Nonlocal nonlinear advection-diffusion equations

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ABSTRACT. We review some results about nonlocal advection-diffusion equations based on lower bounds for the fractional Laplacian.

To Haim, with respect and admiration.

1. Introduction

Nonlocal and nonlinear advection-diffusion equations arise in hydrodynamics and are of general scientific and mathematical interest. In this paper I would like to present concisely some of the results in the area, with enough detail to capture the main ideas, but without all the technical details that might end up by obscuring them (and taking too much space). The equations are of the type

$$\partial_t \theta + u \cdot \nabla \theta + \kappa \Lambda^s \theta = 0 \tag{1}$$

for $\theta(x,t)$ a real scalar-valued function of $x \in \mathbb{R}^d$ and $t \geq 0$, with u given by

$$u = P(\Lambda) R^{\perp} \theta, \tag{2}$$

and where we denote

$$\Lambda = (-\Delta)^{\frac{1}{2}} \tag{3}$$

and

$$R = \nabla \Lambda^{-1} \tag{4}$$

the Riesz transforms. We define $R^{\perp} = MR$ where M is a fixed antisymmetric constant matrix. In most of our discussions below $d = 2$ and $R^{\perp}$ is R rotated counterclockwise by 90 degrees. Note that by construction

$$\nabla \cdot u = 0. \tag{5}$$

The power s and the real function $P(\lambda)$ define the model, and the nonnegative constant κ distinguishes between the inviscid ($\kappa = 0$) and dissipative ($\kappa > 0$) cases. The major examples are, for $d = 2$, $P(\lambda) = \lambda^{-1}$ with $s = 2$, and $P(\lambda) = \lambda^0$ with $s = 1$. The case $P(\lambda) = \lambda^{-1}$, $s = 2$ corresponds to the 2D Euler and, respectively, the Navier-Stokes equations. The case $P(\lambda) = \lambda^0$ and $s = 1$ corresponds to the inviscid Surface Quasi-Geostrophic equation (SQG), and, respectively, to the critical dissipative SQG. The term "critical" here refers to the fact that the pseudodifferential order of the dissipation is the same as the differential order of the nonlinear term. This is criticality in the sense of Goldilocks: the case $s > 1$ is too easy, the case $s < 1$ is too hard, and the case $s = 1$ is just right. There is no reason why criticality in the sense of Goldilocks cannot be true criticality, i.e. a threshold for qualitative change. In the Burgers equation with fractional dissipation, $s = 1$ is a true critical case: blow up behavior occurs for $s < 1$, and does not for $s \geq 1$ ([**1**], [**23**]).

It is well-known that the 2D Euler equations have global smooth solutions if the initial data are smooth and localized. In fact, a slightly more singular constitutive equation $P(\lambda) = \lambda^{-1} \log\log(e + \lambda)$ still gives rise to global smooth solutions ([**5**]). It is not known if solutions with $P(\lambda)$ growing faster at infinity have global smooth solutions.

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SQG appeared as an equation for frontogenesis in meteorology, but its mathematical study was developed because of analogies with 3D incompressible Euler equations ([**9**], [**20**]). The inviscid SQG equation is

$$\partial_t \theta + u \cdot \nabla \theta = 0, \quad u = R^{\perp}\theta. \tag{6}$$

We briefly recall some of the analogies between 2D SQG and 3D Euler equations. The 3D Euler equations are conservative (kinetic energy is conserved) but they could potentially form a first singularity from smooth and localized initial data at time T. This could happen if, and only if, the vorticity $\omega = \nabla \times u$ diverges in L^∞ in such a manner that

$$\int_0^T \|\omega\|_{L^\infty(\mathbb{R}^3)} dt = \infty. \tag{7}$$

This is the celebrated Beale-Kato-Majda criterion ([**2**]). The vorticity has special properties in the 3D Euler equations. It evolves according to

$$\partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u. \tag{8}$$

In this equation, the advecting velocity is one derivative smoother than the advected vector ω. In two dimensions the right-hand side of (8) vanishes identically. In three dimensions this stretching term can produce growth of vorticity magnitude. The equation is geometric, it is the transport equation for tangent fields, and it is equivalent to the commutation relation

$$[D_t, \omega \cdot \nabla] = 0 \tag{9}$$

where $D_t = \partial_t + u \cdot \nabla$ is material derivative. The meaning of this relation is that the integral curves of the vector field $\omega(\cdot, t)$ are transported by the flow: vortex lines are material curves.

