

Radiative Vlasov-Maxwell Equations

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ABSTRACT. The Radiative Vlasov-Maxwell equations model the radiative kinetics of collisionless relativistic plasma. In them the Lorentz force is modified by the addition of radiation reaction forces. The radiation forces produce damping of particle energy but these forces are not divergence-free in momentum space, which has an effect of concentration near zero momentum. We prove unconditional global regularity of solutions for a class of Radiative Vlasov-Maxwell equations with large initial data.

1. Introduction

Radiation reaction forces in plasma capture the irreversible transfer of kinetic energy into radiation as the charged particles accelerate. There are several models of this phenomenon in the physical literature [18], [27] and a formal derivation from microscopic models [17]. These models apply to relativistic plasma often found in high-energy astrophysical systems. Still, a rigorous self-consistent derivation of the particle dynamics and their radiation is fraught with fundamental challenges [30]. Radiative forces are significant for particles at large velocities and are not accounted for in the classical Vlasov-Maxwell equations. In this paper we prove the global regularity of solutions with large initial data for a class of Radiative Vlasov-Maxwell equations. We are not aware of any mathematical analysis of the Radiative Vlasov-Maxwell equations.

In contrast to Radiative Vlasov-Maxwell equations, the problem of global regularity for solutions of the classical Vlasov-Maxwell equations with large data has been studied extensively, but remains unsolved. The Vlasov-Maxwell equations are locally well posed [1]. Small data results have been obtained [12], [28], in which the plasma is initially dilute, the solutions remain small and smooth, disperse and their asymptotic

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behavior is free ([2–5]). This picture holds for nearly neutral data as well ([10],[8]). There are several recent results ([14, 21]) concerning the asymptotic behavior of small perturbations of steady states which do not depend on the space variable. Existence of global weak solutions was obtained in [9].

For smooth large data, the possibility of spontaneous singularity formation has been the focus of many analytical works. In seminal papers, Glassey and Strauss [11, 13] proved that the only way singularities might arise in finite time is through concentration of particle density at very high velocity. Specifically, they proved that if the solution-averaged Lorentz factor $\langle \gamma \rangle$ is uniformly bounded, then no singularities can form in finite time from smooth and localized initial data. The quantity $\langle \gamma \rangle$ is a function of space and time representing the kinetic energy density of the particles. In [15] it was shown using Fourier analysis that the singularities are averted if the electromagnetic fields remain bounded. Several other results are based on Fourier methods [6, 7],[23].

A number of extensions of the results of Glassey and Strauss concern moments of the type $M_{\theta,q} = \|\langle \gamma^\theta \rangle\|_{L^q(dx)}$. In our notation,

$$\gamma = \sqrt{1 + |p|^2} \quad \text{and} \quad \langle \gamma^\theta \rangle = \int_{\mathbb{R}^3} (\sqrt{1 + |p|^2})^\theta f(x, p, t) dp$$

for an exponent θ . The average of the kinetic energy density considered by Glassey and Strauss corresponds to $M_{1,\infty}$. In [22], control of $M_{\theta,q}$ where $\theta > 4/q$ and $6 \leq q \leq \infty$ is shown to be sufficient for regularity. In [29], control of $M_{0,\infty}$ is established as a regularity criterion, and in [24] this result was extended to $M_{0,6}$. The results of [16] imply that finiteness of $M_{3,2}$ is sufficient for regularity. In [20] it is shown that for regularity, if $2 < q \leq \infty$ and $\theta > 2/q$, then control of $M_{\theta,q}$ is sufficient, and if $1 \leq q \leq 2$ and $\theta > 8/q - 3$, then control of $M_{\theta,q}$ is sufficient, and an improvement [25] shows that if $\theta > 3$, then control of $M_{\theta,1}$ is sufficient for regularity. In [19], it is proven that the solutions remain smooth if a plane projection of the momenta is bounded through the evolution. Results of global regularity for cylindrical symmetry are announced in [31].

The Vlasov-Maxwell (VM) equations are formed by the Vlasov equation for the particle distribution function $f = f(x, p, t)$, coupled to the Maxwell equations for the electromagnetic (EM) fields $E = E(x, t)$ and $B = B(x, t)$. The particle dynamics is driven by the Lorentz force

$$F_L = E + v \times B.$$

The Radiative Vlasov-Maxwell (RVM) ¹ are the same equations, except that the particles are moved by a total force

$$F = F_L + F_R$$

where F_R is the radiation reaction force. The RVM equations are not a small perturbation of the nonradiative Vlasov-Maxwell equations. Writing $p = \gamma v$, some of the examples of radiation reaction forces in the physical literature include ([18])

$$F_{LL} = -h\nu\gamma^2(|F_L|^2 - (v \cdot E)^2)$$

and the force due to inverse Compton scattering ([27])

$$F_{IC} = -h\nu\gamma^2(|E|^2 + |B|^2).$$

The parameter $h \geq 0$ measures the relative intensity of the reaction, and is proportional to Planck's constant. We introduce the notation for the EM energy density

$$\mathbf{K}(x, t) = (E(x, t), B(x, t)) \quad \text{and} \quad K^2 = |E|^2 + |B|^2 + 1 = |\mathbf{K}|^2 + 1.$$

These examples grow quadratically with the EM fields and vanish at $p = 0$. The main result of this paper is:

THEOREM 1. *Consider the RVM equations with radiation reaction force*

$$F_R = -h\nu\gamma K - \chi E$$

where $h > 1$ and $\chi = \chi(|p|)$ is a smooth unit bump function around $|p| = 0$ satisfying certain properties. Assume that the initial data $E_0(x)$ and $B_0(x)$ for the electromagnetic fields $E(x, t)$ and $B(x, t)$ and the initial data $f_0(x, p)$ for the particle distribution function $f(x, p, t)$ are smooth, compatible, and decay at spatial infinity. In addition, assume

$$\sup_{x,p} f_0(x, p) \exp(A_0|p|) < \infty$$

holds for some $A_0 > 0$ large enough (the initial particle density decays uniformly exponentially at high momentum). Then, smooth solutions of the RVM equations are globally a priori bounded in $W^{1,\infty}$ and there exist constants C depending explicitly only on the initial data so that

$$|E(x, t)| + |B(x, t)| + |\nabla_x E(x, t)| + |\nabla_x B(x, t)| \leq C \exp(Ct)$$

and

$$f(x, p, t) + |\nabla_x f(x, p, t)| + \sqrt{1 + |p|^2} |\nabla_p f(x, p, t)| \leq C \exp(C \exp(Ct))$$

¹We use the abbreviation RVM throughout this paper for ‘‘Radiative Vlasov-Maxwell equations’’ and not the ‘‘Relativistic Vlasov-Maxwell equations’’, which are the same as the Vlasov-Maxwell equations, for which we use the abbreviation VM.

hold for all x, p and t .

In this paper we address the main problem, which is to obtain unconditional global $W^{1,\infty}$ a priori bounds for smooth solutions with general large data. Persistence of higher regularity follows by classical methods, once these a priori bounds have been established, see Proposition 13. We do not strive for the most economical function spaces, nor provide a construction of solutions. The construction of solutions, asymptotic behavior for small data and analysis of related models will be discussed in forthcoming works. We chose for simplicity the single species model, but the same proof applies to the multispecies model. We also chose to emphasize unconditional results, based on precisely specified reaction forces. Physically motivated conditional results for more general forces, assuming bounds on the EM fields may also be obtained with our methods.

In the present work we use the term $-\chi E$ to mitigate the effect of the electric field at $p = 0$, and the linear growth at infinity of F_R in the EM fields to close an a priori bound on the EM fields using a bootstrap argument. The specific form in Definition 1 below was chosen for its simplicity; many other similar expressions, including modifications of F_{LL} and F_{IC} will provide the same unconditional result.

Because the unmodified expressions F_{LL} and F_{IC} grow quadratically with the size of the EM fields, in these cases our methods provide conditional global regularity for large data, assuming that the EM fields are bounded.

THEOREM 2. *Consider the RVM equations with radiation reaction force F_{IC} or F_{LL} . Assume that the initial data $E_0(x)$ and $B_0(x)$ for the electromagnetic fields $E(x, t)$ and $B(x, t)$ and the initial data $f_0(x, p)$ for the particle distribution function $f(x, p, t)$ are smooth, compatible, and are compactly supported in space. In addition, assume*

$$f_0(x, p) = 0 \quad \text{for } |p| \geq R$$

holds for some $R > 0$. Suppose on some time interval of existence $[0, T)$ for the smooth solution of the RVM equations it holds that

$$\int_0^T \|\mathbf{K}(\cdot, t)\|_{L^\infty}^2 dt < \infty.$$

Then, the smooth solution of RVM can be uniquely extended to a smooth solution beyond T .

Some ideas of the proof and a comparison with the VM equations are given below. Unlike the VM equations, where the total force F_L is

divergence-free in p , $\operatorname{div}_p F_L = 0$, the radiative force's divergence

$$\operatorname{div}_p F_R \neq 0$$

is mostly negative. Thus, unlike the VM case where f is automatically bounded if initially so, in the RVM equations f is not bounded uniformly and can (and will) grow in time. The danger is uncontrolled implosion, because the phase volume is contracting. The radiation reaction force does not act at zero velocity. This is the main reason we introduce χE . On the other hand, the radiation reaction force causes the flux of the kinetic energy density to decay. Thus, the main danger of singularity formation in RVM, as opposed to VM, comes not from high, but from low velocity.

The radiation reaction force in Theorem 1 is used to obtain unconditional a priori bounds on the particle distribution, which blow up like $|p|^{-3}$ near the origin, but decay exponentially at large $|p|$. This is a manifestation of the damping at high momenta, and the price one pays for the negative divergence of forces. The singular bounds on the particle distribution function make it impossible to bound directly the charge density, but they imply unconditional a priori bounds

$$\langle |v|[p]^n \rangle(x, t) = \int_{\mathbb{R}^3} |v|[p]^n f(x, p, t) dp \leq M_n$$

(in our notation the Lorentz factor is $\gamma = [p] = \sqrt{1 + |p|^2}$, with the normalized speed of light $c = 1$, the velocity is $v = p/[p]$ and p is the momentum). These “fluxes of moments” bounds are not, by themselves, bounds on the moments because v vanishes at $p = 0$, but in the next step we deduce new “flux of energy”-type bounds in terms of fluxes of moments and logarithms of gradients of f . To close the bounds we now turn to the Glassey-Strauss method of representing the electromagnetic fields. Using it and the gradient-conditioned moment bounds, we obtain bounds on the EM fields in terms of a choice of logarithms of gradients of f , in other words, in terms of a quantity

$$\min \left\{ \log_+ \|\nabla_p f(t)\|_{L^\infty}, \sup_{s \leq t} \log_+ \|\nabla_x f(s)\|_{L^\infty} \right\}.$$

The Glassey-Strauss representation for gradients is then used together with the EM bounds to obtain a priori estimates of the gradients of the EM fields. Finally, we apply the bounds on the EM fields and their gradients to bound the gradients of f , closing the argument. Ultimately, global regularity is a consequence of superlinear differential inequalities for the gradients of f , with doubly logarithmic nonlinearity. The EM field bound in terms of minimum of two gradient logarithms is crucial

in order to obtain global regularity: without this minimum, our bounds would not be sufficient to rule out finite time blow up.

The paper is organized as follows: After a section on notation and preliminaries (Section 2) where we describe the RVM equations, we make specific the form of the radiation reaction force F_R in Theorem 1 and summarize its properties in Section 3. We recall the Glassey-Strauss representation in Section 4, and in Section 5 we derive moment bounds. In Section 6 we obtain bounds on the EM fields and in Section 7 we derive bounds for their gradients. In Section 8 we obtain the final gradient bounds on f and conclude the proof of Theorem 1. The proof of Theorem 2 is given in Section 9. In Appendix A we verify some properties of the Glassey-Strauss representation and in Appendix B we give the proofs of ODE lemmas. In Appendix C, we calculate $\operatorname{div}_p F_{LL}$.

