Remarks on a Liouville-type theorem for Beltrami flows

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Abstract

We present a simple, short and elementary proof that if v is a Beltrami flow with a finite energy in $\mathbb{R}^3$ then $v = 0$. In the case of the Beltrami flows satisfying $v \in L^\infty_{loc}(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ with $q \in [2, 3)$, or $|v(x)| = O(1/|x|^{1+\varepsilon})$ for some $\varepsilon > 0$, we provide a different, simple proof that $v = 0$.

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1 Introduction

Ideal homogeneous incompressible inviscid fluid flows are governed by the Euler equations:

$$\text{(E)} \begin{cases} \dfrac{\partial v}{\partial t} + (v \cdot \nabla) v = -\nabla p, \quad (x,t) \in \mathbb{R}^3 \times (0,\infty) \\ \text{div } v = 0, \quad (x,t) \in \mathbb{R}^3 \times (0,\infty) \end{cases}$$

where $v = (v_1, \cdots, v_n)$, $v_j = v_j(x,t)$, $j = 1, \cdots, n$, $n \geq 2$, is the velocity of the flow, $p = p(x,t)$ is the scalar pressure. Let R_j, $j = 1, \cdots, n$, denote the Riesz transforms, given by

$$R_j(f)(x) = C_n \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{(x_j - y_j) f(y)}{|x-y|^{n+1}} dy, \quad C_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}.$$

The pressure in the Euler (and Navier-Stokes) equations is given in terms of the velocity up to addition of a harmonic function by

$$p = \sum_{j,k=1}^{n} R_j R_k (v_j v_k). \tag{1.1}$$

This is easily seen by taking the divergence of (E). In [2](see also [1]) the following result is obtained.

Theorem 1.1 *If (v,p) satisfies (1.1) and $|p| + |v|^2 \in L^1(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} v_j v_k dx = -\delta_{jk} \int_{R^n} p\, dx. \tag{1.2}$$

In the next section we present a simple proof of this result using the continuity of the Fourier transform of functions belonging to $L^1(\mathbb{R}^n)$. In order to see the implications of the above theorem for Beltrami flows, let us recall that in the stationary case in $\mathbb{R}^3$, the first equations of (E) can be rewritten as

$$v \times \omega = \nabla(p + \frac{1}{2}|v|^2), \quad \omega = \text{curl}\, v. \tag{1.3}$$

A vector field v is called a Beltrami flow if there exists a function $\lambda = \lambda(x)$ such that

$$\omega = \lambda v. \tag{1.4}$$

Therefore, if v is a Beltrami flow, then the pair (v,p) is a solution of the stationary Euler equations if

$$p+\frac{1}{2}|v|^2=c, \quad c=\text{constant}. \tag{1.5}$$

We call such a solution (v,p) a "Beltrami solution" of the stationary Euler equations. We refer to [3] for a recent interesting result regarding the Beltrami flows. Recently, Nadirashvili proved a Liouville type property of Beltrami flows ([4]). He showed that a Beltrami solution (v,p) satisfying either $v\in L^q(\mathbb{R}^3)$, $2\leq q\leq 3$, or $|v(x)|=o(1/|x|)$ is necessarily trivial, $v=0$. In the case of finite energy Beltrami flows we have the following immediate consequence of Theorem 1.1:

Theorem 1.2 *Let (v,p) be a Beltrami solution of the stationary Euler equations with the λ given in (1.4). If $v\in L^2(\mathbb{R}^3)$, then $v=0$. The same conclusion holds, for instance, if there exists $q\in[\frac{6}{5},\infty]$ such that $v\in L^q(\mathbb{R}^3)$ and $\lambda\in L^{\frac{6q}{5q-6}}(\mathbb{R}^3)$ (if $v\in L^{\frac{6}{5}}(\mathbb{R}^3)$, then we require $\lambda\in L^\infty(\mathbb{R}^3)$).*

We have also the following result for the cases considered in the paper [4], for which we present a different, simple proof.

Theorem 1.3 *Let $v\in L^\infty_{loc}(\mathbb{R}^3)$ be a Beltrami solution of the stationary Euler equations satisfying either $v\in L^q(\mathbb{R}^3)$ for some $q\in[2,3)$, or that there exists $\varepsilon>0$ such that $|v(x)|=O(1/|x|^{1+\varepsilon})$ as $|x|\to\infty$. Then, $v=0$.*

2 Proof of the Theorems

We use the notation for the Fourier transform of $f(x)$

$$\mathcal{F}(f)(\xi)=\widehat{f}(\xi)=\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}f(x)e^{-ix\cdot\xi}dx,$$

whenever the right hand side is defined. In terms of the Fourier transform the Riesz transform is defined as

$$\widehat{R_j(f)}(\xi)=\frac{i\xi_j}{|\xi|}, \quad i=\sqrt{-1}.$$

Proof of Theorem 1.1 Without loss of generality we may restrict ourselves to the stationary case, $v(x,t)=v(x), p(x,t)=p(x)$. By the Fourier transform one has

$$\hat{p}(\xi)=-\sum_{j,k=1}^{n}\frac{\xi_j\xi_k}{|\xi|^2}\widehat{v_jv_k}(\xi). \tag{2.1}$$

