Introduction to PDE

The method of characteristics

1 First Order Quasilinear PDE

We study

$$\sum_{j=1}^{n} a_j(x,u)\frac{\partial u}{\partial x_j} = b(x,u) \tag{1}$$

where a, b are smooth functions of $n+1$ independent variables, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$.

Definition 1 *An integral surface for (1), $\Sigma \subset \mathbb{R}^{n+1}$ is a set*

$$\Sigma = \{\bar{x} = (x, x_{n+1}) \in U \subset \mathbb{R}^{n+1} \mid x_{n+1} = u(x), \quad u \textit{ solves } (1)\}$$

Here U is an open set. Notice that the normal to Σ at (x, x_{n+1}) is

$$\nu(\bar{x}) = ((\nabla u)(x), -1)$$

where $(\nabla u)(x) = (\partial_1 u(x), \ldots, \partial_n u(x))$ and we denote $\partial_i = \frac{\partial}{\partial_{x_i}}$. Thus, (1) can be seen as the requirement that $(a, b) \perp \nu$, i.e. the vector (a, b) is tangent to Σ.

Definition 2 *The direction $(a, b) \in \mathbb{R}^{n+1}$ is called the characteristic direction.*

The characteristic directions define a vector field that is tangent to integral surfaces of (1). The integral curves of this field are called characteristic curves. The characteristic curves are thus solutions of

$$\frac{dx_i}{a_i} = \frac{dx_{n+1}}{b}, \quad i = 1, \ldots, n. \tag{2}$$

Introducing an artificial parameter t we rewrite (2) as

$$\left\{ \begin{array}{l} \frac{dx_i}{dt} = a(x_1, \dots, x_{n+1}) \\ \frac{dx_{n+1}}{dt} = b(x_1, \dots, x_{n+1}) \end{array} \right. \tag{3}$$

Now (3) is an autonomous system of ODEs and from standard facts about ODE it possesses an $n+1$-parameter family of solutions. This family represents an n-parameter family of characteristic curves because replacing t by $t+c$ produces the same characteristic curve.

Proposition 1 *Let*

$$\Sigma_u = \{(x_1, \dots, x_{n+1}) \mid x_{n+1} = u(x_1, \dots x_n)\}$$

be the graph of a C^1 function defined in some open set in $\mathbb{R}^n$. Then Σ_u is an integral surface for (1) if and only if Σ_u is a union of characteristic curves.

Proof. If Σ_u is the graph of u and if through each point $P \in \Sigma$, passes a characteristic curve, then it follows that the tangent to this curve belongs to the tangent space to Σ_u at P. If $P = \overline{x} = (x, x_{n+1})$ this means that $\nu(\bar{x}) \perp (a, b)$ which implies that u solves the equation (1).

On the other hand, assume that Σ_u is an integral surface for (1). Let $P = (x^{(0)}, x^{(0)}_{n+1}) \in \Sigma_u$. Let γ be the characteristic curve passing through P. We need to show that $\gamma \subset \Sigma_u$. Indeed, let us consider the expression $F(x_1, \dots x_{n+1}) = u(x_1, \dots x_n)) - x_{n+1}$ and evaluate it along

$$\gamma(t) = (x_1(t), \dots x_{n+1}(t))$$

where t is a parameter and $P = \gamma(0)$ (with a slight abuse of notation). Let us set $f(t) = F(\gamma(t))$. The fact that $P \in \Sigma_u$ means that $f(0) = 0$. On the other hand,

$$\begin{array}{l} \frac{df}{dt} = \sum_{j=1}^n \frac{\partial u}{\partial x_j}(x(t)) \frac{dx_j}{dt} - \frac{dx_{n+1}}{dt} \\ = \sum_{j=1}^n \frac{\partial u}{\partial x_j}(x(t)) a_j(x(t), x_{n+1}(t)) - b(x(t), x_{n+1}(t)) \end{array}$$

Substracting the equation (1) obeyed by u we have

$$\begin{array}{l} \frac{df}{dt} = \sum_{j=1}^n \partial_j u(x(t)) \left[a_j(x(t), x_{n+1}(t)) - a_j(x(t), u(x(t)))\right] \\ + b(x(t), u(x(t))) - b(x(t), x_{n+1}(t)) \end{array}$$

and thus

$$\begin{array}{l}\frac{df}{dt} = \sum_{j=1}^n \partial_j u(x(t)) \left[a_j(x(t), u(x(t)) - f(t)) - a_j(x(t), u(x(t)))\right] \\ +b(x(t), u(x(t))) - b(x(t), u(x(t)) - f(t))\end{array}$$

The last expression can be viewed as a nice ODE for $f(t)$. Because $f(0) = 0$ and the function $g(t) = 0$ solves the same ODE, it follows from ODE uniqueness theory that $f(t) = 0$ for all t.