2. Preliminaries: notation, the RVM equations

The radiative Vlasov-Maxwell equations are formed with the Vlasov equation

$$\partial_t f + \operatorname{div}_x(vf) + \operatorname{div}_p(Ff) = 0, \quad (1)$$

with $f(x, p, t) \geq 0$, $(x, p, t) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$ and

$$F = F_L + F_R \quad (2)$$

where F_L is the Lorentz force

$$F_L = E + v \times B \quad (3)$$

and F_R is the radiation reaction force, which will be discussed in the next section (see Definition 1). The velocity is denoted by v ,

$$v = \frac{p}{\sqrt{1 + |p|^2}} = \frac{p}{[p]}, \quad (4)$$

and the Lorentz factor γ by $[p]$,

$$[p] = \sqrt{1 + |p|^2}. \quad (5)$$

$E(x, t)$ and $B(x, t)$ are respectively the electric field and the magnetic field. They solve the Maxwell equations,

$$\begin{cases} \partial_t E - \nabla_x \times B = -j, \\ \operatorname{div}_x E = \rho \\ \partial_t B + \nabla_x \times E = 0 \\ \operatorname{div}_x B = 0, \end{cases} \quad (6)$$

together with

$$\rho = \int f dp = \langle 1 \rangle \quad \text{and} \quad j = \int v f dp = \langle v \rangle. \quad (7)$$

Throughout the paper, for a function $\phi(x, p, t)$, we denote the solution average

$$\langle \phi \rangle(x, t) = \int \phi(x, p, t) f(x, p, t) dp. \quad (8)$$

The RVM equations are comprised of (1) with (2) and (6) with (7). Smooth solutions of RVM require the following compatibility conditions to be satisfied by the initial data: $f_0 \geq 0$,

$$\operatorname{div}_x E_0 = \int f_0 dp \quad \text{and} \quad \operatorname{div}_x B_0 = 0. \quad (9)$$

3. Properties of the radiation reaction force

We write

$$\mathbf{K}(x, t) = (E(x, t), B(x, t)) \quad (10)$$

and

$$K^2 = |E|^2 + |B|^2 = |\mathbf{K}|^2. \quad (11)$$

DEFINITION 1. In Theorem 1, the *radiation reaction force* is

$$F_R(x, p, t) = -\chi(|p|)E(x, t) - MpK(x, t)$$

with $M > 2$ a constant. Here $0 \leq \chi \leq 1$ is a smooth cutoff,

$$\chi(r) = 1 \text{ for } r \leq R_0 \text{ and } \chi(r) = 0 \text{ for } r \geq R_1, |\chi'(r)| \leq 2.$$

The effect of this radiation reaction force as it pertains to regularity is as follows. Writing $\hat{p} = p/|p|$, we find

$$\begin{aligned} F \cdot \hat{p} &= (1 - \chi(|p|))E \cdot \hat{p} - MK|p| \\ &\leq -K(x, t)(M|p| - (1 - \chi(|p|))) \leq 0 \end{aligned} \quad (12)$$

holds because

$$M \geq 2, \quad 1 - \chi(r) \leq 2r. \quad (13)$$

We note that

$$\operatorname{div}_p F_L = 0, \quad (14)$$

however, $\operatorname{div}_p F_R \neq 0$; in fact

$$-\operatorname{div}_p F = 3MK(x, t) + \chi'(|p|)E \cdot \hat{p}. \quad (15)$$

We show in Section 5 that for large enough positive constants A ,

$$\left(\frac{3}{|p|} + A \right) F \cdot \hat{p} - \operatorname{div}_p F \leq 0 \quad (16)$$

holds. This is a key property of F .

Observe that

$$|F(x, p, t)| \leq (M + 2)|p|K(x, t), \quad (17)$$

and differentiating, we find

$$|\nabla_p F(x, p, t)| + |\nabla_p \nabla_p F(x, p, t)| \leq C(M + 2)K(x, t). \quad (18)$$

Moreover,

$$|\nabla_x F(x, p, t)| \leq C|p|(|\nabla_x E| + |\nabla_x B| + K(x, t)), \quad (19)$$

and

$$|\nabla_p \nabla_x F(x, p, t)| \leq C(|\nabla_x E| + |\nabla_x B| + K(x, t)). \quad (20)$$

The properties (16)-(20) are sufficient to obtain global regularity.

REMARK 1. The requirement that F_R grows linearly in the size of the electromagnetic fields is only required at large K . So, in the form of F_R , replacing K by $\psi(K^2)$ where $\psi = 1$ for $K^2 \leq C$ and $\psi = K$ for $K^2 \geq 2C$ leads to the same persistence of regularity.

4. On the Glassey-Strauss representation

Differentiating the Maxwell equations results in the wave equations

$$\square E = -\partial_t j - \nabla_x \rho, \quad (21)$$

and

$$\square B = \nabla_x \times j. \quad (22)$$

We write

$$\square^{-1} g = \int_{|x-y| \leq t} \frac{1}{|x-y|} g(y, t - |x-y|) dy. \quad (23)$$

We consider the tangential derivatives T_i

$$T_i = \partial_i - \omega_i \partial_t \quad (24)$$

with $\omega = (y - x)/|y - x|$, which differentiate in directions parallel to the light cone,

$$T_i g = \frac{\partial}{\partial y_i} (g(y, t - |x - y|)), \quad (25)$$

and the derivative

$$V = \partial_t - \omega \cdot \nabla_y \quad (26)$$

which differentiates in the running time s along the light cone,

$$\frac{d}{ds} g(x + (t - s)\omega, s) = (Vg)(x + (t - s)\omega, s). \quad (27)$$

We note that

$$\omega \cdot T + V = 0. \quad (28)$$

Now we note that, if $g = Lh$ where L is a vector field belonging to the linear span of T_i and V , and if h is bounded, then $\square^{-1}g$ is bounded. This is done by integration by parts, using the representation (23) for Vh

and Th . The linear span have variable coefficients depending smoothly on ω .

Glasse and Strauss [11] represent E and B using the linear wave equations and expressing ∂_t and ∇_y as linear combinations of S and T_i where

$$S = \partial_t + v \cdot \nabla_y \quad (29)$$

is the streaming derivative, and where T_i is the tangential derivative given in (24). The linear combinations are

$$\partial_i = T_i + \frac{\omega_i}{1 + \omega \cdot v} (S - v \cdot T) \quad (30)$$

and

$$\partial_t = \frac{S - v \cdot T}{1 + \omega \cdot v}. \quad (31)$$

This procedure results in two sets of expressions, one coming from the streaming derivative S and one coming from the tangential derivatives T_i . The overall form is

$$\mathbf{K}(x, t) = (\mathbf{K}_T + \mathbf{K}_S)(x, t) + O(1) \quad (32)$$

where $O(1)$ represents a smooth function of (x, t) which depends explicitly on the initial data. For the expressions coming from S , we have

$$\begin{aligned} \mathbf{K}_S(x, t) &= \int_{|x-y| \leq t} a_S(\omega, v)(Sf)(y, p, t - |x - y|) dp \frac{dy}{|x - y|} \\ &= \int_0^t (t - s) ds \int_{|\omega|=1} a_S(\omega, v)(Sf)(x + (t - s)\omega, p, s) dp dS(\omega) \end{aligned} \quad (33)$$

where the kernel $a_S = a_S(\omega, v)$ is an explicit analytic tensor valued function satisfying

$$|\nabla_p a_S| \leq C[p]. \quad (34)$$

The expressions coming from T are

$$\begin{aligned} \mathbf{K}_T(x, t) &= \int_{|x-y| \leq t} a_T(\omega, v)f(y, p, t - |x - y|) dp \frac{dy}{|x - y|^2} \\ &= \int_0^t ds \int_{|\omega|=1} a_T(\omega, v)f(x + (t - s)\omega, p, s) dp dS(\omega) \end{aligned} \quad (35)$$

where the kernel $a_T = a_T(\omega, v)$ is an explicit analytic tensor valued function satisfying

$$|a_T| \leq C[p]. \quad (36)$$

For the gradient of the field, the representation ([11] Theorem 4), which is obtained via a similar procedure, has the form

$$\nabla_x \mathbf{K}(x, t) = ((\nabla_x \mathbf{K})_{TT} + (\nabla_x \mathbf{K})_{TS} + (\nabla_x \mathbf{K})_{SS})(x, t) + O(1) \quad (37)$$

where $O(1)$ represents a smooth function of (x, t) which depends explicitly on the initial data. The terms are

$$(\nabla_x \mathbf{K})_{TT}(x, t) = \int_{|x-y|\leq t} a_{TT}(\omega, v) f(y, p, t - |x - y|) dp \frac{dy}{|x - y|^3} \quad (38)$$

$$(\nabla_x \mathbf{K})_{TS}(x, t) = \int_{|x-y|\leq t} a_{TS}(\omega, v) (Sf)(y, p, t - |x - y|) dp \frac{dy}{|x - y|^2} \quad (39)$$

$$(\nabla_x \mathbf{K})_{SS}(x, t) = \int_{|x-y|\leq t} a_{SS}(\omega, v) (S^2 f)(y, p, t - |x - y|) dp \frac{dy}{|x - y|}. \quad (40)$$

Above, the kernels a_{TT} , a_{TS} and a_{SS} are explicit tensor valued analytic functions which satisfy various properties (see [13] Lemma 4). In particular, their derivatives in y and p are bounded by powers of $[p]$.

5. Moment bounds

In this section we use the radiation reaction force to obtain bounds for moments

$$m_n(x, t) = \langle [p]^n \rangle = \int [p]^n f(x, p, t) dp. \quad (41)$$

The charge density ρ corresponds to $m_0(x, t)$ and, as a consequence of the Vlasov equation (1), it obeys the conservation equation

$$\partial_t \rho + \operatorname{div}_x j = 0. \quad (42)$$

For higher moments, from the Vlasov equation (1), we have

$$\frac{\partial}{\partial t} m_n + \operatorname{div}_x \langle v [p]^n \rangle = n \langle (v \cdot F) [p]^{n-1} \rangle, \quad (43)$$

where we used

$$v = \nabla_p [p] \quad (44)$$

and integrated by parts in $\int [p]^n \operatorname{div}_p (Ff) dp$. A key element of the proof is provided by the unconditional a priori control of the fluxes vm_n of the moments m_n ,

$$vm_n(x, t) = \int f(x, p, t) |v| [p]^n dp = \langle |v| [p]^n \rangle \quad (45)$$

in terms of the initial data.

PROPOSITION 1. *Let (f, E, B) be a smooth solution of the RVM equations on $[0, T]$. Assume that there exists a constant C_0 such that*

$$0 \leq |p|^3 f_0(x, p) \exp(A|p|) \leq C_0$$

holds for some

$$A \geq \frac{3 + 2R_0}{(M - 2)(R_0)^2}.$$

Then, for any $n \geq 0$

$$\sup_{0 \leq t \leq T} \|vm_n(\cdot, t)\|_{L^\infty} \leq M_n$$

holds with constants M_n depending explicitly only on n , A and C_0 .

Proposition 1 is a corollary of the a priori estimate:

PROPOSITION 2. *Let (f, E, B) be a smooth solution of the RVM equations on $[0, T]$. Assume that there exists a constant C_0 such that*

$$0 \leq |p|^3 f_0(x, p) \exp A|p| \leq C_0 \quad (46)$$

holds for some

$$A \geq \frac{3 + 2R_0}{(M - 2)(R_0)^2}. \quad (47)$$

Then,

$$0 \leq f(x, p, t) \leq C_0 |p|^{-3} \exp(-A|p|) \quad (48)$$

holds for $t \leq T$.

