \times \mathbb{T}^2$ *if* c_i^* *is independent of the spatial variables* y *and* z, *independent of time, and solves the one dimensional problem (179)-(183).*

REMARK 7. *A solution* $c_i^*(x)$ *to the one dimensional system (179)-(183), seen as a three dimensional function on* Ω, *independent of* y, z, *together with* $u^* \equiv 0$, *is indeed a solution to the three dimensional steady* *state NPNS system because*

$$\begin{aligned}\partial_x c_1^* + c_1^* \partial_x \Phi^* &= j_1 \\ \partial_x c_2^* - c_2^* \partial_x \Phi^* &= j_2\end{aligned} \tag{200}$$

for constants j_i. It follows from this that

$$\rho^* \nabla \Phi^* = -\nabla(c_1^* + c_2^*) + (j_1 + j_2, 0, 0) = -\nabla(c_1^* + c_2^* - (j_1 + j_2)x), \tag{201}$$

and thus $u^*(x, y, z) \equiv 0$ and $p^*(x, y, z) = K(c_1^*(x) + c_2^*(x) - (j_1 + j_2)x)$ solve the Stokes equations (113)-(114).

In general, it is not known whether solutions to the one dimensional system (179)-(183) are unique, and therefore it is also unknown whether any given one dimensional steady current solution $(c_1^*, c_2^*, u^* \equiv 0)$ to the three dimensional NPNS system is unique (in general one cannot rule out the existence of other one dimensional steady current solutions nor the existence of solutions that depend also on y and/or z).

For the remainder of this subsection, we study the stability of a *fixed* one dimensional steady current solution. However, using the a priori estimates of Section 4.1 and under a *weak current* or *small perturbation from equilibrium* assumption, c.f. (216), we obtain the global stability of the fixed steady current solution (Theorem 4). As a consequence of stability it follows that the fixed steady current solution is the unique steady state solution of the full three dimensional system (110)-(114) with boundary conditions (176)-(178) (Theorem 5).

The main tool in proving global stability is the following log-Sobolev type inequality, which is also used in [**14**] in an equilibrium setting.

LEMMA 5. *Suppose f_i, g_i, $i = 1, 2$ and p^f, p^g are smooth real valued functions on a bounded domain $\Omega \subset \mathbb{R}^3$ satisfying the bounds*

$$0 < f_1, f_2 \le M_f, \quad 0 < g_1, g_2 \le M_g. \tag{202}$$

and the relations

$$\begin{aligned} f_{1|\partial\Omega} &= g_{1|\partial\Omega} \\ f_{2|\partial\Omega} &= g_{2|\partial\Omega} \\ p^f{}_{|\partial\Omega} &= p^g{}_{|\partial\Omega} \\ -\epsilon \Delta p^f &= f_1 - f_2 \\ -\epsilon \Delta p^g &= g_1 - g_2, \quad 1 \le i \le 2. \end{aligned} \tag{203}$$

Then the functions

$$\pi_i^f = \log f_i + z_i p^f, \quad \pi_i^g = \log g_i + z_i p^g, \quad 1 \le i \le 2 \tag{204}$$

where $z_1 = 1 = -z_2$, satisfy the bound

$$\frac{\omega}{l^2}\left(\sum_{i=1}^{2} \frac{1}{2}\int_\Omega g_i \psi\left(\frac{f_i}{g_i}\right) dV + \epsilon \int_\Omega |\nabla(p^f - p^g)|^2 \, dV\right) \le \sum_{i=1}^{2} \int_\Omega |\nabla(\pi_i^f - \pi_i^g)|^2 \, dV \tag{205}$$

where

$$\psi(s) = s \log s - s + 1, \quad s > 0 \tag{206}$$

$$\omega = \frac{2}{\max\{M_f, M_g\}} \tag{207}$$

and l can be chosen to be the height of any infinite slab in $\mathbb{R}^3$ that contains Ω (i.e. $\Omega \subset \{x_0 + s_1 e_1 + s_2 e_2 + s_3 e_3 \,|\, s_1 \in (0, l),\ s_2, s_3 \in (-\infty, \infty)\}$ for some $x_0 \in \mathbb{R}^3$ and orthonormal basis $\{e_i\}$ of $\mathbb{R}^3$).

Prior to proving Lemma 5, we first establish an interpolation inequality that interpolates L^2 between $L \log L$ and L^∞.