1.1 The Cauchy problem for the quasilinear equation

We want to construct integral surfaces by the method of characteristics. Because integral surfaces are union of characteristics, we are lead naturally to the following procedure to construct them: Consider an arbitrary $n-1$ dimensional surface in $\mathbb{R}^{n+1}$. At each point on it, consider the characteristic curve passing through the point. The union of these curves should be (in most cases) an integral surface. Let $\Gamma \subset \mathbb{R}^{n+1}$ be a nice $n-1$-dimensional surface in $\mathbb{R}^{n+1}$ given locally by

$$\Gamma = \{(x_1, \dots, x_{n+1}) \mid \quad x_j = f_j(s),\ j = 1, \dots n+1,\ s \in D \subset \mathbb{R}^{n-1}\}$$

where D is an open set and f_j are smooth enough (C^2 is sufficient). Fix $s \in D$. Let us solve the characteristic equations

$$\left\{ \begin{array}{l} \frac{dx_i}{dt} = a_j(\bar{x}), \quad j = 1, \dots n \\ \frac{dx_{n+1}}{dt} = b(\bar{x}) \end{array} \right. \tag{4}$$

with $P = (f_1(s), \dots f_{n+1}(s))$ as initial datum. We obtain the function $(x_1(s,t), \dots x_{n+1}(s,t))$ satisfying (4) and

$$x_j(s, 0) = f_j(s), \quad j = 1, \dots, n+1. \tag{5}$$

Consider I_s the maximal interval of existence for $\bar{x}(s, \cdot)$ and let

$$\Sigma = \{y \in \mathbb{R}^{n+1} \mid \quad y_j = x_j(s,t),\ s \in D,\ t \in I_s\} \tag{6}$$

In order for Σ to be locally the graph of a function of $(x_1, \dots x_n)$ we need the matrix $(\partial_s x_j, \partial_t x_j)_{j=1,\dots n}$ to be invertible. Locally, that is all that is needed:

Theorem 1 *Let $f_1, \dots f_{n+1} : D \subset \mathbb{R}^{n-1} \to \mathbb{R}$ be C^2. Assume that at $s_0 \in D$ we have*

$$Det \begin{pmatrix} \frac{\partial f_1}{\partial s_1}(s_0), & \dots & \frac{\partial f_1}{\partial s_{n-1}}(s_0), & a_1^{(0)} \\ \dots & \dots & \dots & . \\ \frac{\partial f_n}{\partial s_1}(s_0), & \dots & \frac{\partial f_n}{\partial s_{n-1}}(s_0), & a_n^{(0)} \end{pmatrix} \neq 0, \tag{7}$$

where $a_j^{(0)} = a_j(f_1(s_0), \dots f_{n+1}(s_0))$. Then there exists a neighborhood D_0 of s_0, an open set $U \subset \mathbb{R}^n$ containing $x^{(0)} = (f_1(s_0), \dots, f_n(s_0))$ and a solution $u(x)$ of (1) defined for $x \in U$ so that

$$\Gamma_0 = \{y \in \mathbb{R}^{n+1} | \quad x_j = f_j(s), \ s \in D_0\}$$

is included in $\Sigma = \{x \in U \times \mathbb{R} | \quad x_{n+1} = u(x)\}$.