PROOF. The path map is defined by the ordinary differential equations

$$\left\{ \begin{array}{l} \frac{dX}{dt}(a, \pi, t) = v(P(a, \pi, t)), \quad X(a, \pi, 0) = a, \\ \frac{dP}{dt}(a, \pi, t) = F(X(a, \pi, t), P(a, \pi, t), t), \quad P(a, \pi, 0) = \pi. \end{array} \right. \quad (49)$$

These represent the characteristic curves of the operator

$$D_t = \partial_t + v \cdot \nabla_x + F \cdot \nabla_p. \quad (50)$$

Note that

$$|X(a, \pi, t) - a| < t, \quad (51)$$

because $|v| < 1$. This property implies that the decay of f at spatial infinity is controlled for finite time, as long as F is Lipschitz continuous.

We fix a single characteristic $X(a, \pi, t)$ and $P(a, \pi, t)$. The equation (1) implies

$$\begin{aligned} \frac{d}{dt} f(X(a, \pi, t), P(a, \pi, t), t) = \\ -(\operatorname{div}_p F(X(a, \pi, t), P(a, \pi, t), t)) f(X(a, \pi, t), P(a, \pi, t), t). \end{aligned} \quad (52)$$

For the purpose of economy of notation, let us write

$$r(t) = |P(a, \pi, t)|, \quad (53)$$

for the momentum magnitude,

$$k(t) = K(X(a, \pi, t), t), \quad (54)$$

for the field strength and

$$f(t) = f(X(a, \pi, t), P(a, \pi, t), t) \quad (55)$$

for the probability density on characteristics. These quantities depend on initial data a and π .

In view of (15), whenever $f(0)$ is nonzero (52) results in

$$\frac{d}{dt} \log f(t) \leq (3M + |\chi'(r(t))|)k(t). \quad (56)$$

For further economy, we suppress that r, k, f are evaluated at t . Using (12) we have

$$\begin{aligned} \frac{dr}{dt} &\leq -Mkr + (1 - \chi(r))k \\ &\leq -k(Mr - (1 - \chi(r))) \\ &\leq 0 \end{aligned} \quad (57)$$

where we use the facts that $M \geq 2$ and $(1 - \chi(r)) \leq |\chi'(r)|r \leq 2r$. Let us consider the function

$$\Phi(r) = Ar + \log r^3. \quad (58)$$

We have that

$$\begin{aligned} \frac{d}{dt} (\Phi(r) + \log f) &= \Phi'(r) \frac{dr}{dt} + \frac{d}{dt} \log f \\ &\leq -\left(A + \frac{3}{r}\right) (Mkr - (1 - \chi)k) + 3Mk + |\chi'|k \\ &\leq -A(Mkr - (1 - \chi)k) + \frac{3}{r}(1 - \chi)k + |\chi'|k \\ &\leq 0. \end{aligned} \quad (59)$$

The last inequality follows because A is large enough (47). Indeed, the supports of χ' and of $(1 - \chi)$ are included in $r \geq R_0$, and

$$Mkr - (1 - \chi)k \geq k(M - 2)r \geq k(M - 2)R_0 \quad (60)$$

there, while $k((1 - \chi)3/r + |\chi'|) \leq k(3/R_0 + 2)$. We deduce that

$$\frac{d}{dt}(r^3 f \exp Ar) \leq 0. \quad (61)$$

This bound also holds in case $f(0) = 0$, trivially. We obtained on each characteristic

$$\begin{aligned} |P(a, \pi, t)|^3 f(X(a, \pi, t), P(a, \pi, t), t) \\ \leq (f_0(a, \pi) |\pi|^3 \exp A|\pi|) \exp(-A|P(a, \pi, t)|). \end{aligned} \quad (62)$$

Straightforward from (52) and $f_0 \geq 0$ is

$$f(X(a, \pi, t), P(a, \pi, t), t) \geq 0. \quad (63)$$

Reading (62) and (63) at $x = X(a, \pi, t)$, $p = P(a, \pi, t)$ where (x, p, t) is arbitrary in view of the fact that the flow map is invertible (due to the inverse map theorem of Hadamard, see e.g. [26]) we deduce (48). \square

We show that bounds on moment fluxes imply bounds on moments which depend logarithmically on gradients of f in either x or p . We define

$$G_1(t) = \sup_{0 \leq s \leq t} \sup_{x, p} |\nabla_x f(x, p, s)| + 2, \quad (64)$$

and

$$G_2(t) = \sup_{x, p} |\nabla_p f(x, p, t)| + 2. \quad (65)$$

PROPOSITION 3. *Let (f, E, B) be a smooth solution of the RVM equations on $[0, T]$. Assume that the initial data $f_0(x, p)$ obeys the decay condition (48).*

Then,

$$m_n(x, t) \leq CM_n + C_n \log G_2(t)$$

holds for $t \leq T$ with a constant C_n depending continuously and explicitly only on n and initial data.

PROOF. Fixing (x, t) , for any $R > 0$ we may pick p_0 with $|p_0| = R$. For $|p| \leq R$, we may write

$$f(x, p, t) = (f(x, p, t) - f(x, p_0, t)) + f(x, p_0, t) \quad (66)$$

and so it follows that

$$f(x, p, t) \leq RG_2 + \frac{C_0}{R^3} e^{-AR} \quad (67)$$

where for the last summand (48) was used. Then, we write

$$\int f(x, p, t) dp = \int_{|p| \leq R} (f(x, p, t)) dp + \int_{|p| \geq R} f(x, p, t) dp. \quad (68)$$

From (67), we have the bound

$$\int_{|p| \leq R} f(x, p, t) dp \leq \frac{4\pi}{3} (R^4 G_2 + C_0). \quad (69)$$

Now using (48) which implies that $\int_{|p| \geq R} f(x, p, t) dp \leq C_0 \log \frac{1}{R} + \frac{C_0}{A}$, and optimizing in R we obtain

$$\rho(x, t) \leq C \log G_2(t). \quad (70)$$

We have proved the claim for $m_0 = \rho$. For higher moments, we observe

$$\begin{aligned} m_n(x, t) &\leq \sqrt{2}(vm_n(x, t)) + (\sqrt{2})^n \int_{|p| \leq 1} f(x, p, t) dp \\ &\leq \sqrt{2}(vm_n(x, t)) + (\sqrt{2})^n \rho(x, t). \end{aligned} \quad (71)$$

So, the bound on m_0 implies bounds on all higher moments, in view of Proposition 1. \square

We estimate in terms of G_1 the space-time average of m_n ,

$$\bar{m}_n(x, t) = \frac{1}{4\pi t} \int_0^t \int_{|\omega|=1} m_n(x + (t-s)\omega, s) dS(\omega) ds. \quad (72)$$

Let us denote the region

$$\Gamma(x, t) = \{(y, s) : 0 \leq s \leq t, |x - y| \leq t - s\}. \quad (73)$$

Fixing n and the vertex (x, t) , we consider the quantity

$$Q(s) = \int_{|x-y| \leq t-s} m_n(y, s) \frac{dy}{|x-y|^2} \quad (74)$$

and take the time derivative. Differentiating, we find

$$\frac{dQ}{ds} = -\frac{1}{(t-s)^2} \int_{|x-y|=t-s} m_n(y, s) dS(y) + \int_{|x-y| \leq t-s} \frac{\partial m_n}{\partial s}(y, s) \frac{dy}{|x-y|^2}. \quad (75)$$

Then by the moment evolution law (43) and the property (17) of F

$$\begin{aligned}
& \int_{|x-y|\leq t-s} \frac{\partial m_n}{\partial s}(y, s) \frac{dy}{|x-y|^2} \\
&= \int_{|x-y|\leq t-s} n \langle (v \cdot F)[p]^{n-1} \rangle(y, s) - \operatorname{div}_y \langle v[p]^n \rangle(y, s) \frac{dy}{|x-y|^2} \\
&\leq \int_{|x-y|\leq t-s} C_n K(y, s) - \operatorname{div}_y \langle v[p]^n \rangle(y, s) \frac{dy}{|x-y|^2}
\end{aligned} \tag{76}$$

where C_n depends only on n and the a priori moment flux bound M_n in Proposition 1. Then, integrating by parts

$$\begin{aligned}
& - \int_{|x-y|\leq t-s} \operatorname{div}_y \langle v[p]^n \rangle(y, s) \frac{dy}{|x-y|^2} \\
&= - \frac{1}{(t-s)^2} \int_{|x-y|=t-s} \omega \cdot \langle v[p]^n \rangle(y, s) dS(y) \\
&+ P.V. \int_{|x-y|\leq t-s} \frac{2}{|y-x|^3} \omega \cdot \langle v[p]^n \rangle(y, s) dy \\
&+ \lim_{\varepsilon \rightarrow 0} \int_{|\omega|=1} \langle v[p]^n \rangle(x + \varepsilon \omega, s) \cdot \omega dS(\omega).
\end{aligned} \tag{77}$$

We observe that on the right hand side of the equality above, the first term is bounded by M_n , and the last term vanishes. Then integrating the equation (75) with respect to ds with (77) and (76) in hand yields upon dividing by $4\pi t$

$$\begin{aligned}
& \bar{m}_n(x, t) \\
&\leq C_n + C_n \frac{1}{t} \int_0^t \|\mathbf{K}(s)\|_{L^\infty} ds + \frac{1}{2\pi t} P.V. \int_{\Gamma(x,t)} \frac{1}{|x-y|^3} \omega \cdot \langle v[p]^n \rangle dy ds.
\end{aligned} \tag{78}$$

Here, $t > 0$ and C_n depends only on n and the initial data. Indeed, $(4\pi t)^{-1} Q(0)$ is bounded uniformly in (x, t) for smooth data.

PROPOSITION 4. *Let (f, E, B) be a smooth solution of the RVM equations on $[0, T]$. Assume that the initial data $f_0(x, p)$ obeys the decay*

condition (46). Then

$$\bar{m}_n(x, t) \leq C_n \frac{1}{t} \int_0^t \|\mathbf{K}(s)\|_{L^\infty} ds + C_T(1 + \log G_1(t))$$

and

$$\bar{m}_n(x, t) \leq C_n(1 + \log G_2(t))$$

holds for $t \leq T$ with constant C_n depending continuously and explicitly only on n and initial data and C_T depending on n , initial data and T .

PROOF. The bound for \bar{m}_n in terms of G_2 is an immediate consequence of Proposition 3.

To show the bound for \bar{m}_n in terms of G_1 , we estimate the principal value integral in (78) as follows. For fixed s , we split the spatial integral into the regions $|x - y| \leq \delta$ and $\delta \leq |x - y| \leq t - s$. The value δ is chosen below.