LEMMA 6. *For positive, real valued, bounded, measurable functions f, g defined on Ω, we have*

$$\int_\Omega (f-g)^2\,dV \le \max\{\|f\|_{L^\infty}, \|g\|_{L^\infty}\} \int_\Omega g\psi\left(\frac{f}{g}\right)\,dV \tag{208}$$

with ψ defined in (206).

PROOF. Taylor expanding $\psi(s)$ around $s=1$, we have

$$\psi(s) \ge \min\{1, s^{-1}\}(s-1)^2 \Rightarrow (s-1)^2 \le \max\{1,s\}\psi(s), \tag{209}$$

so taking $s = f/g$ we have

$$\begin{aligned} &\left(\frac{f}{g}-1\right)^2 \le \max\left\{1, \frac{f}{g}\right\}\psi\left(\frac{f}{g}\right) \\ \Rightarrow &(f-g)^2 \le \max\{f,g\}g\psi\left(\frac{f}{g}\right) \end{aligned} \tag{210}$$

and thus the lemma follows after integrating over Ω. □

Now we prove Lemma 5.

PROOF. We consider the following expression

$$\sum_{i=1}^{2}\left\langle f_i - g_i, \pi_i^f - \pi_i^g\right\rangle. \tag{211}$$

On one hand we have, using the Poisson equation (203),

$$\begin{aligned} \sum_i \left\langle f_i - g_i, \pi_i^f - \pi_i^g\right\rangle &= \sum_i \left\langle f_i - g_i, \log\frac{f_i}{g_i} + z_i(p^f - p^g)\right\rangle \\ &= \sum_i \int_\Omega g_i\left(\frac{f_i}{g_i}-1\right)\log\frac{f_i}{g_i}\,dV + \epsilon\int_\Omega |\nabla(p^f-p^g)|^2\,dV \\ &\ge \sum_i \int_\Omega g_i\psi\left(\frac{f_i}{g_i}\right)\,dV + \epsilon\int_\Omega |\nabla(p^f-p^g)|^2\,dV \end{aligned} \tag{212}$$

where in the last inequality we used the inequality

$$(s-1)\log s \ge \psi(s), \quad s>0. \tag{213}$$

On the other hand, we have due to Young's inequality, Poincaré's inequality, and Lemma 6,

$$\begin{aligned} \sum_i \left\langle f_i - g_i, \pi_i^f - \pi_i^g\right\rangle &\le \sum_i \frac{\omega}{4}\|f_i - g_i\|_{L^2}^2 + \sum_i \frac{1}{\omega}\|\pi_i^f - \pi_i^g\|_{L^2}^2 \\ &\le \sum_i \frac{\omega\max\{M_f, M_g\}}{4}\int_\Omega g_i\psi\left(\frac{f_i}{g_i}\right)\,dV + \sum_i \frac{l^2}{\omega}\|\nabla(\pi_i^f - \pi_i^g)\|_{L^2}^2. \end{aligned} \tag{214}$$

Therefore, choosing $\omega > 0$ as in (207), we have, combining (212) and (214),

$$\frac{\omega}{l^2}\left(\sum_i \frac{1}{2}\int_\Omega g_i\psi\left(\frac{f_i}{g_i}\right)\,dV + \epsilon\int_\Omega |\nabla(p^f-p^g)|^2\,dV\right) \le \sum_i \int_\Omega |\nabla(\pi_i^f - \pi_i^g)|^2\,dV. \tag{215}$$

□

We now state the global stability theorem.

THEOREM 4. *Suppose $(c_1^*, c_2^*, u^* \equiv 0)$ is a one dimensional steady current solution to the NPS system (110)-(114) with boundary conditions (176)-(178). Suppose furthermore that the corresponding **p-current** j_1 and **n-current** j_2 (c.f. (200)) satisfy*

$$\max_i |j_i| L G_i < \frac{1}{\sqrt{2}} \tag{216}$$

where

$$G_i = \sqrt{\frac{1}{D}\left(\frac{D_i \overline{\gamma}^2}{2\underline{\gamma}^4} + \frac{K L^2 \overline{\gamma}^2}{\nu \underline{\gamma}^3}\right)} \tag{217}$$

and $D = \min_i D_i$. Then $(c_1^*, c_2^*, u^* \equiv 0)$ is globally asymptotically stable. That is, for any smooth initial conditions $c_1(0) \geq 0, c_2(0) \geq 0, u(0)$ (with $\operatorname{div} u(0) = 0$), the corresponding solution (c_1, c_2, u) to (110)-(114) satisfies

$$\sum_{i=1}^{2} \int_\Omega c_i^* \psi\left(\frac{c_i(t)}{c_i^*}\right) dV + \int_\Omega |\nabla(\Phi - \Phi^*)|^2 \, dV + \int_\Omega |u(t)|^2 \, dV \to 0 \quad \text{as } t \to \infty. \tag{218}$$

Furthermore, there exists $T^* > 0$, depending on initial and boundary data and the parameters of the system, such that after time $t = T^*$, the rate of convergence in (218) is exponential in time.