Remark 1 *If condition (7) is violated, it may still happen that one can solve the problem $\Gamma_0 \subset \Sigma$. If this is possible, the necessarily (a, b) is tangent to Γ. Indeed, if we have a solution u of (1) and if*

$$f_{n+1}(s) = u(f_1(s), \dots f_n(s))$$

differentiating we obtain

$$\frac{\partial f_{n+1}}{\partial s_k} = \sum_{j=1}^{n} (\partial_j u) \frac{\partial f_j}{\partial s_k}, \quad k = 1, \dots, n-1. \tag{8}$$

The fact that (7) is not true implies that there exist $(\alpha_1, \dots, \alpha_n) \neq 0$ such that

$$\sum_{k=1}^{n-1} \alpha_k \frac{\partial f_j}{\partial s_k} + \alpha_n a_j = 0, \ \ j = 1, \dots, n \tag{9}$$

Multiplying (8) by α_k and summing we obtain, using (9) and the equation (1)

$$\begin{aligned} &\textstyle\sum_{k=1}^{n-1} \alpha_k \frac{\partial f_{n+1}}{\partial s_k} = \sum_{j=1}^{n} (\partial_j u) \sum_{k=1}^{n-1} \alpha_k \frac{\partial f_j}{\partial s_k} \\ &= -\alpha_n \textstyle\sum_{j=1}^{n} (\partial_j u) a_j = -\alpha_n b \end{aligned} \tag{10}$$

Thus, in view of (9) and (10), the tangent vector

$$\sum_{k=1}^{n-1} \alpha_k \frac{\partial f_j}{\partial s_k}, \quad j = 1, \dots, n+1$$

is equal to $-\alpha_n(a, b)$, so it is characteristic.

In general, if the condition (7) is violated and there exists a solution, there exist infinitely many solutions.

A particular case of a Cauchy problem is when $x_n = t$ is singled out, and $\Gamma = \{(x_1, \ldots, x_{n-1}, 0)\}$. Then the parameterization is given by $s = x$, functions $f_j(x) = x_j$ for $j = 1, \ldots, n-1$, $f_n = 0$ and $f_{n+1}(x) = u_0(x)$. The problem then becomes: solve

$$a_n(x, t, u)\partial_t u(x, t) + \sum_{j=1}^{n-1} a_j(x, t, u(x, t))\partial_j u(x, t) = b(x, t, u(x, t)) \tag{11}$$

with initial datum $u(x, 0) = u_0(x)$. The theorem above says that this has a local solution around $x_0 \in \mathbb{R}^{n-1}$ if $a_n(x_0, 0, u_0(x_0)) \neq 0$.

2 The Cauchy problem for general first order equations

We are concerned with the equation

$$F(x, u(x), \nabla u(x)) = 0 \tag{12}$$

where $x \in \mathbb{R}^n$, $u \in C^1$ (or smoother) and $F(x, z, p)$ is assumed to be smooth enough in its variables in $\mathbb{R}^{2n+1}$. The theory is local, but for simplicity of exposition we will not carry around excessive notation regarding the domains of F and u. If we want to formulate (12) in geometric terms we need to consider n -dimensional integral surfaces in $\mathbb{R}^{2n+1}$:

$$\Sigma = \{(x, z, p) \in \mathbb{R}^{2n+1} \mid \; z = u(x), \;\; u \text{ solves } (12), \; p_j = \partial_j u(x), \; j = 1, \ldots n.\} \tag{13}$$

Then, considering the zero level set

$$Z = \{(x, z, p) \mid \; F(x, z, p) = 0\} \tag{14}$$

the equation (12) means $\Sigma \subset Z$. Surfaces like Σ (i.e. with $z = u(x)$, $p = \nabla u(x)$ for some u) satisfy some restrictions. Let us consider a point (x, z, p) on such a surface and let $(\delta x, \delta z, \delta p)$ denote the components of a tangent vector to Σ at (x, z, p). Then, obviously

$$\delta z = \sum_{i=1}^{n} (\partial_i u)\delta x_i = \sum_{i=1}^{n} p_i \delta x_i \tag{15}$$

where p_j are the components of p. Also

$$\delta p_i = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j} \delta x_j \tag{16}$$

We see that the n components δx determine the rest of the components $(\delta z, \delta p)$, and any choice of δx will produce via (15) and (16) a tangent vector to Σ. The fact that the matrix $(\frac{\partial^2 u}{\partial x_j \partial x_k})$ is symmetric (if $u \in C^2$) is reflected in the following property of the tangent space at (x, z, p). If $(\delta x, \delta z, \delta p)$ and $(\overline{\delta x}, \overline{\delta z}, \overline{\delta p})$ are two tangent vectors, then

$$\sum_{i=1}^{n} \left(\delta p_i \overline{\delta x_i} - \overline{\delta p_i} \delta x_i\right) = 0 \tag{17}$$