The integral on $\delta \leq |x - y| \leq t - s$ is bounded by

$$\left| \int_{\delta \leq |x-y| \leq t-s} \frac{1}{|x-y|^3} \omega \cdot \langle v[p]^n \rangle(s) dy \right| \leq CM_n \log \left(\frac{t-s}{\delta} \right). \quad (79)$$

For $|x - y| \leq \delta$ and $|p| \geq |x - y|^{-\kappa}$, we evaluate

$$\int_{|p| \geq |x-y|^{-\kappa}} |v[p]^n f(y, p, s)| dp \leq |x - y|^{k\kappa} v m_{n+k}(y, s) \quad (80)$$

for some $k > 0$ and thus the contribution of this term is bounded,

$$\begin{aligned} & \left| \int_{|x-y| \leq \delta} \frac{1}{|x-y|^3} \omega \cdot \int_{|p| \geq |x-y|^{-\kappa}} v[p]^n f(y, p, s) dp dy \right| \\ & \leq CM_{n+k} \int_{|x-y| \leq \delta} |x-y|^{k\kappa-3} dy \\ & \leq CM_{n+k} \delta^{k\kappa}. \end{aligned} \quad (81)$$

We are left with the integral for $|x - y| \leq \delta$ and $|p| \leq |x - y|^{-\kappa}$. Because the unit sphere average $\int_{|\omega|=1} (\omega \cdot v) f(x, p, s) dS(\omega)$ vanishes, we have

$$\begin{aligned} & \left| \int_{|x-y| \leq \delta} \frac{1}{|x-y|^3} \omega \cdot \int_{|p| \leq |x-y|^{-\kappa}} v[p]^n f(y, p, s) dp dy \right| \\ & \leq C \sup_{y,p} |\nabla_y f(s)| \int_{|x-y| \leq \delta} \frac{dy}{|x-y|^2} \int_{|p| \leq |x-y|^{-\kappa}} [p]^n dp \\ & \leq C \sup_{y,p} |\nabla_y f(s)| \delta^{1-(n+3)\kappa}. \end{aligned} \quad (82)$$

By choosing $0 < \kappa < \frac{1}{n+3}$ and $\delta = (t-s)/(2 + \sup_{y,p} |\nabla_y f(s)|)$, we find that the time average of the principal value integral is bounded as

$$\left| \frac{1}{2\pi t} \int_0^t P.V. \int_{|x-y| \leq t-s} \frac{1}{|x-y|^3} \omega \cdot \langle v[p] \rangle(s) dy ds \right| \leq C_T (1 + \log G_1(t)). \quad (83)$$

With this estimate and inequality (78), we have shown the bound in terms of G_1 . \square

6. Electromagnetic field bounds

PROPOSITION 5. *Let (f, E, B) be a smooth solution of the RVM equations on $[0, T]$. Assume that the initial data $f_0(x, p)$ obeys the decay condition (46). Let*

$$K_\infty(t) = \sup_{0 \leq s \leq t} \|\mathbf{K}(\cdot, s)\|_{L^\infty}.$$

Then

$$K_\infty(t) \leq C_1 (1 + \min\{\log G_1(t), \log G_2(t)\})$$

holds for $t \leq T$ with a constant C_1 depending continuously and explicitly only on initial data and T .

PROOF. We use the Glassey-Strauss representation (32) for \mathbf{K} and bound the integrals \mathbf{K}_S and \mathbf{K}_T .

To bound the integral \mathbf{K}_S with kernel a_S , we first use the Vlasov equation (1), $Sf = -\operatorname{div}_p(Ff)$, to integrate by parts in p , so

$$\int a_S Sf dp = \int (\nabla_p a_S) Ff dp \quad (84)$$

pointwise in (y, s) . Then, properties (17) and (34) imply

$$\begin{aligned} \left| \int a_S Sf dp \right| &\leq C \int [p] |p| |\mathbf{K}| f dp \\ &\leq CM_2 \|\mathbf{K}(s)\|_{L^\infty} \end{aligned} \quad (85)$$

because $|p|[p] = |v|[p]^2$. Therefore, \mathbf{K}_S has the bound

$$\begin{aligned} \left| \int_{|x-y| \leq t} a_S Sf dp \frac{dy}{|x-y|} \right| &\leq CM_2 \int_0^t (t-s) \|\mathbf{K}(s)\|_{L^\infty} ds \\ &\leq CM_2 T \int_0^t \|\mathbf{K}(s)\|_{L^\infty} ds. \end{aligned} \quad (86)$$

To bound the integral \mathbf{K}_T with kernel a_T , we use Proposition 4 because $\langle a_T \rangle$ does not generally have a pointwise bound by a moment

flux. In particular, property (36) implies

$$\left| \int_{|x-y|\leq t} a_T f dp \frac{dy}{|x-y|^2} \right| \leq CT \bar{m}_1(x, t) \quad (87)$$

pointwise in (x, t) and then we apply Proposition 4 for $n = 1$ using the bound in terms of either G_1 or G_2 . Using the bound in terms of G_2 , we obtain

$$\left| \int_{|x-y|\leq t} a_T f dp \frac{dy}{|x-y|^2} \right| \leq C_1 M_1 T \log G_2(t). \quad (88)$$

On the other hand, from the bound in terms of G_1 we have

$$\left| \int_{|x-y|\leq t} a_T f dp \frac{dy}{|x-y|^2} \right| \leq C_1 \int_0^t \|\mathbf{K}(s)\|_{L^\infty} ds + C_1 \log G_1(t). \quad (89)$$

To conclude, we apply the estimate (86) for \mathbf{K}_S with either estimate (88) or (89) for \mathbf{K}_T in the Glassey-Strauss representation, and use the Grönwall inequality. \square

7. Gradient bounds for electromagnetic fields

Now that we know the bounds for the moments in Proposition 1 and the uniform L^∞ bound on \mathbf{K} in Proposition 5, we can use the Glassey-Strauss representations (37) for the spatial gradients of E and B which we denote by $\nabla_x \mathbf{K}$.

PROPOSITION 6. *Let (f, E, B) be a smooth solution of the RVM equations on $[0, T]$. Assume that the initial data $f_0(x, p)$ obeys the decay condition (46). Then*

$$\|\nabla_x \mathbf{K}(\cdot, t)\|_{L^\infty} \leq C_1 \log G_1(t) \log G_2(t)$$

holds for $t \leq T$ with a constant C_1 depending continuously and explicitly only on initial data and T .

PROOF. We use the representation (37) for the gradient $\nabla_x \mathbf{K}$ and bound the integrals $(\nabla_x \mathbf{K})_{TT}$, $(\nabla_x \mathbf{K})_{TS}$ and $(\nabla_x \mathbf{K})_{SS}$.

The simplest term to bound is the integral $(\nabla_x \mathbf{K})_{TS}$ whose kernel a_{TS} satisfies ([13], Lemma 4)

$$|\nabla_p a_{TS}| \leq C[p]^4. \quad (90)$$

After using $Sf = -\operatorname{div}_p(Ff)$ to integrate by parts, we find

$$\int a_{TS} Sf dp = \int (\nabla_p a_{TS}) Ff dp. \quad (91)$$

The properties (17) and (90) then imply

$$\begin{aligned} \left| \int a_{TS} S f dp \right| &\leq C \int [p]^4 |p| |\mathbf{K}| f dp \\ &\leq CM_5 \|\mathbf{K}(s)\|_{L^\infty} \end{aligned} \quad (92)$$

with the fact $|p|[p]^4 = |v|[p]^5$. Therefore, $(\nabla_x \mathbf{K})_{TS}$ has the bound

$$\begin{aligned} \left| \int_{|x-y|\leq t} a_{TS} S f dp \frac{dy}{|x-y|^2} \right| &\leq CM_5 \int_0^t \|\mathbf{K}(s)\|_{L^\infty} ds \\ &\leq CM_5 TK_\infty(t). \end{aligned} \quad (93)$$

In order to bound $(\nabla_x \mathbf{K})_{SS}$, we first rewrite $S^2 f$ appealing twice to the Vlasov equation $Sf = -\operatorname{div}_p(Ff)$. Pointwise,

$$\begin{aligned} S(Sf) &= -S(\operatorname{div}_p(Ff)) \\ &= \nabla_x(Ff) : \nabla_p v - \operatorname{div}_p(S(Ff)) \\ &= \nabla_x(Ff) : \nabla_p v - \operatorname{div}_p(fSF) - \operatorname{div}_p(FSf) \\ &= \nabla_x(Ff) : \nabla_p v - \operatorname{div}_p(fSF) + \operatorname{div}_p(F \operatorname{div}_p(Ff)). \end{aligned} \quad (94)$$

We thus have three terms entering the expression of $(\nabla_x \mathbf{K})_{SS}$. For $n = 0, 1, 2$, the kernel a_{SS} satisfies ([13], Lemma 4)

$$|\nabla_p^n a_{SS}| \leq C[p]^4. \quad (95)$$

For the last term, integrating by parts in p twice gives

$$\int a_{SS} \operatorname{div}_p(F \operatorname{div}_p(Ff)) dp = \int F \cdot \nabla_p(F \cdot \nabla_p a_{SS}) f dp. \quad (96)$$

Then, properties (17), (18) and (95) imply

$$\begin{aligned} \left| \int a_{SS} \operatorname{div}_p(F \operatorname{div}_p(Ff)) dp \right| &\leq C \int |p|[p]^4 (|\nabla_p F| + |F|) |\mathbf{K}| f dp \\ &\leq C \int |v|[p]^5 |\mathbf{K}|^2 f dp \\ &\leq CM_5 \|\mathbf{K}(s)\|_{L^\infty}^2. \end{aligned} \quad (97)$$

The bound for the last term is therefore

$$\begin{aligned} \left| \int_{|x-y|\leq t} a_{SS} \operatorname{div}_p(F \operatorname{div}_p(Ff)) dp \frac{dy}{|x-y|} \right| \\ \leq CM_5 \int_0^t (t-s) \|\mathbf{K}(s)\|_{L^\infty}^2 ds \\ \leq CM_5 (TK_\infty(t))^2 \end{aligned} \quad (98)$$

For the second term, integration by parts in p yields

$$- \int a_{SS} \operatorname{div}_p(fSF) dp = \int (\nabla_p a_{SS})(SF) f dp. \quad (99)$$

From the Maxwell equations,

$$\begin{aligned} SE &= v \cdot \nabla_x E + \nabla_x \times B - j, \\ SB &= v \cdot \nabla_x B - \nabla_x \times E, \end{aligned} \quad (100)$$

and so from property (95), noting that $S\chi = 0$,

$$\begin{aligned} \left| \int a_{SS} \operatorname{div}_p(fSF) dp \right| &\leq C \int |p| |p|^4 (|SE| + |SB|) f dp \\ &\leq CM_5 (M_0 + \|\nabla_x \mathbf{K}(s)\|_{L^\infty}). \end{aligned} \quad (101)$$

The bound for the second term is then

$$\begin{aligned} \left| \int_{|x-y|\leq t} a_{SS} \operatorname{div}_p(fSF) dp \frac{dy}{|x-y|} \right| &\leq CM_5 \int_0^t (t-s) (M_0 + \|\nabla_x \mathbf{K}(s)\|_{L^\infty}) ds \\ &\leq CM_5 T \int_0^t (M_0 + \|\nabla_x \mathbf{K}(s)\|_{L^\infty}) ds \end{aligned} \quad (102)$$

For the first term, to integrate by parts the quantity

$$\nabla_x(Ff) : \nabla_p v = \partial_i(F_j f) \frac{\partial v_i}{\partial p_j} \quad (103)$$

where $\partial_i = \partial/\partial x_i$, we recall the decomposition of derivatives

$$\partial_i = T_i + \left(\frac{\omega_i}{1 + v \cdot \omega} \right) (v \cdot T - S). \quad (104)$$

Repeated indices indicate summation. We then write

$$a_{SS}(\nabla_x(Ff) : \nabla_p v) = A^{ij} T_i(F_j f) + b^j S(F_j f) \quad (105)$$

as the sum of two expressions.