A consequence of Theorem 4 is the following uniqueness theorem.

THEOREM 5. *Under the same hypotheses as in Theorem 4, the one dimensional steady current solution $(c_1^*, c_2^*, u^* \equiv 0)$ is the unique steady state solution to the NPS system (110)-(114) with boundary conditions (176)-(178).*

REMARK 8. *We note that the currents j_i are solution dependent constants and the condition* (216) *is not explicitly written solely in terms of the boundary data. Writing (200) in terms of the electrochemical potentials and the Slotboom variables (c.f. (184), (185)) we have*

$$c_i^* \partial_x \mu_i^* = j_i, \quad e^{-z_i \Phi^*} \partial_x \eta_i^* = j_i, \quad 1 \leq i \leq 2 \tag{219}$$

and thus

$$j_i = \frac{\mu_i^*(L) - \mu_i^*(0)}{\int_0^L \frac{1}{c_i^*} \, dx} = \frac{\eta_i^*(L) - \eta_i^*(0)}{\int_0^L e^{z_i \Phi^*} \, dx}, \quad 1 \leq i \leq 2. \tag{220}$$

Then, using the uniform bounds from Proposition 3, we see that explicit sufficient conditions in terms of the boundary data which imply the smallness conditions (216) are given by

$$\begin{cases} \left|\log \frac{\alpha_1}{\beta_1} + V\right| \overline{\gamma} G_1 < \frac{1}{\sqrt{2}} \\ \left|\log \frac{\alpha_2}{\beta_2} - V\right| \overline{\gamma} G_2 < \frac{1}{\sqrt{2}} \end{cases} \tag{221}$$

or

$$\begin{cases} \left|\alpha_1 - \beta_1 e^{-V}\right| e^{v} G_1 < \frac{1}{\sqrt{2}} \\ \left|\alpha_2 - \beta_2 e^{V}\right| e^{\mathcal{V}} G_2 < \frac{1}{\sqrt{2}} \end{cases} \tag{222}$$

where $v, \mathcal{V}$ are defined in Proposition 3.

Now we prove Theorem 4.

PROOF. First suppose that the initial conditions satisfy

$$0 < \underline{\gamma}_\delta \leq c_i(0) \leq \overline{\gamma}_\delta, \quad 1 \leq i \leq 2 \tag{223}$$

where

$$\underline{\gamma}_\delta = \underline{\gamma} - \delta, \quad \overline{\gamma}_\delta = \overline{\gamma} + \delta \tag{224}$$

for some small $\delta > 0$, to be determined below (c.f. (240)). Then by Proposition 3 and Theorem 3 (I), the time dependent solution (c_1, c_2, u) and the one dimensional steady current $(c_1^*, c_2^*, u^* \equiv 0)$ satisfy the bounds

$$\underline{\gamma}_\delta \le c_i(t) \le \overline{\gamma}_\delta, \quad \underline{\gamma} \le c_i^* \le \overline{\gamma}, \quad 1 \le i \le 2. \tag{225}$$

Next, writing (110), (111) in terms of the electrochemical potentials

$$\mu_i = \log c_i + z_i \Phi, \quad 1 \le i \le 2 \tag{226}$$

and in terms of the differences $c_i - c_i^*$, $\mu_i - \mu_i^*$, we have

$$\begin{aligned} \partial_t(c_1 - c_1^*) &= -u \cdot \nabla c_1 + D_1 \mathrm{div}\,(c_1 \nabla(\mu_1 - \mu_1^*) + (c_1 - c_1^*)\nabla \mu_1^*) \\ \partial_t(c_2 - c_2^*) &= -u \cdot \nabla c_2 + D_2 \mathrm{div}\,(c_2 \nabla(\mu_2 - \mu_2^*) + (c_2 - c_2^*)\nabla \mu_2^*). \end{aligned} \tag{227}$$