Inviscid SQG in 2D has all these properties: it is conservative (kinetic energy is conserved) and the vector $\omega = \nabla^{\perp}\theta$ (not to be confused with the curl of the SQG velocity u) obeys the same transport equation (8) by a velocity u that is one derivative smoother, and the same commutation relation (9) holds. The transported integral lines are level sets of the scalar θ. The same Beale-Kato-Majda criterion (7) applies. Both the Euler equations and SQG have a geometric depletion of nonlinearity that reduces the order of the nonlinear stretching if the direction field $\frac{\omega}{|\omega|}$ is regular in regions of high $|\omega|$.

There are differences between the two equations: SQG has more known conservation laws: the whole distribution function of θ is conserved, and the $\dot{H}^{-\frac{1}{2}}$ norm is conserved as well. Nevertheless, the blow up problem is open for both the 3D Euler and the 2D SQG equations.

The 2D SQG equations have a form of weak continuity of the nonlinearity that permits the construction of weak solutions in L^2 from arbitrary initial data ([**25**]). In fact, local existence of smooth solutions and global existence of weak solutions holds for inviscid equations with $P(\lambda) = \lambda^{-1+\beta}$ for $1 \le \beta \le 2$ ([**6**]).

The critical dissipative SQG has global smooth solutions. This was proved independently by Caffarelli and Vasseur ([**4**]) and by Kiselev, Nazarov and Volberg ([**21**]). The proofs are different in spirit. The proof of ([**4**]) uses a harmonic extension and a de Giorgi methodology of zooming in. The proof of ([**21**]) uses an invariant family of moduli of continuity. Other proofs exist ([**22**]). An extension of an inequality of Córdoba and Córdoba ([**14**]) providing a nonlinear lower bound for the fractional Laplacian ([**11**]) was used for yet a different proof. The proof I describe below appeared in ([**10**]) and was used to study long time behavior of forced critical SQG. Global regularity can be obtained also for critical modified SQG equations ([**8**]) and for slightly supercritical SQG equations ([**16**], [**17**], [**29**]). The problem of global existence of smooth solutions for supercritical SQG, by which we mean equations (1) with u given in (2) with $P(\lambda) = \lambda^0$ and with $\kappa > 0$ but $0 < s < 1$, is open. Solutions of supercritical $s < 1$ drift diffusion equations with $u \in C^\alpha$, with $\alpha = 1 - s$ are Hölder continuous with small exponent ([**13**], [**26**]). This condition is sharp in the sense that there exist linear drift diffusion equations with drift of lower regularity than C^{1-s} for which the solutions loose continuity in finite time ([**28**]). Higher regularity is obtained if $u \in C^\alpha$ with $\alpha > 1 - s$ ([**12**], [**19**]). Thus, in the critical case $s = 1$, any C^α regularity with $\alpha > 0$ implies full regularity. All weak solutions the supercritical SQG become regular after a finite time ([**18**], [**24**], [**27**], [**15**]).

In this paper we discuss a method of proof based on lower bounds for the fractional Laplacian. We discuss results in the whole space $\mathbb{R}^d$, and comment on counterparts in the case of bounded domains. In order to make the presentation simple, we discuss only the critical dissipative SQG.