We note that $\widehat{p}(\xi)$ and $\widehat{v_jv_k}(\xi)$, $j,k=1,\cdots,n$, are continuous at $\xi=0$ from the hypothesis, $|p|+|v|^2\in L^1(\mathbb{R}^n)$. Let w be a given constant vector with $|w|=1$. We put $\xi=\rho w$ in (2.1), and pass $\rho\to 0$ to obtain

$$\int_{\mathbb{R}^n} p\,dx=-\int_{\mathbb{R}^n}(v\cdot w)^2dx. \tag{2.2}$$

If we plug $w=\mathbf{e}^j$ in (2.2), where $\mathbf{e}^j$ is the canonical basis of $\mathbb{R}^n$ with its components given by $(\mathbf{e}^j)_k=\delta_{jk}$, then we have

$$\int_{\mathbb{R}^n} p\,dx=-\int_{\mathbb{R}^n} v_j^2dx \quad \forall j=1,\cdots,n.$$

On the other hand, for $j\neq k$, if we put $w=\frac{\mathbf{e}^j+\mathbf{e}^k}{\sqrt{2}}$ in (2.2), we obtain $\int_{\mathbb{R}^n} v_jv_kdx=0$. $\square$

Proof of Theorem 1.2 Since (v,p) is a Beltrami solution of stationary Euler equations, we have $p-c=-\frac{1}{2}|v|^2:=\tilde{p}$ for some constant c. In the case $v\in L^2(\mathbb{R}^3)$ we find that $|v|^2+|\tilde{p}|\in L^1(\mathbb{R}^3)$, and by Theorem 1.1 we obtain

$$\int_{\mathbb{R}^3}\tilde{p}\,dx=-\frac{1}{3}\int_{\mathbb{R}^3}|v|^2dx=-\frac{1}{2}\int_{\mathbb{R}^3}|v|^2dx,$$

which implies that $v=0$, and $\tilde{p}=\sum_{j,k=1}^n R_jR_k(v_jv_k)=0$. On the other hand, if $v\in L^q(\mathbb{R}^3)$ and $\lambda\in L^{\frac{6q}{5q-6}}(\mathbb{R}^3)$ with $\frac{6}{5}<q\leq\infty$, or $v\in L^{\frac{6}{5}}(\mathbb{R}^3)$ and $\lambda\in L^\infty(\mathbb{R}^3)$, then we estimate

$$\begin{aligned}\|v\|_{L^2} &\leq C\|\nabla v\|_{L^{\frac{6}{5}}}\leq C\|\omega\|_{L^{\frac{6}{5}}}=C\|\lambda v\|_{L^{\frac{6}{5}}}\\ &\leq C\|\lambda\|_{L^{\frac{6q}{5q-6}}}\|v\|_{L^q}<\infty,\end{aligned}$$

and we reduce to the above case of $v\in L^2(\mathbb{R}^3)$. $\square$

Proof of Theorem 1.3 We first observe that our hypothesis implies that

$$\int_{\mathbb{R}^3}|v|^2|x|^{\mu-2}dx<\infty. \tag{2.3}$$

for some $\mu \in (1,2)$. Indeed, the case $|v(x)| = O(1/|x|^{1+\varepsilon})$ as $|x| \to \infty$ is obvious, while in the case $v \in L^q(\mathbb{R}^3)$ for some $q \in [2,3)$, we have the following estimate,

$$\int_{\{|x|\geq 1\}} |v|^2 |x|^{\mu-2} dx \leq C\|v\|_{L^q}^2 \left(\int_1^\infty r^{2+\frac{q(\mu-2)}{q-2}} dr \right)^{\frac{q-2}{q}} < \infty$$

for μ with $1 < \mu < \frac{6}{q} - 1$. Let us introduce a standard radial cut-off function $\sigma \in C_0^\infty(\mathbb{R}^N)$ such that

$$\sigma(|x|) = \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 2, \end{cases} \tag{2.4}$$

and $0 \leq \sigma(x) \leq 1$ for $1 < |x| < 2$. Then, for each $R > 0$, we define $\sigma\left(\frac{|x|}{R}\right) := \sigma_R(|x|) \in C_0^\infty(\mathbb{R}^N)$. A Beltrami solution (v,p) with $p = -\frac{1}{2}|v|^2 + C$ satisfies

$$\sum_{j,k=1}^3 \int_{\mathbb{R}^3} v_j v_k \partial_j \partial_k \varphi dx = \frac{1}{2} \int_{\mathbb{R}^3} |v|^2 \Delta\varphi dx \quad \forall \varphi \in C_0^2(\mathbb{R}^3). \tag{2.5}$$

We choose our test function $\varphi(x) = \varphi_{\delta,R}(x) = (|x|^{2\mu} + \delta)^{\frac{1}{2}} \sigma_R$ for $\delta, R > 0$ in (2.5), which is an approximation of $\varphi = |x|^\mu$, and passing first $\delta \to 0$, and then $R \to \infty$, using continuity of integrals and the dominated convergence theorem, taking (2.3) into account, we obtain easily that

$$(\mu - 1) \int_{\mathbb{R}^3} |v|^2 |x|^{\mu-2} dx = 2(\mu - 2) \int_{\mathbb{R}^3} (v \cdot x)^2 |x|^{\mu-4} dx. \tag{2.6}$$

The fact that $\mu \in (1,2)$ in (2.6) implies $v = 0$. □

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