We multiply the above equations by $\mu_1 - \mu_1^*$ and $\mu_2 - \mu_2^*$, respectively, and integrate by parts. On the left hand side, we obtain, after summing in i,

$$\begin{aligned} \sum_i \langle \partial_t(c_i - c_i^*), \mu_i - \mu_i^* \rangle &= \sum_i \left(\frac{d}{dt} \int_\Omega c_i^* \psi\left(\frac{c_i}{c_i^*}\right) dV + z_i \langle \partial_t(c_i - c_i^*), \Phi - \Phi^* \rangle \right) \\ &= \sum_i \frac{d}{dt} \int_\Omega c_i^* \psi\left(\frac{c_i}{c_i^*}\right) dV + \langle \partial_t(\rho - \rho^*), \Phi - \Phi^* \rangle \\ &= \sum_i \frac{d}{dt} \int_\Omega c_i^* \psi\left(\frac{c_i}{c_i^*}\right) dV + \frac{\epsilon}{2}\frac{d}{dt} \|\nabla(\Phi - \Phi^*)\|_{L^2}^2. \end{aligned} \tag{228}$$

On the right hand side, for $i = 1$, we have, using Lemma 6, (225), and (219),

$$\begin{aligned} &\langle -u \cdot \nabla c_1 + D_1 \mathrm{div}\,(c_1 \nabla(\mu_1 - \mu_1^*) + (c_1 - c_1^*)\nabla \mu_1^*), \mu_1 - \mu_1^* \rangle \\ &= -\langle u \cdot \nabla c_1, \mu_1 - \mu_1^* \rangle - D_1 \int_\Omega c_1 |\nabla(\mu_1 - \mu_1^*)|^2 \, dV - D_1 \langle (c_1 - c_1^*)\nabla \mu_1^*, \nabla(\mu_1 - \mu_1^*) \rangle \\ &\le -\frac{D_1}{2} \int_\Omega c_1 |\nabla(\mu_1 - \mu_1^*)|^2 \, dV - \langle u \cdot \nabla c_1, \mu_1 - \mu_1^* \rangle + \frac{D_1}{2} \int_\Omega \frac{(c_1 - c_1^*)^2}{c_1} (\partial_x \mu_1^*)^2 \, dV \\ &\le -\frac{D_1}{2} \int_\Omega c_1 |\nabla(\mu_1 - \mu_1^*)|^2 \, dV - \langle u \cdot \nabla c_1, \mu_1 - \mu_1^* \rangle + \frac{D_1 \overline{\gamma}_\delta}{2 \underline{\gamma}_\delta \underline{\gamma}^2} j_1^2 \int_\Omega c_1^* \psi\left(\frac{c_1}{c_1^*}\right) dV. \end{aligned} \tag{229}$$

We take a closer look at the term involving u, integrating by parts and using $\operatorname{div} u = 0$,

$$
\begin{aligned}
-\langle u \cdot \nabla c_1, \mu_1 - \mu_1^* \rangle &= -\langle u \cdot \nabla c_1, \log c_1 + \Phi \rangle + \langle u \cdot \nabla c_1, \mu_1^* \rangle \\
&= -\langle u, \nabla (c_1 \log c_1 - c_1) \rangle + \int_\Omega u c_1 \cdot \nabla \Phi \, dV \\
&\quad + \langle u \cdot \nabla (c_1 - c_1^*), \mu_1^* \rangle + \langle u \cdot \nabla c_1^*, \mu_1^* \rangle \\
&= \int_\Omega u c_1 \cdot \nabla \Phi \, dV - \langle u (c_1 - c_1^*), \nabla \mu_1^* \rangle \\
&\quad + \langle u, \nabla (c_1^* \log c_1^* - c_1^*) \rangle - \int_\Omega u c_1^* \cdot \nabla \Phi^* \, dV \\
&\le \int_\Omega u c_1 \cdot \nabla \Phi \, dV - \int_\Omega u c_1^* \cdot \nabla \Phi^* \, dV \\
&\quad + \frac{\nu}{4KL^2} \|u\|_H^2 + \frac{KL^2}{\nu} \int_\Omega |c_1^* \partial_x \mu_1^*|^2 \frac{(c_1 - c_1^*)^2}{(c_1^*)^2} \, dV \\
&\le \int_\Omega u c_1 \cdot \nabla \Phi \, dV - \int_\Omega u c_1^* \cdot \nabla \Phi^* \, dV \\
&\quad + \frac{\nu}{4K} \|u\|_V^2 + \frac{KL^2 \overline{\gamma}_\delta}{\nu \underline{\gamma}^2} j_1^2 \int_\Omega c_1^* \psi \left(\frac{c_1}{c_1^*} \right) \, dV
\end{aligned}
\tag{230}
$$