Now (15) and (17) can be written simply in terms of differential forms. Considering the one-form $\alpha = pdx - dz$ in $\mathbb{R}^{2n+1}$ (where $pdx = \sum_{i=1}^{n} p_i dx^i$), then (15) means that

$$\alpha = pdx - dz = 0 \text{ at } \Sigma, \tag{18}$$

and (17) means that $d(dz - pdx) = 0$ at Σ i.e.

$$\omega = \sum dp_i \wedge dx_i = 0 \ \text{ at } \Sigma. \tag{19}$$

Let us emphasize that the forms α and ω are respectively a one-form and a two-form defined in all $\mathbb{R}^{2n+1}$ and do not depend on Σ. Let us consider now a solution u of (12). Differentiating in the direction x_i we obtain

$$\frac{\partial F}{\partial x_i} + (\partial_i u)\left(\frac{\partial F}{\partial z}\right) + \sum_{j=1}^{n} \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right) \frac{\partial F}{\partial p_j} = 0. \tag{20}$$

If $(x, z, p) \in \Sigma$ we can write (20) as

$$\frac{\partial F}{\partial x_i} + p_i \left(\frac{\partial F}{\partial z}\right) + \sum_{j=1}^{n} \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right) \frac{\partial F}{\partial p_j} = 0. \tag{21}$$

We look at the group $\sum_{j=1}^{n} \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right) \frac{\partial F}{\partial p_j}$ and compare it to the rule (16) that gives the δp components in terms of the δx components. It is fairly natural (but very far from obvious) to consider the tangent vector at (x, z, p) whose components are given by

$$\delta x_i = \frac{\partial F}{\partial p_i}(x, z, p) \tag{22}$$

and then (16) provides

$$\delta p_i = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial F}{\partial p_j} \tag{23}$$

Thus (21) becomes

$$\frac{\partial F}{\partial x_i} + p_i \frac{\partial F}{\partial z} + \delta p_i = 0 \tag{24}$$

and (15) provides

$$\delta z = \sum_{i=1}^{n} p_i \frac{\partial F}{\partial p_i}. \tag{25}$$

The absolutely remarkable fact about (22), (24), (25) is that they give you a tangent vector $(\delta x, \delta z, \delta p)$ to an integral surface Σ at a point (x, z, p) in terms only of (x, z, p) and the equation they solve, $F(x, z, p) = 0$, without having to know the solution u of (12). This fortunate choice of $\delta x = \nabla_p F$ merits a brief discussion. So, what would be some good reasons to even consider this recipe? Obviously, this choice is a generalization from the quasilinear situation $F = a(x, z)p - b(x, z)$. However, in general $\nabla_p F$ depends on p, and in the quasilinear case it does not. (By the way, this is why we got away in the quasilinear case with studying integral surfaces $\Sigma = \{z = u(x)\}$, i.e. graphs, and did not have to go to the cotangent space). Another justification of the choice is that the geometric significance of $\mathbb{R}^n_x \times \mathbb{R}_z \times \mathbb{R}^n_p$ is that of the product (switching order) $T^*X \times \mathbb{R}$ of the cotangent space with the target space, and p are local coordinates in the cotangent fiber at $x \in X$. Then a vector $\nabla_p F$ defines indeed a tangent vector field to X, and not many such are available if we know only F, and the point (x, z, p). This is one of those instances in which the only general thing available is pretty good. Another justification comes from a geometric insight given by the Monge cones. Let us consider $n = 2$ and fix x and z. The equation $F(x, z, p) = 0$ can be viewed as an equation for the two direction numbers p_1 and p_2 of the tangent plane to the graph of u. The equation of the plane, at some point $(x, z) \in \Sigma' = \{(x, z) \mid \ z = u(x)\}$ is

$$\zeta - z = p_1(\xi - x_1) + p_2(\eta - x_2)$$

Let us assume that we can eliminate, say p_2 from $F(x, z, p) = 0$ and get $p_2 = p_2(p_1)$. (With a slight abuse of notation, now p_2 is a function of x, z and p_1, but we keep x and z fixed). The family of possible tangent planes to Σ' at (x, z) forms thus a one-parameter family whose "envelope" is in general a cone, the Monge cone. This is obained by solving the system

$$\begin{cases} \zeta - z = p_1(\xi - x_1) + p_2(p_1)(\eta - x_2) \\ 0 = (\xi - x_1) + \frac{dp_2}{dp_1}(p_1)(\eta - x_2) \end{cases}$$