2. Lower Bounds

The motivation for the lower bounds is as follows: we consider the equation

$$\partial_t \theta + u \cdot \nabla \theta + \Lambda \theta = 0, \quad u = R^{\perp} \theta \tag{10}$$

with initial data θ_0. Suppose the problem is set in $\mathbb{R}^d$ and we would like to prove that smooth initial data (in appropriate sense) give rise to solutions that remain smooth for all time. It is known that the equation has weak solutions with a weak maximum principle ([**25**])

$$\sup_t \|\theta\|_{L^\infty} \le \|\theta_0\|_{L^\infty}. \tag{11}$$

Differentiating the equation results formally in

$$\partial_t g + u \cdot \nabla g = -\Lambda g - (\nabla u)^T g \tag{12}$$

for $g = \nabla \theta$. Because $\partial_t + u \cdot \nabla$ is pure transport, it does not add size to g. We would like Λg to win the battle with $(\nabla u) g$. This leads us to consider Λg for functions g which are gradients of bounded functions.

It is quite remarkable that Λg, as it turns out, does in fact beat $(\nabla u) g$, because $\nabla u \sim g$, and for large data the battle looks hopeless.

The fractional Laplacian has an explicit kernel in $\mathbb{R}^d$,

$$\Lambda^s f(x) = cPV \int_{\mathbb{R}^d} \frac{f(x) - f(y)}{|x-y|^{d+s}} dy \tag{13}$$

and it is this explicit form that was used in ([**11**]) to prove the lower bound that we are discussing. We are also interested in the same problem in bounded domains, where Λ_D is the Dirichlet Laplacian. This is defined in terms of the eigenfunction expansion, and the kernel is not explicit. Let us consider a bounded open domain $\Omega \subset \mathbb{R}^d$ with smooth boundary. Let Δ denote the Laplacian operator with homogeneous Dirichlet boundary conditions. Its $L^2(\Omega)$ - normalized eigenfunctions are denoted w_j, and its eigenvalues counted with their multiplicities are denoted λ_j:

$$-\Delta w_j = \lambda_j w_j. \tag{14}$$

It is well known that $0 < \lambda_1 \le \ldots \le \lambda_j \to \infty$ and that $-\Delta$ is a positive selfadjoint operator in $L^2(\Omega)$ with domain $\mathcal{D}(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$. Functional calculus can be defined using the eigenfunction expansion. In particular

$$(-\Delta)^\alpha f = \sum_{j=1}^{\infty} \lambda_j^\alpha f_j w_j \tag{15}$$

with

$$f_j = \int_\Omega f(y) w_j(y) dy$$

for $f \in \mathcal{D}\left((-\Delta)^\alpha\right) = \{f \mid \; (\lambda_j^\alpha f_j) \in \ell^2(\mathbb{N})\}$. We denote by

$$\Lambda_D^s = (-\Delta)^\alpha, \quad s = 2\alpha \tag{16}$$

the fractional powers of the Dirichlet Laplacian, with $0 \le \alpha \le 1$ and with $\|f\|_{s,D}$ the norm in $\mathcal{D}\left(\Lambda_D^s\right)$:

$$\|f\|_{s,D}^2 = \sum_{j=1}^{\infty} \lambda_j^s f_j^2. \tag{17}$$

Note that in view of the identity

$$\lambda^\alpha = c_\alpha \int_0^\infty (1 - e^{-t\lambda}) t^{-1-\alpha} dt, \tag{18}$$

with

$$1 = c_\alpha \int_0^\infty (1 - e^{-s}) s^{-1-\alpha} ds,$$

valid for $0 \le \alpha < 1$, we have the representation

$$\left((-\Delta)^\alpha f\right)(x) = c_\alpha \int_0^\infty \left[f(x) - e^{t\Delta} f(x)\right] t^{-1-\alpha} dt \tag{19}$$

for $f \in \mathcal{D}\left((-\Delta)^\alpha\right)$. The same representation holds in the whole space (using the Fourier transform). The kernel of the heat semigroup in the whole space is explicit