The latter expression is

$$b^j S(F_j f) = \frac{\partial v_i}{\partial p_j} \left(\frac{\omega_i}{1 + v \cdot \omega} \right) a_{SS} S(F_j f) \quad (106)$$

which becomes

$$b^j S(F_j f) = b^j F_j S f + b^j f S F_j. \quad (107)$$

Observe that each term on the right hand side above may be treated in a similar fashion to terms previously discussed; we use the Vlasov equation to integrate by parts in p and use property (17) to deduce

$$\left| \int_{|x-y|\leq t} b^j F_j S f dp \frac{dy}{|x-y|} \right| \leq C M_6 T^2 K_\infty(t), \quad (108)$$

and we use properties (18) and (100) to arrive at

$$\left| \int_{|x-y|\leq t} b^j f S F_j dp \frac{dy}{|x-y|} \right| \leq C M_6 T \int_0^t (M_0 + \|\nabla_x \mathbf{K}(s)\|_{L^\infty}) ds \quad (109)$$

The former expression is

$$A^{ij} T_i(F_j f) = \frac{\partial v_i}{\partial p_j} \left(T_i + \frac{\omega_i}{1 + v \cdot \omega} v \cdot T \right) (F_j f). \quad (110)$$

Each T_i is a total y derivative, and so integrating by parts in y gives

$$\int_{|x-y|\leq t} A^{ij} T_i(F_j f) dp \frac{dy}{|x-y|} = - \int_{|x-y|\leq t} \tilde{A}^j(F_j f) dp \frac{dy}{|x-y|^2} + O(1) \quad (111)$$

where $O(1)$ represents a function of (x, t) which depends explicitly on the initial data. On the right hand side is the kernel $\tilde{A}^j = r^2 \partial / \partial y_i (A^{ij} / r)$ where $r = |x - y|$, which in particular satisfies $|\tilde{A}^j| \leq C[p]^4$ (see [13] Lemma 4). The estimate for this expression is then by property (17)

$$\begin{aligned} \left| \int_{|x-y|\leq t} A^{ij} T_i(F_j f) dp \frac{dy}{|x-y|} \right| &\leq C \int_{|x-y|\leq t} |v| [p]^5 |\mathbf{K}| dp \frac{dy}{|x-y|^2} + O(1) \\ &\leq C M_5 \int_0^t \|\mathbf{K}(s)\|_{L^\infty} ds + O(1) \\ &\leq C_0 (1 + M_5 T K_\infty(t)) \end{aligned} \quad (112)$$

where C_0 depends only on the initial data. Taking together (112), (108) and (109) gives us a bound on the first term entering the expression of $(\nabla_x \mathbf{K})_{SS}$, while the second and last term have bounds (102) and (98).

Therefore, $(\nabla_x \mathbf{K})_{SS}$ has the bound

$$\left| \int_{|x-y|\leq t} a_{SS}(S^2 f) dp \frac{dy}{|x-y|} \right| \leq C_T \left(1 + K_\infty(t)^2 + \int_0^t \|\nabla_x \mathbf{K}(s)\|_{L^\infty} ds \right) \quad (113)$$

where C_T depends only on the initial data and T .

To bound $(\nabla_x \mathbf{K})_{TT}$, we write the integral as

$$(\nabla_x \mathbf{K})_{TT}(x, t) = \int_0^t \frac{ds}{t-s} \int_{|\omega|=1} a_{TT}(\omega, v) f(x + (t-s)\omega, p, s) dp dS(\omega) \quad (114)$$

We split the integral on the backwards light cone into two pieces: the base piece on $0 \leq s \leq t - \delta$, and tip piece on $t - \delta \leq s \leq t$, where δ is chosen below. The properties of the kernel a_{TT} ([13], Lemma 4),

$$|a_{TT}| \leq C[p]^3 \quad (115)$$

and

$$\int_{|\omega|=1} a(v, \omega) dS(\omega) = 0, \quad (116)$$

imply for the base piece

$$\left| \int_0^{t-\delta} \frac{ds}{t-s} \int_{|\omega|=1} a_{TT} f dp dS(\omega) \right| \leq C(M_4 + \log G_2(t)) \log \left(\frac{t}{\delta} \right). \quad (117)$$

Here, we used the estimates of Proposition 3. For the tip piece, we first note

$$\begin{aligned} \left| \int_{|p| \geq (t-s)^{-\kappa}} a_{TT} f dp \right| &\leq C \int_{|p| \geq (t-s)^{-\kappa}} [p]^3 f dp \\ &\leq C(t-s)^\alpha \int_{|p| \geq (t-s)^{-\kappa}} |p|^{\frac{\alpha}{\kappa}} [p]^3 f dp \\ &\leq CM_n (t-s)^\alpha \end{aligned} \quad (118)$$

where $n = \lceil 4 + \alpha/\kappa \rceil$, and α, κ are numbers chosen freely. We let $\alpha > 0$ so that

$$\left| \int_{t-\delta}^t \frac{ds}{t-s} \int_{|\omega|=1} \int_{|p| \geq (t-s)^{-\kappa}} a_{TT} f dp dS(\omega) \right| \leq CM_n \delta^{1-\alpha}. \quad (119)$$

Then, we choose $\kappa < \frac{1}{6}$ such that

$$\begin{aligned} \left| \int_{t-\delta}^t \frac{ds}{t-s} \int_{|\omega|=1} \int_{|p| \leq (t-s)^{-\kappa}} a_{TT} f dp dS(\omega) \right| \\ \leq \sup_{s \leq t} \sup_{x, p} |\nabla_x f(x, p, s)| \int_{t-\delta}^t \int_{|p| \leq (t-s)^{-\kappa}} [p]^3 dp \\ \leq \delta^{1-6\kappa} \sup_{s \leq t} \sup_{x, p} |\nabla_x f(x, p, s)|. \end{aligned} \quad (120)$$

With $\alpha = 1$, we choose here $\delta = t(2 + \sup_{s \leq t, x, p} |\nabla_x f(x, p, s)|)^{-1/(1-6\kappa)}$ in view of the above.

Therefore, we have the following bound for $(\nabla_x \mathbf{K})_{TT}$

$$\left| \int_0^t \frac{ds}{t-s} \int_{|\omega|=1} a_{TT} f dp dS(\omega) \right| \leq C_T \log G_1(t) \log G_2(t). \quad (121)$$

Putting together estimates (93), (113) and (121), we obtain

$$|\nabla_x \mathbf{K}(x, t)| \leq C_T \left(\log G_1(t) \log G_2(t) + \int_0^t \|\nabla_x \mathbf{K}(s)\|_{L^\infty} ds \right) \quad (122)$$

where we chose to bound K_∞^2 by the product $C \log G_1 \log G_2$ in view of Proposition 4. Using the Grönwall inequality, we conclude the proof. \square

8. Proof of Theorem 1

PROPOSITION 7. *Let (f, E, B) be a smooth solution of the RVM equations on $[0, T]$. Assume that the initial data $f_0(x, p)$ obeys the decay condition (46). Then*

$$\|\nabla_x f(\cdot, t)\|_{L^\infty} + \|[p]|\nabla_p f(\cdot, t)\|_{L^\infty} \leq C \exp(C \exp(Ct))$$

holds for $t \leq T$ with a constant C depending continuously and explicitly only on initial data.

PROOF. We consider the quantities

$$W(t) = \sup_{s \leq t} \|\nabla_x f(s)\|_{L^\infty} + 3 \quad (123)$$

and

$$Z(t) = \sup_{s \leq t} \|[p]|\nabla_p f(s)| + (1 + |p|)f(s)\|_{L^\infty} + 3. \quad (124)$$

Below we show W and Z obey the certain differential inequalities. We write (1) as

$$D_t f = -(\operatorname{div}_p F) f \quad (125)$$

and take derivatives in x and in p :

$$D_t(\partial_{x_i} f) = -(\partial_{x_i} F) \cdot \nabla_p f - (\operatorname{div}_p F)(\partial_{x_i} f) - (\partial_{x_i}(\operatorname{div}_p F)) f \quad (126)$$

and

$$D_t(\partial_{p_i} f) = -(\partial_{p_i} v) \cdot \nabla_x f - (\operatorname{div}_p F)(\partial_{p_i} f) - (\partial_{p_i} F) \cdot \nabla_p f - (\partial_{p_i}(\operatorname{div}_p F)) f \quad (127)$$

We deduce inequalities for quantities

$$w = |\nabla_x f| + 3 \quad (128)$$

and

$$z = (1 + |p|)f + |p||\nabla_p f| + 3. \quad (129)$$

Using the estimates (17), (18), (19) and (20), we find that

$$D_t w \leq C(Kw + (K + |\nabla_x \mathbf{K}|)z) \quad (130)$$

and

$$D_t z \leq C(w + Kz) \quad (131)$$

from equations (126) and (127).

To see this, first multiply the equation (126) by $\partial_{x_i} f / |\nabla_x f|$ and add in i to obtain,

$$\begin{aligned} D_t |\nabla_x f| &\leq |\operatorname{div}_p F| |\nabla_x f| + |\nabla_x F| |\nabla_p f| + |\nabla_x \operatorname{div}_p F| f \\ &\leq C(K |\nabla_x f| + (K + |\nabla_x \mathbf{K}|)(|p| |\nabla_p f| + f)) \\ &\leq C(Kw + (K + |\nabla_x \mathbf{K}|)z). \end{aligned} \quad (132)$$

This implies (130). Then, we multiply (127) by $|p| \partial_{p_i} f / |\nabla_p f|$ and add in i to obtain

$$\begin{aligned} |p| D_t |\nabla_p f| &\leq 2 |\nabla_x f| + |p| |\operatorname{div}_p F| |\nabla_p f| + |p| |\nabla_p \operatorname{div}_p F| f \\ &\leq 2 |\nabla_x f| + C(K |p| |\nabla_p f| + K |p| f) \\ &\leq C(w + Kz). \end{aligned} \quad (133)$$

We used $|p| |\nabla_p v| < 2$, which is immediate from (197) and $|v| < 1$. This implies, with (125), the estimate (131). Now we have that

$$W(t) = \sup_{s \leq t} \|w(s)\|_{L^\infty}, \quad (134)$$

and

$$Z(t) = \sup_{s \leq t} \|z(s)\|_{L^\infty}. \quad (135)$$

Taking the supremum in time of (130) and (131), we find

$$\begin{aligned} &\sup_{s \leq t} \|D_t w(s)\|_{L^\infty} \\ &\leq C \left(K_\infty(t) W(t) + \left(K_\infty(t) + \sup_{s \leq t} \|\nabla_x \mathbf{K}(s)\|_{L^\infty} \right) Z(t) \right) \end{aligned} \quad (136)$$

and

$$\sup_{s \leq t} \|D_t z(s)\|_{L^\infty} \leq C(K_\infty(t) Z(t) + W(t)) \quad (137)$$

We use now

LEMMA 1. *Let $g = g(t)$ be a positive Lipschitz function of $t \in [0, T]$ and let $G(t) = \sup_{s \leq t} g(s)$. Then, $G = G(t)$ is Lipschitz and*

$$\limsup_{h \rightarrow 0} \frac{G(t+h) - G(t)}{h} \leq \liminf_{\varepsilon \rightarrow 0} \sup_{s \leq t+\varepsilon} |g'(s)|.$$

By Lemma 1, differentiation under $\sup_{s \leq t}$ for Lipschitz functions of time is permissible. We have thus

$$\frac{dW}{dt} \leq \liminf_{\varepsilon \rightarrow 0} \sup_{s \leq t+\varepsilon} \|D_t w(s)\|_{L^\infty} \quad (138)$$

and

$$\frac{dZ}{dt} \leq \liminf_{\varepsilon \rightarrow 0} \sup_{s \leq t + \varepsilon} \|D_t z(s)\|_{L^\infty} \quad (139)$$

holds for almost all t . Then, using Proposition 5 and Proposition 6 and the continuity of the upper bounds allowing to set $\varepsilon = 0$, we arrive at the ODE system

$$\frac{dW}{dt} \leq C((\log W)W + (\log W)(\log Z)Z) \quad (140)$$

and

$$\frac{dZ}{dt} \leq C((\log Z)Z + W). \quad (141)$$

We apply Lemma 2:

LEMMA 2. *Let $W = W(t)$ and $Z = Z(t)$ be nondecreasing, Lipschitz functions of $t \geq 0$. Let $W(0) = W_0$ and $Z(0) = Z_0$ and suppose*

$$\min \{\log W_0, \log Z_0\} \geq 1.$$

Assume that $W(t)$ and $Z(t)$ obey differential inequalities

$$\frac{dW}{dt} \leq C((\log W)W + (\log W)(\log Z)Z)$$

and

$$\frac{dZ}{dt} \leq C((\log Z)Z + W).$$

Then the functions W and Z satisfy

$$W + Z \leq C \exp(C \exp(Ct))$$

where C depends only on W_0 and Z_0 .