and thus returning to (229), we have

$$
\begin{aligned}
&\langle -u \cdot \nabla c_1 + D_1 \operatorname{div} (c_1 \nabla (\mu_1 - \mu_1^*) + (c_1 - c_1^*) \nabla \mu_1^*), \mu_1 - \mu_1^* \rangle \\
\le &- \frac{D_1}{2} \int_\Omega c_1 |\nabla (\mu_1 - \mu_1^*)|^2 \, dV + \left(\frac{D_1 \overline{\gamma}_\delta}{2 \underline{\gamma}_\delta \underline{\gamma}^2} + \frac{KL^2 \overline{\gamma}_\delta}{\nu \underline{\gamma}^2} \right) j_1^2 \int_\Omega c_1^* \psi \left(\frac{c_1}{c_1^*} \right) \, dV \\
&+ \frac{\nu}{4K} \|u\|_V^2 + \int_\Omega u c_1 \cdot \nabla \Phi \, dV - \int_\Omega u c_1^* \cdot \nabla \Phi^* \, dV.
\end{aligned}
\tag{231}
$$

Similarly for $i = 2$ we obtain

$$
\begin{aligned}
&\langle -u \cdot \nabla c_2 + D_2 \operatorname{div} (c_2 \nabla (\mu_2 - \mu_2^*) + (c_2 - c_2^*) \nabla \mu_2^*), \mu_2 - \mu_2^* \rangle \\
\le &- \frac{D_2}{2} \int_\Omega c_2 |\nabla (\mu_2 - \mu_2^*)|^2 \, dV + \left(\frac{D_2 \overline{\gamma}_\delta}{2 \underline{\gamma}_\delta \underline{\gamma}^2} + \frac{KL^2 \overline{\gamma}_\delta}{\nu \underline{\gamma}^2} \right) j_2^2 \int_\Omega c_2^* \psi \left(\frac{c_2}{c_2^*} \right) \, dV \\
&+ \frac{\nu}{4K} \|u\|_V^2 - \int_\Omega u c_2 \cdot \nabla \Phi \, dV + \int_\Omega u c_2^* \cdot \nabla \Phi^* \, dV.
\end{aligned}
\tag{232}
$$

Collecting our estimates thus far and using the fact that $\rho^* \nabla \Phi^* = -\nabla (c_1^* + c_2^* - (j_1 + j_2) x)$ is a gradient, we have

$$
\begin{aligned}
&\frac{d}{dt} \mathcal{E} + \sum_i \frac{D_i}{2} \int_\Omega c_i |\nabla (\mu_i - \mu_i^*)|^2 \, dV \\
\le &\sum_i M_i^\delta j_i^2 \int_\Omega c_i^* \psi \left(\frac{c_i}{c_i^*} \right) \, dV + \int_\Omega u \rho \cdot \nabla \Phi \, dV - \int_\Omega u \rho^* \cdot \nabla \Phi^* \, dV + \frac{\nu}{2K} \|u\|_V^2 \\
= &\sum_i M_i^\delta j_i^2 \int_\Omega c_i^* \psi \left(\frac{c_i}{c_i^*} \right) \, dV + \int_\Omega u \rho \cdot \nabla \Phi \, dV + \frac{\nu}{2K} \|u\|_V^2
\end{aligned}
\tag{233}
$$

where (see [**6**])

$$\mathcal{E} = \sum_i \int_\Omega c_i^* \psi\left(\frac{c_i}{c_i^*}\right) dV + \frac{\epsilon}{2}\|\nabla(\Phi - \Phi^*)\|_{L^2}^2$$
$$M_i^\delta = \frac{D_i \overline{\gamma}_\delta}{2\underline{\gamma}_\delta \underline{\gamma}^2} + \frac{KL^2\overline{\gamma}_\delta}{\nu\underline{\gamma}^2}, \quad 1 \le i \le 2. \tag{234}$$

Now we take a look at the Stokes equations,

$$\partial_t u + \nu A u = -K\mathbb{P}(\rho\nabla\Phi). \tag{235}$$

Multiplying by $\frac{u}{K}$ and integrating by parts, we obtain using the self-adjointness of $\mathbb{P}$,

$$\frac{1}{2K}\frac{d}{dt}\|u\|_H^2 + \frac{\nu}{K}\|u\|_V^2 = -\int_\Omega u\cdot\mathbb{P}(\rho\nabla\Phi)\,dV = -\int_\Omega u\rho\cdot\nabla\Phi\,dV. \tag{236}$$