$$G_t(z) = (4\pi t)^{-\frac{d}{2}} e^{-\frac{|z|^2}{4t}} \tag{20}$$

and this, together with the fact that $\int_{\mathbb{R}^d} G_t(z) dz = 1$ gives

$$\Lambda^s f(x) = c \int_0^\infty t^{-1-\frac{s}{2}} \int_{\mathbb{R}^d} G_t(z)(f(x) - f(x-z)) dz,$$

which yields (13). It is known that the kernel $H_D(x, y, t)$ of the Dirichlet heat semigroup in bounded domains,

$$H_D(x, y, t) = \sum_{j=1}^\infty e^{-t\lambda_j} w_j(x) w_j(y), \tag{21}$$

is positive and nonsingular for $t > 0$, and this is enough to prove the analogue of the Córdoba-Córdoba inequality ([**14**]) in the case of bounded domains as well ([**7**]):

PROPOSITION 1. *Let Φ be a C^2 convex function satisfying $\Phi(0) = 0$. Let $f \in C_0^\infty(\Omega)$ and let $0 \le s \le 2$. Then*

$$\Phi'(f)\Lambda^s f - \Lambda^s(\Phi(f)) \ge 0 \tag{22}$$

holds pointwise almost everywhere in Ω.

In order to go beyond this inequality, more information about the kernel is needed. Let us explain the case of $\mathbb{R}^d$, $\Phi(f) = \frac{1}{2} f^2$ and $s = 1$. We define $D_2(g)$,

$$D_2(g)(x) = g(x)\Lambda g(x) - \frac{1}{2}\Lambda g^2(x), \tag{23}$$

and estimate it for a scalar valued function $g = \partial_1 f$ where ∂_1 is a partial derivative, and f is a bounded function. We use the explicit representation (13) and compute

$$D_2(g)(x) = \frac{c}{2} \int_{\mathbb{R}^d} \frac{(g(x) - g(y))^2}{|x-y|^{d+1}} dy \tag{24}$$

We take a smooth radial cutoff function $\psi(r)$ obeying $0 \le \psi(r) \le 1$ with $\psi(r) = 0$ for $r \in [0, \frac{1}{2}]$ $\psi(r) = 1$ on $r \in [1, \infty)$. We take an arbitrary length ℓ (to be chosen later) and write

$$D_2(g)(x) \ge \frac{c}{2} \int_{\mathbb{R}^d} \psi\left(\frac{|x-y|}{\ell}\right) \frac{(g(x) - g(y))^2}{|x-y|^{d+1}} dy \tag{25}$$

We open brackets and ignore one positive term

$$\begin{aligned} D_2(g)(x) &\ge \tfrac{c}{2} g^2(x) \textstyle\int_{\mathbb{R}^d} \psi\left(\frac{|x-y|}{\ell}\right) \frac{1}{|x-y|^{d+1}} dy - c g(x) \int_{\mathbb{R}^d} \psi\left(\frac{|x-y|}{\ell}\right) \frac{g(y)}{|x-y|^{d+1}} dy \\ &= G(x) - B(x) \end{aligned} \tag{26}$$

It is time to remember that $g = \partial_1 f$. We integrate by parts in the bad term $B(x)$ and bound from above:

$$|B(x)| \le C_1 |g(x)| \ell^{-2} \|f\|_{L^\infty}$$

We bound the good term $G(x)$ below:

$$G(x) \ge C_2 g^2(x) \ell^{-1}.$$

Now we choose ℓ so that $|B(x)| \le \frac{1}{2}G(x)$, i.e.