REMARK 2. In contrast, the ODE

$$\frac{dY}{dt} = Y(\log Y)^2$$

blows up in finite time.

The proof of Lemmas 1 and 2 are given in Appendix B. □

The proof of Theorem 1 is completed now by applying the bounds of Proposition 7 to the bounds on the EM fields in Propositions 5 and 6.

9. Proof of Theorem 2

PROPOSITION 8. Consider the RVM equations with radiation reaction force F_{IC} or F_{LL} and let (f, E, B) be a smooth solution of the RVM equations on $[0, T]$. Assume there is some $A > 0$ such that

$$\sup_{x,p} e^{A|p|} f_0(x, p) < \infty \quad (142)$$

and that

$$\int_0^T \|\mathbf{K}(t)\|_{L^\infty}^2 dt =: \Gamma < \infty. \quad (143)$$

Then, there exists a constant C depending only on A , $A\Gamma$ and Γh so that

$$f(x, p, t) \leq C e^{-A|p|}$$

holds for all $t \leq T$.

PROOF. Let us consider the case $F_R = F_{IC}$ first. We observe that

$$-\operatorname{div}_p F_{IC} = 3h[p](1 + |v|^2)(|E|^2 + |B|^2) \quad (144)$$

and so

$$-\operatorname{div}_p F_{IC} \leq 6h[p]|\mathbf{K}|^2. \quad (145)$$

On the other hand,

$$\hat{p} \cdot F_{IC} = -h|p|[p](|E|^2 + |B|^2), \quad (146)$$

and so

$$\hat{p} \cdot (F_L + F_{IC}) \leq |\mathbf{K}| - h|p|^2|\mathbf{K}|^2. \quad (147)$$

It follows that on the corresponding characteristics $X = X(a, \pi, t)$ and $P = P(a, \pi, t)$, in the shorthand notation $r(t) = |P(a, \pi, t)|$,

$$k(t) = |\mathbf{K}(X(a, \pi, t), t)| \quad \text{and} \quad f(t) = f(X(a, \pi, t), P(a, \pi, t), t), \quad (148)$$

we have the estimate

$$\begin{aligned} \frac{d}{dt} \log(f \exp Ar) &= A \frac{dr}{dt} + \frac{d}{dt} \log f \\ &\leq Ak + h(6\sqrt{1+r^2} - Ar^2)k^2 \\ &\leq Ak + h(1 + 9A^{-2})k^2 \end{aligned} \quad (149)$$

Taking the supremum over labels on the right hand side, and integrating in time

$$\log(f(t) \exp Ar(t)) - \log(f_0 \exp Ar(0)) \leq A\Gamma + h(1 + 9A^{-2})\Gamma \quad (150)$$

where we used Cauchy-Schwarz for the integral with factor A . Explicitly, we have obtained

$$f(X(a, \pi, t), P(a, \pi, t), t) \leq C_A f_0(a, \pi) \exp(A|\pi|) \exp(-A|P(a, \pi, t)|) \quad (151)$$

where $C_A = \exp(A + h(1 + 9A^{-2}))\Gamma$. Reading at $x = X(a, \pi, t)$ and $p = P(a, \pi, t)$, the result then follows for the case $F_R = F_{IC}$.

Now consider the case $F_R = F_{LL}$. We calculate

$$-\operatorname{div}_p F_{LL} = h[p](5 - |v|^2)(|F_L|^2 - (v \cdot E)^2) - 2h[p]^{-1}(E \cdot F_L). \quad (152)$$

The computation is given in Appendix C. Thus

$$-\operatorname{div}_p F_{LL} \leq 5h[p](|F_L|^2 - (v \cdot E)^2) + h|\mathbf{K}|^2. \quad (153)$$

We have directly

$$\hat{p} \cdot F_{LL} = -h|p|[p](|F_L|^2 - (v \cdot E)^2) \quad (154)$$

so that

$$\hat{p} \cdot (F_L + F_{LL}) \leq |\mathbf{K}| - h|p|^2(|F_L|^2 - (v \cdot E)^2). \quad (155)$$

On the corresponding characteristics $X = X(a, \pi, t)$ and $P = P(a, \pi, t)$ here, we use again the shorthands $r(t), k(t)$ and $f(t)$ and define

$$q(t) = \left| E(X(t), t) + \frac{P(t)}{[P(t)]} \times B(X(t), t) \right|^2 - \left(\frac{P(t)}{[P(t)]} \cdot E(X(t), t) \right)^2 \quad (156)$$

for each label (a, π) . Differentiating in time, we deduce

$$\begin{aligned} \frac{d}{dt} \log(f \exp Ar) &= A \frac{dr}{dt} + \frac{d}{dt} \log f \\ &\leq Ak + h(5\sqrt{1+r^2} - Ar^2)q + hk^2 \\ &\leq Ak + h(2 + 9A^{-2})k^2 \end{aligned} \quad (157)$$

We used $q \leq k^2$. From here, we conclude as in the other case. \square

REMARK 3. The conclusion of Proposition 8 implies there exists a constant C_n such that $m_n(x, t) \leq C_n$ holds, for every $n \geq 0$.

PROPOSITION 9. *Consider the RVM equations with radiation reaction force F_{IC} or F_{LL} and let (f, E, B) be a smooth solution of the RVM equations on $[0, T]$. Assume there is some $A > 0$ such that*

$$\sup_{x,p} e^{A|p|} f_0(x, p) < \infty$$

and that

$$\int_0^T \|\mathbf{K}(t)\|_{L^\infty}^2 dt =: \Gamma < \infty.$$

Then, there exists a constant K_∞ depending on $A, A\Gamma, \Gamma h$, the initial data and T so that

$$\|\mathbf{K}(t)\|_{L^\infty} \leq K_\infty$$

holds for all $t \leq T$.

PROOF. Let us recall the Glassey-Strauss representation

$$\mathbf{K}(x, t) = (\mathbf{K}_T + \mathbf{K}_S)(x, t) + O(1) \quad (158)$$

where $O(1)$ represents a smooth function bounded by the initial data. Estimating \mathbf{K}_T is independent of the choice of F_R ,

$$|\mathbf{K}_T(x, t)| \leq C_1 T. \quad (159)$$

The term \mathbf{K}_S depends on the nonlinearity $Ff = (F_L + F_R)f$. In either case $F_R = F_{IC}$ or $F_R = F_{LL}$, we have the estimate

$$|F|f \leq 2|\mathbf{K}|f + 2h[p]^2|\mathbf{K}|^2f. \quad (160)$$

This implies directly

$$\begin{aligned} \left| \int a_S S f dp \right| &= \left| \int (\nabla_p a_S) F f dp \right| \leq C \int [p](|\mathbf{K}| + h[p]^2|\mathbf{K}|^2)f dp \\ &\leq C_1 |\mathbf{K}| + hC_3 |\mathbf{K}|^2 \end{aligned} \quad (161)$$

and so

$$\begin{aligned} |\mathbf{K}_S(x, t)| &\leq C \int_0^t (t-s)(C_1 \|\mathbf{K}(s)\|_{L^\infty} + C_3 \|\mathbf{K}(s)\|_{L^\infty}^2) ds \\ &\leq CT\Gamma(C_1 + hC_3) \end{aligned} \quad (162)$$

where we used Cauchy-Schwarz on the time integral with C_1 . \square

REMARK 4. In Proposition 9, we needed only bounds for Γ and m_n for $n \leq 3$ to hold.

With uniform control of \mathbf{K} and m_n in $L^\infty(dx)$, controlling derivatives is straightforward. This is because in either model F_{IC} or F_{LL} for the radiation reaction force, one observes the pointwise bounds

$$|\nabla_x F| \leq C(1 + h[p]^2|\mathbf{K}|)|\nabla_x \mathbf{K}| \quad (163)$$

and

$$|\partial_t F| \leq C(1 + h[p]^2|\mathbf{K}|)(m_0 + |\nabla_x \mathbf{K}|) \quad (164)$$

where C is an independent constant. In the second bound we recalled the Maxwell equations. Following the classical analysis of the representation for $\nabla_x \mathbf{K}$ (e.g., [13]), is the estimate

PROPOSITION 10. Consider the RVM equations with radiation reaction force F_{IC} or F_{LL} and let (f, E, B) be a smooth solution of the RVM equations on $[0, T]$. Assume there is some $A > 0$ such that

$$\sup_{x,p} e^{A|p|} f_0(x, p) < \infty$$

and that

$$\int_0^T \|\mathbf{K}(t)\|_{L^\infty}^2 dt =: \Gamma < \infty.$$

Then, there exists a constant C depending only on $A, A\Gamma, \Gamma h$, the initial data and T so that

$$\|\nabla_x \mathbf{K}(t)\|_{L^\infty} \leq C(1 + \log G_1(t))$$

holds for all $t \leq T$.

REMARK 5. In Proposition 10, we needed only bounds for K_∞ and m_n for $n \leq 8$ to hold.

PROPOSITION 11. Consider the RVM equations with radiation reaction force F_{IC} or F_{LL} and let (f, E, B) be a smooth solution of the RVM equations on $[0, T]$. Assume there is some $A > 0$ such that

$$\sup_{x,p} e^{A|p|} f_0(x, p) < \infty$$

and

$$\sup_{x,p} e^{A|p|} (|\nabla_x f_0(x, p)| + [p]^2 |\nabla_p f_0(x, p)|) < \infty,$$

and assume that

$$\int_0^T \|\mathbf{K}(t)\|_{L^\infty}^2 dt =: \Gamma < \infty.$$

Then, there exists constants C depending only on $A, A\Gamma$ and Γh , the initial data and T such that

$$|\nabla_x E(x, t)| + |\nabla_x B(x, t)| \leq C \exp(Ct)$$

and, for any $\delta > 0$,

$$|\nabla_x f(x, p, t)| + [p]^2 |\nabla_p f(x, p, t)| \leq C e^{(-A+\delta)[p]} \exp(C \exp(Ct))$$

holds for all x, p and $t \leq T$.

PROOF. We write (1) as

$$D_t f = -(\operatorname{div}_p F_R) f \tag{165}$$

and take derivatives in x and in p :

$$D_t(\partial_{x_i} f) = -(\partial_{x_i} F) \cdot \nabla_p f - (\operatorname{div}_p F_R)(\partial_{x_i} f) - (\partial_{x_i}(\operatorname{div}_p F_R)) f \tag{166}$$

and

$$D_t(\partial_{p_i} f) = -(\partial_{p_i} v) \cdot \nabla_x f - (\operatorname{div}_p F_R)(\partial_{p_i} f) - (\partial_{p_i} F) \cdot \nabla_p f - (\partial_{p_i}(\operatorname{div}_p F_R))f \quad (167)$$

In either case $F_R = F_{IC}$ or $F_R = F_{LL}$, it holds

$$|\nabla_x \operatorname{div}_p F_R| \leq hC[p]|\mathbf{K}||\nabla_x \mathbf{K}| \quad (168)$$

and

$$|\nabla_p \operatorname{div}_p F_R| \leq hC|\mathbf{K}|^2 \quad (169)$$

where C is an independent constant. We consider here $h = 1$, without loss of generality. We will show the quantity

$$w = |\nabla_x f| + [p]^2 |\nabla_p f| \quad (170)$$

remains controlled.