Now we combine the estimates (233) and (236) to obtain

$$\frac{d}{dt}\left(\mathcal{E} + \frac{1}{2K}\|u\|_H^2\right) + \sum_i \frac{D_i}{2}\int_\Omega c_i|\nabla(\mu_i - \mu_i^*)|^2\,dV + \frac{\nu}{2K}\|u\|_V^2 \le \sum_i M_i^\delta j_i^2 \int_\Omega c_i^*\psi\left(\frac{c_i}{c_i^*}\right) dV. \tag{237}$$

Now applying Lemma 5 to the dissipation term

$$\mathcal{D} = \sum_i \frac{D_i}{2}\int_\Omega c_i|\nabla(\mu_i - \mu_i^*)|^2\,dV \tag{238}$$

we obtain

$$\mathcal{D} \ge \frac{D\underline{\gamma}_\delta}{\overline{\gamma}_\delta L^2}\left(\sum_i \frac{1}{2}\int_\Omega c_i^*\psi\left(\frac{c_i}{c_i^*}\right) dV + \epsilon\|\nabla(\Phi - \Phi^*)\|_{L^2}^2\right) \ge \frac{D\underline{\gamma}_\delta}{2\overline{\gamma}_\delta L^2}\mathcal{E} \tag{239}$$

where $D = \min_i D_i$. Next, defining

$$\kappa_i^\delta = \frac{D\underline{\gamma}_\delta}{2\overline{\gamma}_\delta L^2} - M_i^\delta j_i^2, \quad 1 \le i \le 2 \tag{240}$$

we have, due to (216), $\kappa_i^\delta > 0$ for each i for small enough $\delta > 0$. Therefore, after an application of Poincaré's inequality to $\|u\|_V^2$ in (237), we obtain, for small enough δ,

$$\frac{d}{dt}\mathcal{F} \le -\kappa^\delta \mathcal{F} \tag{241}$$

for

$$\mathcal{F} = \mathcal{E} + \frac{1}{K}\|u\|_H^2 \tag{242}$$

and

$$\kappa^\delta = \min\{\kappa_1^\delta, \kappa_2^\delta, \nu/(2L^2)\} > 0. \tag{243}$$

It follows that

$$\mathcal{F}(t) \le \mathcal{F}(0)e^{-\kappa^\delta t}. \tag{244}$$

Now, for general initial conditions, it suffices to observe that due to Theorem 3 (II), there exists some time $T^* > 0$ such that $\underline{\gamma}_\delta \le c_1(T^*), c_2(T^*) \le \overline{\gamma}_\delta$, and then the convergence result follows from the preceding analysis by taking $c_i(T^*)$ to be the initial conditions. This completes the proof of the theorem. □

Lastly we prove Theorem 5.

PROOF. Suppose $(\overline{c}_1^*, \overline{c}_2^*, \overline{u}^*) \ne (c_1^*, c_2^*, 0)$ is a steady state solution of the NPS system (110)-(114) with boundary conditions (176)-(178). Then by taking initial conditions $c_i(0) = \overline{c}_i^*$, $u(0) = \overline{u}^*$, we find that the corresponding energy $\mathcal{F}(t)$, defined in (242), is constant in time and positive, $\mathcal{F}(t) = \mathcal{F}(0) > 0$. On the other hand, Theorem 4 implies that $\mathcal{F}(t) \to 0$ as $t \to \infty$, and thus we have a contradiction. This contradiction completes the proof. □

REMARK 9. *For the proof of Theorem 5 above, uniqueness is shown by contradiction by considering the time dependent system. Alternatively, we can use an energy estimates argument directly to the difference* $(\bar{c}_1^* - c_1^*, \bar{c}_2^* - c_2^*, \bar{u}^*)$ *using the a priori* L^∞ *bounds for steady state solutions to NPNS/NPS (Remark 6). This yields computations in the spirit of* (229) *and obtains the same result. In a sense, such an approach seems more natural as it appears to not use any analysis of the time dependent system. However, as remarked in Remark 6, the a priori* L^∞ *bounds for steady states are themselves obtained from analysis of the time dependent system, thus the two approaches are not very different.*

Acknowledgment. The work of PC was partially supported by NSF grant DMS- 2106528.

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