In general, if we have a one-parameter family of surfaces in $\mathbb{R}^3$ given by $G(\xi, \eta, \zeta, \lambda) = 0$, ($\lambda$ being the parameter), the envelope is obtained by solving simultaneousy $G = 0$ and $\partial_\lambda G = 0$ which yields (two equations for four unknowns) a surface, the envelope. This surface has the property that it is tangent to the one-parameter family of surfaces along the curves γ_λ obtained by solving the same system with λ fixed (two equations for three unknowns). In our case, the tangent line to the Monge cone lying in tangent plane to Σ obeys

$$\begin{cases} \delta z = p_1 \delta x_1 + p_2(p_1)\delta x_2 \\ 0 = \delta x_1 + \frac{dp_2}{dp_1}(p_1)\delta x_2 \end{cases}$$

Now if the function $p_2(p_1)$ came from solving $F = 0$ then (differentiating) it follows that

$$\frac{dp_2}{dp_1} = -\frac{\partial_{p_1} F}{\partial_{p_2} F}$$

and so

$$\delta x_1 = \left(\frac{\partial_{p_1} F}{\partial_{p_2} F}\right) \delta x_2$$

so the tangent line with direction numbers $(\delta x_1, \delta x_2)$ is the same as the one with direction numbers $(\frac{\partial F}{\partial p_1}, \frac{\partial F}{\partial p_2})$. The tangent vector $\delta x = \nabla_p F$ is therefore the common tangent to the Monge cone and Σ'.

Ok, so motivated somewhat the choice of the tangent direction $\delta x = \nabla_p F$. Let us return to the business of integrating the system. We have from (22), (24) and (25) a tangent field to Σ at (x, z, p) given by

$$\begin{cases} \delta x = \nabla_p F(x, z, p) \\ \delta z = p \cdot \nabla_p F(x, z, p) = \sum_{i=1}^n p_i \frac{\partial F}{\partial p_i} \\ \delta p = -\nabla_x F - p \partial_z F \end{cases}$$

These are called the bicharacteristic directions, the associated ODE system

$$\begin{cases} \frac{dx}{dt} = \nabla_p F(x, z, p) \\ \frac{dz}{dt} = p \cdot \nabla_p F(x, z, p) \\ \frac{dp}{dt} = -\nabla_x F - p\partial_z F \end{cases} \tag{26}$$

the bicharacteristic system, and its solutions, bicharacteristics. We have a $2n$ family of bicharacteristics. Let us start by observing that the value of F does not change along bicharacteristics:

$$\begin{aligned} &\tfrac{d}{dt}F(x(t), z(t), p(t)) = (\nabla_x F)\tfrac{dx}{dt} + (\partial_z F)\tfrac{dz}{dt} + (\nabla_p F)\tfrac{dp}{dt} \\ &= (\nabla_x F)(\nabla_p F) + (\partial_z F)(p\nabla_p F) - (\nabla_p F)(\nabla_x F + p\partial_z F) \\ &= 0 \end{aligned}$$

This implies that if a bicharacteristic starts in $Z = \{(x, z, p) \mid \ F = 0\}$, it stays in Z. So, we are lead to the following procedure to find integral surfaces. We start with an $n-1$ parameter surface Γ in $\mathbb{R}^{2n+1}$ included in Z and on which $\alpha = pdx - dz$ equals to zero (so it can be part of a graph). This means that we start with $2n+1$ functions $(f_1(s), \dots, f_n(s), u_0(s), g_1(s), \dots, g_n(s))$ of $n-1$ parameters satisfying

$$F(f(s), u_0(s), g(s)) = 0 \tag{27}$$

and

$$\frac{\partial u_0}{\partial s_k} = \sum_{j=1}^{n} g_j(s) \frac{\partial f_j}{\partial s_k}, \quad k = 1, \dots, n-1. \tag{28}$$

We assume that the $n-1$ "horizontal" components (δx) of the tangent plane to Γ, together with the δx component of the bicharacteristic direction are linearly independent, i.e.