We multiply (166) by $\partial_{x_i} f / |\nabla_x f|$ and add in i to obtain,

$$\begin{aligned} D_t |\nabla_x f| &\leq |\nabla_x F| |\nabla_p f| + |\operatorname{div}_p F_R| |\nabla_x f| + |\nabla_x \operatorname{div}_p F_R| f \\ &\leq (|\operatorname{div}_p F_R| + [p]^{-2} |\nabla_x F|)g + |\nabla_x \operatorname{div}_p F_R| f \\ &\leq CK_\infty([p]K_\infty + |\nabla_x \mathbf{K}|)g + CK_\infty |\nabla_x \mathbf{K}| [p]f \end{aligned} \quad (171)$$

where we used estimate (168). Then, we multiply (167) by $[p]^2 \partial_{p_i} f / |\nabla_p f|$ and add in i to obtain

$$\begin{aligned} [p]^2 D_t |\nabla_p f| &\leq 2[p] |\nabla_x f| + [p]^2 |\operatorname{div}_p F| |\nabla_p f| + [p]^2 |\nabla_p \operatorname{div}_p F| f \\ &\leq (2[p] + |\operatorname{div}_p F|)g + [p]^2 |\nabla_p \operatorname{div}_p F| f \\ &\leq [p](2 + CK_\infty^2)g + C[p]^2 K_\infty^2 f. \end{aligned} \quad (172)$$

By assumption, the estimates of Proposition 8 and Proposition 9 hold. In particular K_∞ is finite and there exists a constant C_K depending only on K_∞ so that

$$D_t w \leq C_K([p] + |\nabla_x \mathbf{K}|)(w + [p]f) \quad (173)$$

where we combined estimates (171) and (172) and used $D_t [p] \leq K_\infty$.

Denote now

$$z = (w + \alpha e^{-\beta[p]})e^{A_1[p]} \quad (174)$$

where we choose positive α, β as follows. Choosing $\beta \geq A_1$, we have

$$D_t z \leq C_K([p] + |\nabla_x \mathbf{K}|)(z + e^{A_1[p]}[p]f) + 1. \quad (175)$$

Then for any $A_1 < A$, we apply the estimate of Proposition 8 so

$$D_t z \leq C_0([p] + |\nabla_x \mathbf{K}|)(z + 1) \quad (176)$$

where C_0 depends only on A, Γ and the initial data. Now we express $[p]$ in terms of $\log z$. In fact,

$$A_1 [p] = \log z - \log(w + \alpha e^{-\beta[p]}) \leq \log z + [p]\beta \log \frac{1}{\alpha} \quad (177)$$

and if we take $\beta \log \frac{1}{\alpha} = \frac{A_1}{2}$ we may write (176) as

$$D_t \log(z + 1) \leq \frac{C_0}{A_1} \log(z + 1) + C_0 |\nabla_x \mathbf{K}|. \quad (178)$$

Defining now

$$Z(t) = \sup_{s \leq t} \|z(s)\|_{L^\infty}. \quad (179)$$

Taking the supremum in time of (178), we find

$$\sup_{s \leq t} \|D_t \log(z(s) + 1)\|_{L^\infty} \leq \frac{C_0}{A_1} \log(Z(t) + 1) + C_0 \sup_{s \leq t} \|\nabla_x \mathbf{K}(s)\|_\infty. \quad (180)$$

By Lemma 1, we have that

$$\frac{d}{dt} \log(Z + 1) \leq \liminf_{\varepsilon \rightarrow 0} \sup_{s \leq t + \varepsilon} \|D_t \log(z(s) + 1)\|_{L^\infty} \quad (181)$$

holds for almost all t . On the other hand, by Proposition 10

$$\sup_{s \leq t} \|\nabla_x \mathbf{K}(s)\|_\infty \leq C_T \log(Z(t) + 1). \quad (182)$$

where C_T depends only on A , Γ , the initial data and T . Then we conclude after applying Grönwall to

$$\frac{d}{dt} \log(Z + 1) \leq C \log(Z + 1). \quad (183)$$

□

PROPOSITION 12. *Consider the RVM equations with radiation reaction force F_{IC} or F_{LL} and let (f, E, B) be a smooth solution of the RVM equations on $[0, T]$. Assume that the initial data are compactly supported in space and momentum, i.e., there exists $R_0 > 0$ such that $f_0(x, p) = 0$ for $|x| + |p| > R_0$, $|E_0(x)| + |B_0(x)| = 0$ for $|x| > R_0$. Assume that the initial data are smooth, $f_0 \in H^s(\mathbb{R}^6)$, $(E_0, B_0) \in H^s(\mathbb{R}^3)^2$, for $s > 4$. Assume that (142) is satisfied, and that*

$$\int_0^T \|\mathbf{K}(t)\|_{L^\infty}^2 dt =: \Gamma < \infty.$$

Then, there exists a constant R_T depending on Γ , R_0 and T such that the solution (f, E, B) is compactly supported, $f(x, p, t) = 0$ for $|x| + |p| > R_0 + R_T$, $|E(x, t)| + |B(x, t)| = 0$ for $|x| > R_0 + R_T$. Moreover, for each $s > 4$ there exists a constant C_s depending on the initial data, Γ , R_0 and T such that

$$\sup_{0 \leq t \leq T} (\|f(t)\|_{H^s(\mathbb{R}^6)} + \|(E, B)(t)\|_{H^s(\mathbb{R}^3)^2}) \leq C_s < \infty.$$

PROOF. Because the initial data for f are compactly supported in p , the condition (142) is satisfied. In view of Proposition 9 we know that the electromagnetic fields are bounded, $|\mathbf{K}(x, t)| \leq K_\infty$. Then, on characteristics, because $F_R \cdot v \leq 0$ we have that $[P(a, \pi, t)] \leq R_0 + tK_\infty$. This implies that the support of f remains included in $|x| \leq R_0 + T$ (the speed of X on characteristics is bounded by 1) and $|p| \leq R_0 + R_T$ because the speed of P is bounded by a constant (as $|F| \leq C(1 + |p|^2)K_\infty^2$). We take now a smooth cutoff $\chi(p)$ that is identically 1 for $|p| \leq R_0 + R_T + 1$ and is compactly supported, and define variables $U(x, p, t)$,

$$U(x, p, t) = (\chi(p)E(x, t), \chi(p)B(x, t), f(x, p, t)) \in \mathbb{R}^7 \quad (184)$$

In view of Propositions 8, 10 and 11 we know

$$\sup_{0 \leq t \leq T} \|U(\cdot, t)\|_{W^{1,\infty}(\mathbb{R}^6)^7} \leq G < \infty \quad (185)$$

Now the RVM equations imply that U solves

$$\partial_t U + (A_j(U, y)\partial_j U = S(y, t) \quad (186)$$

where $y = (x, p) \in \mathbb{R}^6$, $j = 1, \dots, 6$, $A_j(U, y) = A_j(U, y)^T$ are symmetric 7×7 matrices depending smoothly on U and y . For each $j = 1, 2, 3$ the 6×6 block corresponding to the electromagnetic fields has constant entries. The only nonzero entry on the seventh row is at the SE corner and corresponds to v_j written in terms of the variables y . For $j = 4, 5, 6$ the 6×6 block corresponding to the EM fields vanishes, and the only nonzero entry on the seventh row is at the SE corner and is the total force F_j which is a quadratic expression in U and y . The source term corresponds to the undifferentiated terms in RVM. These include the contribution of the current j from the form $-\chi j$, the terms coming from $(\nabla_p \chi)(E, B)$, and the term $-(\nabla_p \cdot F_R)f$. Because of the fact that A_j are symmetric the system (186) is a forced quasilinear hyperbolic system. We took $s > 1 + \frac{6}{2}$ so that $H^s(\mathbb{R}^6) \subset W^{1,\infty}(\mathbb{R}^6)$ is a Banach algebra. Because the nonlinearity in the RVM is polynomial (at most cubic in fact) and in view of well-known calculus inequalities in H^s

$$\|\phi\psi\|_{H^s} \leq C\|\phi\|_{L^\infty}\|\psi\|_{H^s} + \|\psi\|_{L^\infty}\|\phi\|_{H^s} \quad (187)$$

and the bounds (185) we have that

$$\begin{aligned} & \frac{d}{dt} \|U(t)\|_{H^s(\mathbb{R}^6)^7} \\ & \leq (\|\operatorname{div} A(U, y)\|_{L^\infty} + C(G^2 + 1)) \|U(t)\|_{H^s(\mathbb{R}^6)^7} + \|S(t)\|_{H^s(\mathbb{R}^6)^7} \\ & \leq C_1(G^2 + 1) \|U(t)\|_{H^s(\mathbb{R}^6)^7} + \|S(t)\|_{H^s(\mathbb{R}^6)^7}, \end{aligned} \quad (188)$$

where $\operatorname{div}A(u) = \sum_j \partial_j(A_j(U, y))$. In (188) we used the fact due to (185) that we have

$$\|\operatorname{div}A(U, y)\|_{L^\infty} \leq C(G^2 + 1). \quad (189)$$

Because of the compact support of (f, E, B) , we have

$$\|S(t)\|_{H^s(\mathbb{R}^6)^7} \leq C\|U(t)\|_{H^s(\mathbb{R}^6)^7} \quad (190)$$

Applying a Grönwall inequality to (188) we conclude that $\|U(t)\|_{H^s(\mathbb{R}^6)^7}$ is bounded a priori on $[0, T]$. \square

PROPOSITION 13. *Let (f, E, B) be a smooth solution of RVM with the radiation reaction force F_R given in Definition 1. Assume that the initial data are compactly supported in space and momentum, i.e., there exists $R_0 > 0$ such that $f_0(x, p) = 0$ for $|x| + |p| > R_0$, $|E_0(x)| + |B_0(x)| = 0$ for $|x| > R_0$. Assume that the initial data are smooth, $f_0 \in H^s(\mathbb{R}^6)$, $(E_0, B_0) \in H^s(\mathbb{R}^3)^2$, for $s > 4$. Then, there exists a constant R_T depending on K_∞ , R_0 and T such that the solution (f, E, B) is compactly supported, $f(x, p, t) = 0$ for $|x| + |p| > R_0 + R_T$, $|E(x, t)| + |B(x, t)| = 0$ for $|x| > R_0 + R_T$. Moreover, for each $s > 4$ there exists a constant C_s depending on the initial data, K_∞ , R_0 and T such that*

$$\sup_{0 \leq t \leq T} (\|f(t)\|_{H^s(\mathbb{R}^6)} + \|(E, B)(t)\|_{H^s(\mathbb{R}^3)^2}) \leq C_s < \infty. \quad (191)$$

PROOF. The proof is identical with the proof of Proposition 12. The main point here is that the bounds are unconditional, because of Theorem 1. \square

Appendix A. Checking the nonlinear Glassey-Strauss representation

Here we derive (33) and (35) and check the properties (34) and (36). The expressions for E coming from S , E_S are [11] p.63,

$$\begin{aligned} & (E_S)_i \\ &= - \int dp \int_0^t \int_{|\omega|=1} \left(\frac{\omega_i + v_i}{1 + (\omega \cdot v)} \right) (Sf)(x - r\omega, p, t - r) r dr dS(\omega) \end{aligned} \quad (192)$$

where $\omega = \widehat{y - x}$. Using the equation (1), denoting

$$N(y, p, s) = F(y, p, s)f(y, p, s), \quad (193)$$

and integrating by parts in (192) we obtain

$$\begin{aligned} & (E_S)_i \\ &= - \int dp \int_0^t \int_{|\omega|=1} \partial_{p_j} \left(\frac{\omega_i + v_i}{1 + (\omega \cdot v)} \right) N_j(x - r\omega, p, t - r) r dr dS(\omega) \end{aligned} \quad (194)$$

which we write as

$$\begin{aligned} & (E_S)_i \\ &= - \int dp \int_0^t \frac{1}{t-s} \int_{|x-y|=t-s} N(y, s) \cdot \nabla_p \left(\frac{\omega_i + v_i}{1 + (\omega \cdot v)} \right) dS(y) ds \end{aligned} \quad (195)$$