$$Det \begin{pmatrix} \frac{\partial f_1}{\partial s_1}, & \dots & \frac{\partial f_1}{\partial s_{n-1}} & \frac{\partial F}{\partial p_1} \\ \dots & \dots & \dots & \dots \\ \frac{\partial f_n}{\partial s_1}, & \dots & \frac{\partial f_n}{\partial s_{n-1}} & \frac{\partial F}{\partial p_n} \end{pmatrix} \neq 0 \tag{29}$$

at some s_0. We form then the bicharacteristic curves $X(s, t)$, $Z(s, t)$, $P(s, t)$, solving

$$\begin{cases} \frac{dX}{dt} = \nabla_p F, \qquad X(s, 0) = f(s) \\ \frac{dZ}{dt} = P\nabla_p F, \qquad Z(s, 0) = u_0(s) \\ \frac{dP}{dt} = -\nabla_x F - P\partial_z F, \qquad P(s, 0) = g(s) \end{cases} \tag{30}$$

Then, locally, the n-dimensional surface given parametrically by

$$\Sigma = \{(x, z, p) \mid x = X(s, t), z = Z(s, t), p = P(s, t)\} \tag{31}$$

is the graph of a function u

$$\Sigma = \{(x, z, p) \mid z = u(x), p = \nabla u\}$$

satisfying $F(x, u(x), \nabla_x u(x)) = 0$. The function $u(x)$ is defined by $u(x) = Z(s, t)$ where $x = X(s, t)$. Let us sketch informally the proof. The Jacobian condition (29) means that for small $|s - s_0| + |t|$ we can invert the relation $x = X(s, t)$ and find s, t as functions of x. Then the recipe

$$u(x) = Z(s, t) \tag{32}$$

defines a function $u(x)$ for x near $x_0 = f(s_0) = X(s_0, 0)$. Differentiating $u(X(s, t)) = Z(s, t)$ we obtain

$$\begin{cases} \sum_{j=1}^n \frac{\partial X_j}{\partial s_k}(\partial_j u)(X) = \frac{\partial Z}{\partial s_k}, \quad k = 1, \dots, n-1 \\ \sum_{j=1}^n \frac{\partial X_j}{\partial t}(\partial_j u)(X) = \frac{\partial Z}{\partial t} \end{cases} \tag{33}$$

On the other hand, we claim that

$$\begin{cases} \sum_{j=1}^n \frac{\partial X_j}{\partial s_k} P_j = \frac{\partial Z}{\partial s_k}, \quad k = 1\ , \dots, n-1 \\ \sum_{j=1}^n \frac{\partial X_j}{\partial t} P_j = \frac{\partial Z}{\partial t} \end{cases} \tag{34}$$

hold. Admitting this for a moment, let us note that the matrix $(\nabla_s X, \partial_t X)$ is injective for $|s - s_0| + |t|$ small, and therefore we deduce that

$$(\nabla u)(x) = P(s, t) \quad \text{when } x = X(s, t). \tag{35}$$

This implies that u solves (12) because $F(X(s, t), Z(s, t), P(s, t)) = 0$. So, it remains only to verify that (34) holds. The second equation in (34) is part of the bicharacteristic equations (30). The proof that

$$F_k(s, t) = \sum_{j=1}^n P_j(s, t) \frac{\partial X_j}{\partial s_k}(s, t) - \frac{\partial Z}{\partial s_k}(s, t) = 0$$

is done by considering the evolution in t at fixed s. First of all, $F_k(s,0) = 0$ because of (28) and the facts that $P_j(s,0) = g_j(s)$, $X_j(s,0) = f_j(s)$, $\frac{\partial Z}{\partial s_k}(s,0) = \frac{\partial u_0}{\partial s_k}$, $\frac{\partial X_j}{\partial s_k}(s,0) = \frac{\partial f_j}{\partial s_k}$. Now we differentiate:

$$\begin{aligned}
&\tfrac{dF_k}{dt} = \textstyle\sum_{j=1}^n \frac{dP_j}{dt}\frac{\partial X_j}{\partial s_k} + \sum_{j=1}^n P_j \frac{\partial^2 X_j}{\partial t \partial s_k} - \frac{\partial}{\partial s_k}\left(\frac{\partial Z}{\partial t}\right) \\
&= -\textstyle\sum_{j=1}^n \left(\frac{\partial F}{\partial x_j} + \frac{\partial F}{\partial z}P_j\right)\frac{\partial X_j}{\partial s_k} + \sum_{j=1}^n P_j \frac{\partial}{\partial s_k}\left(\frac{\partial F}{\partial p_j}\right) - \frac{\partial}{\partial s_k}\left(P \cdot \frac{\partial F}{\partial p}\right) \\
&= -\textstyle\sum_{j=1}^n \left(\frac{\partial F}{\partial x_j} + \frac{\partial F}{\partial z}P_j\right)\frac{\partial X_j}{\partial s_k} - \sum_{j=1}^n \left(\frac{\partial P_j}{\partial s_k}\right)\left(\frac{\partial F}{\partial p_j}\right) \\
&= -\tfrac{\partial}{\partial s_k} F(X(s,t), Z(s,t), P(s,t)) + \left(\frac{\partial F}{\partial z}\right)\left(\frac{\partial Z}{\partial s_k}\right) - \frac{\partial F}{\partial z}\textstyle\sum_{j=1}^n P_j \frac{\partial X_j}{\partial s_k} \\
&= -\tfrac{\partial}{\partial s_k} F - \frac{\partial F}{\partial z} F_k
\end{aligned} \tag{36}$$

We are almost done: we note that, because $F(X(s,t), Z(s,t), P(s,t)) = 0$, we know that $\frac{\partial}{\partial s_k} F = 0$. Thus, at fixed s, F_k satisfies the ODE

$$\begin{cases} \frac{dF_k}{dt} = -\frac{\partial F}{\partial z} F_k \\ F_k(s,0) = 0 \end{cases} \tag{37}$$

and thus F_k is identically 0.

3 Exercises

1. Solve

$$\partial_t u + c\partial_x u = 0$$

with $x \in \mathbb{R}$, $t \in \mathbb{R}$, $c \in \mathbb{R}$ and initial condition $u(x,0) = u_0(x)$.

2. Solve

$$\sum_{j=1}^n x_j \partial_j u = mu$$

with $m \in \mathbb{R}$ and with the boundary condition

$$u(x_1, \ldots x_{n-1}, 1) = g(x_1, \ldots x_{n-1})$$

3. Take $m \geq 1$, $0 < \alpha < 1$. Show that the equation

$$\alpha x \partial_x u + y \partial_y u = mu$$

with $x \in \mathbb{R}, y > 0$ initial value $u(x,0) = 0$ has infinitely many solutions.

Hint: Consider

$$u(x,y) = \int_0^y \eta^{m-1} \left(\int_0^{x\eta^{-\alpha}} f(\xi) d\xi \right) d\eta$$

for arbitrary nice f. Check the homogeneity

$$u(\lambda^\alpha x, \lambda y) = \lambda^m u,$$

differentiate in λ and set $\lambda = 1$.

4. Solve

$$\partial_t u + a(u) \partial_x u = 0$$

with initial data $u(x,0) = u_0(x)$. How do the characteristic curves look like? If u_0 is a smooth function (say C^1) with compact support, does there exist a time $T > 0$ so that $u(x,t)$ exists for all $x \in \mathbb{R}$ and $0 \le t < T$, and is smooth? Can the solution stay smooth for all time if u_0 is like that and if $a(u)$ is non-degenerate (say $a'(u) \neq 0$ in places of interest)?

5. Consider $\bar{x} = (x_1, \ldots x_n, t)$, where we denoted x_{n+1} by t. Consider the equation (12) for the case

$$F(\bar{x}, z, p) = p_{n+1} + H(\bar{x}, p),$$

where $p = (p_1, \ldots p_n)$. In other words, consider the Hamilton-Jacobi equation

$$\partial_t u + H(x, t, \nabla_x u) = 0$$

Prove that the bicharacteristics are given by the classical Hamiltonian system of ODEs

$$\begin{cases} \frac{dx}{dt} = \nabla_p H \\ \frac{dp}{dt} = -\nabla_x H \end{cases}$$

together with

$$\frac{dz}{dt} = p \nabla_p H - H$$

6. Consider the eikonal equation

$$c^{-2}\left(\frac{\partial \psi}{\partial t}\right)^2 = \sum_{j=1}^{n}\left(\frac{\partial \psi}{\partial x_j}\right)^2$$

where $c > 0$ is a constant. Solve the bicharacteristic equations for the initial value problem for ψ with data $\psi(x,0) = \psi_0(x)$. (Do not expect to explicitly compute the solution $\psi(x,t)$).