The expressions (195) for E_S are nonlinear because they employ (1). The expression for E_T [11] p. 63 is

$$\begin{aligned} & (E_T)_i \\ &= - \int dp \int_0^t \frac{1}{(t-s)^2} \int_{|x-y|=t-s} f(y, s) \frac{1}{[p]^2} \left(\frac{\omega_i + v_i}{(1 + (\omega \cdot v))^2} \right) dS(y) ds \end{aligned} \quad (196)$$

Note that E_T is linear in f , because it comes without use of the equation of evolution of f . There are analogous representations for B . The main point here is to verify (34) and (36). We observe that

$$\partial_{p_i} v_k = \frac{1}{\sqrt{1 + |p|^2}} (\delta_{ik} - v_i v_k) = [p]^{-1} (\mathbb{I} - v \otimes v)_{ik} \quad (197)$$

and

$$|v|^2 = 1 - \frac{1}{[p]^2}. \quad (198)$$

We note the following facts. First,

$$\partial_{p_j} \left(\frac{1}{1 + \omega \cdot v} \right) = \frac{v_j}{[p](1 + \omega \cdot v)} - \frac{\omega_j + v_j}{[p](1 + \omega \cdot v)^2} \quad (199)$$

and

$$\partial_{p_j} \left(\frac{\omega_i + v_i}{1 + (\omega \cdot v)} \right) = \frac{1}{[p]} \frac{(\delta_{ij} + v_j \omega_i)}{1 + (\omega \cdot v)} - \frac{1}{[p]} \frac{(\omega_i + v_i)(\omega_j + v_j)}{(1 + (\omega \cdot v))^2}. \quad (200)$$

These are done by direct calculation, inserting $\omega + v$ terms. The second observation is that

$$\frac{|\omega + v|^2}{(1 + (\omega \cdot v))^2} = \frac{(1 - |v|)^2 + 2|v|\delta}{(1 - |v|)^2 + |v|^2 \delta^2 + 2(1 - |v|)|v|\delta} \quad (201)$$

where

$$\delta = 1 + \omega \cdot \hat{p} = 1 + \cos \theta. \quad (202)$$

Multiplying the numerator by $1 - |v|$ and using $(1 - |v|)^3 \leq (1 - |v|)^2$ in the numerator we see that the resulting fraction is less than 1, and therefore, after taking square roots we have,

$$\frac{|\omega + v|}{1 + (\omega \cdot v)} \leq \sqrt{2}[p] \quad (203)$$

where we used

$$(1 - |v|)^{-1} = [p]^2(1 + |v|) \leq 2[p]^2. \quad (204)$$

Also, from $1 + (\omega \cdot v) = 1 + |v| \cos \theta \geq 1 - |v|$ and (204) we have that

$$0 \leq \frac{1}{1 + \omega \cdot v} \leq 2[p]^2. \quad (205)$$

Thus, the second term in (200) obeys

$$\left| \frac{1}{[p]} \frac{(\omega_i + v_i)(\omega_j + v_j)}{(1 + (\omega \cdot v))^2} \right| \leq 2[p] \quad (206)$$

and the first term in (200) is bounded in view of (205) by $4[p]$. This implies

$$\left| \partial_{p_j} \left(\frac{(\omega_i + v_i)}{1 + (\omega \cdot v)} \right) \right| \leq 6[p]. \quad (207)$$

Note also that

$$\left| \frac{1}{[p]^2} \left(\frac{\omega_i + v_i}{(1 + (\omega \cdot v))^2} \right) \right| \leq 2\sqrt{2}[p]. \quad (208)$$

We verified thus the bounds (34) and (36) in the representation of the electric field. After use of the equation (1) and integration by parts, the magnetic field representation [11] p.63, yields

$$B_S = \int dp \int_0^t \frac{1}{t-s} \int_{|x-y|=t-s} N(y, s) \cdot \nabla_p \left(\frac{\omega \times v}{1 + (\omega \cdot v)} \right) dS(y) ds \quad (209)$$

From (199) and the inequalities (203) and (205) and because $|\omega \times v| \leq |\omega + v|$ we have

$$\left| \nabla_p \left(\frac{\omega \times v}{1 + (\omega \cdot v)} \right) \right| \leq 10[p] \quad (210)$$

Finally, the representation of B_T from [11] is

$$B_T = \int dp \int_0^t \frac{1}{(t-s)^2} \int_{|x-y|=t-s} \left(\frac{\omega \times v}{[p]^2(1 + (\omega \cdot v))^2} \right) f(y, s) dS(y) ds \quad (211)$$

and we have

$$\left| \frac{\omega \times v}{[p]^2(1 + (\omega \cdot v))^2} \right| \leq 2\sqrt{2}[p], \quad (212)$$

concluding the verification of the inequalities (34) and (36).

Appendix B. Proof of ODE Lemmas

We prove here Lemma 1 and Lemma 2.

PROOF OF LEMMA 1. If $G(t) = g(s)$ with $s < t$, then $g(s') \leq g(s)$ for all $s \leq s' \leq t$ (otherwise, $G(t)$ would have been attained at s' not at s) and therefore $G(s') = g(s)$ for $s' \in [s, t]$ and the left derivative of $G'(t - 0)$ of G at t vanishes. If $g(t) < G(t)$ then $G(s) = G(t)$ for a small interval of $s > t$ and so $G'(t) = 0$.

If $g(t) = G(t)$ then for any $\varepsilon > 0$ we have

$$g(s) - g(t) \leq (s - t)L_\varepsilon \quad (213)$$

for all $t < s \leq t + \varepsilon$, where $L_\varepsilon = \sup_{t \leq s \leq t + \varepsilon} |g'(s)|$. We take $0 < h < \varepsilon$, write $g(s) \leq g(t) + hL_\varepsilon$ for $s \leq t + h$ and take the supremum in s to deduce

$$G(t + h) \leq G(t) + hL_\varepsilon. \quad (214)$$

Thus $G'(t + 0) \leq L_\varepsilon$. Because $\varepsilon > 0$ is arbitrary, we have

$$G'(t + 0) \leq \liminf_{\varepsilon \rightarrow 0} L_\varepsilon.$$

Finally, if $G(t) = g(t)$ and $g(s) < G(t)$ for all $s < t$ then

$$G(s_0) \leq G(t) + \sup_{s' \leq t} |g'(s')|(t - s_0) \quad (215)$$

holds by taking supremum of

$$g(s) \leq g(t) + \sup_{s' \leq t} |g'(s')|(t - s) \quad (216)$$

for $s \leq s_0 < t$. This concludes the argument. \square

PROOF OF LEMMA 2. Consider $\tilde{Z} = Z \log Z$. The differential inequality for W then reads

$$\frac{dW}{dt} \leq C((\log W)W + (\log W)\tilde{Z}) \quad (217)$$

and from the differential inequality for Z we have

$$\frac{d\tilde{Z}}{dt} \leq C(\log Z + 1)(W + \tilde{Z}). \quad (218)$$

Now take $\overline{W} = W + \tilde{Z}$. Because $\tilde{Z} \geq Z$ we have $\log Z \leq \log \tilde{Z} \leq \log \overline{W}$. We also have $\log W \leq \log \overline{W}$, so we obtain

$$\frac{d\overline{W}}{dt} \leq C(\log \overline{W} + 1)\overline{W} \quad (219)$$

and thus \overline{W} is bounded by a double exponential of time. \square

Appendix C. Divergence of the Landau-Lifshitz force

Here we compute the divergence in p of the radiation reaction force

$$F_{LL} = -hv[p]^2(|F_L|^2 - (v \cdot E)^2). \quad (220)$$

Setting $h = 1$, we calculate

$$\begin{aligned} -\operatorname{div}_p F_{LL} &= \frac{\partial}{\partial p_i} ([p]^2(|F_L|^2 - (v \cdot E)^2)v_i) \\ &= \left(\frac{\partial v_i}{\partial p_i} + v_i \frac{\partial}{\partial p_i} \right) ([p]^2(|F_L|^2 - (v \cdot E)^2)) \\ &= \left([p](3 - |v|^2) + v_i \frac{\partial}{\partial p_i} [p]^2 + [p]^2 v_i \frac{\partial}{\partial p_i} \right) (|F_L|^2 - (v \cdot E)^2). \end{aligned} \quad (221)$$

The first directional derivative is

$$\begin{aligned} v \cdot \nabla_p [p]^2 &= v_i \frac{\partial}{\partial p_i} [p]^2 \\ &= 2[p]v_i \frac{\partial}{\partial p_i} [p] \\ &= 2[p]|v|^2. \end{aligned} \quad (222)$$

The others two terms are

$$(v \cdot \nabla_p)(|F_L|^2 - (v \cdot E)^2) = (v \cdot \nabla_p)|F_L|^2 - (v \cdot \nabla_p)(v \cdot E)^2. \quad (223)$$

The first term is

$$\begin{aligned} (v \cdot \nabla_p)|F_L|^2 &= 2F_L \cdot (v \cdot \nabla_p)F_L \\ &= 2(F_L)_j v_i \frac{\partial}{\partial p_i} (F_L)_j \\ &= 2(F_L)_j v_i \frac{\partial}{\partial p_i} (\epsilon_{jmn} v_m B_n) \\ &= \frac{2}{[p]} (F_L)_j v_i \epsilon_{jmn} (\delta_{im} - v_i v_m) B_n \\ &= \frac{2}{[p]} (F_L)_j \epsilon_{jmn} (1 - |v|^2) v_m B_n \\ &= \frac{2}{[p]^3} F_L \cdot (v \times B). \end{aligned} \quad (224)$$

The second term is

$$\begin{aligned}
(v \cdot \nabla_p)(v \cdot E)^2 &= 2(v \cdot E)(v \cdot \nabla_p)(v \cdot E) \\
&= 2(v \cdot E)v_i \frac{\partial}{\partial p_i}(v_j E_j) \\
&= 2(v \cdot E)v_i \frac{\partial v_j}{\partial p_i} E_j \\
&= \frac{2}{[p]}(v \cdot E)v_i(\delta_{ij} - v_i v_j)E_j \\
&= \frac{2}{[p]}(v \cdot E)(1 - |v|^2)v_j E_j \\
&= \frac{2}{[p]^3}(v \cdot E)^2.
\end{aligned} \tag{225}$$

Thus we find

$$(v \cdot \nabla_p)(|F_L|^2 - (v \cdot E)^2) = \frac{2}{[p]^3}(F_L \cdot (v \times B) - (v \cdot E)^2) \tag{226}$$

$$= \frac{2}{[p]^3}(|F_L|^2 - (v \cdot E)^2 - E \cdot F_L). \tag{227}$$

We simplify

$$[p](3 - |v|^2) + 2[p]|v|^2 = [p](4 - (1 - |v|^2)) = [p]\left(4 - \frac{1}{[p]^2}\right). \tag{228}$$

Then adding the terms yields

$$-\operatorname{div}_p F_{LL} = [p]\left(4 + \frac{1}{[p]^2}\right)(|F|^2 - (v \cdot E)^2) - \frac{2}{[p]}(E \cdot F_L). \tag{229}$$

Data Availability. No datasets were generated or analyzed in this paper.

Conflict of Interest. The authors declare that they have no conflict of interest.

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