$$\ell^{-1} \le C_3|g(x)|\|f\|_{L^\infty}^{-1}$$

This proves ([**11**])

$$D_2(g)(x) \ge C\|f\|_{L^\infty}^{-1}|g(x)|^3. \tag{27}$$

This is effectively a cubic lower bound for a quadratic form, given the existing information on f. The fact that the kernel was precisely a power was not important. What we used was: the translation invariance (the kernel is a function of $x-y$), the positivity of the kernel, the fact that the kernel is not integrable near the origin, and the fact that the kernel is integrable at infinity. The translation invariance requirement can be relaxed. In fact, a similar lower bound can be obtained in the case of the fractional Laplacian with Dirichlet boundary conditions Λ_D ([**7**]):

$$D_2(g)(x) \ge C\|f\|_{L^\infty}^{-1}|g_d(x)|^3. \tag{28}$$

where $g_d(x) = g(x)$ if $|g(x)| \ge \frac{\|f\|_{L^\infty}}{(dist(x,\partial\Omega))}$ and $g_d = 0$ otherwise. The proof of this fact requires a different treatment, because we don't have in general explicit representations of the kernel of the fractional Laplacian. We use instead the heat kernel representation (19) and precise lower bounds on the heat kernel and upper bounds on its gradient.

There are many possible variants of the arguments above and lower bounds, corresponding to the available information on g. A useful variant concerns finite differences, when

$$g(x) = (\delta_h f)(x) = f(x+h) - f(x)$$

where h is vector in $\mathbb{R}^d$. Then, in the case of $\mathbb{R}^d$ we obtain

$$D_2(\delta_h f)(x) \ge C|h|^{-1}\|f\|_{L^\infty}^{-1}|\delta_h f(x)|^3. \tag{29}$$

3. Hölder regularity

The velocity advecting the scalar θ in (10) is given explicitly by

$$u(x,t) = cP.V.\int_{\mathbb{R}^d} \frac{(x-y)^\perp}{|x-y|^{d+1}}\theta(y,t)dy. \tag{30}$$

We take a finite difference $g = \delta_h\theta$ and compute its evolution:

$$(\partial_t + u\cdot\nabla_x + \delta_h(u)\cdot\nabla_h + \Lambda)\, g = 0. \tag{31}$$

We used here the fact that $\delta_h(u)\cdot\nabla_x\theta(x+h) = \delta_h(u)\cdot\nabla_h(\delta_h u)(x)$. Let us denote by L the operator

$$L = (\partial_t + u\cdot\nabla_x + \delta_h(u)\cdot\nabla_h + \Lambda) \tag{32}$$

and note that it has a weak maximum principle. The easiest way to see this is by time-splitting: the short time evolution under the pure transport term does not add size, and the short time evolution under the dissipative semigroup does not add size either. We multiply (31) by g in order to have nonnegative quantities, obtain

$$\frac{1}{2}L(g^2) + D_2(g) = 0, \tag{33}$$

and then we divide by $|h|^{2\alpha}$:

$$\frac{1}{2}L(|h|^{-2\alpha}g^2) + |h|^{-2\alpha}D_2(g) = \frac{1}{2}\left(\delta_h(u)\cdot\nabla|h|^{-2\alpha}\right)g^2. \tag{34}$$

The right-hand side is bounded by

$$\left|\frac{1}{2}\left(\delta_h(u)\cdot\nabla|h|^{-2\alpha}\right)g^2\right| \le \alpha|\delta_h(u)||h|^{-2\alpha-1}g^2. \tag{35}$$

The job of $|h|^{-2\alpha}D_2(g)$ is to be larger than this bound of the right hand side. As usual in critical cases, constants do matter. Nevertheless, we use the same name C for all constants; they are explicitly computable and universal, and the order in which they are computed can be easily unraveled by the interested reader. We know from (29) that

$$|h|^{-2\alpha}D_2(g) \geq C\|\theta\|_{L^\infty}^{-1}|h|^{-2\alpha-1}g^3. \tag{36}$$

Now $\delta_h u \sim g$ is true in spirit, but not in flesh (pointwise). We use the representation (30) and split

$$\delta_h u = \delta_h u_{in} + \delta_h u_{out} \tag{37}$$

with

$$\delta_h u_{in} = cP.V.\int_{\mathbb{R}^d}\left(1-\psi\left(\frac{|x-y|}{\ell}\right)\right)\frac{(x-y)^\perp}{|x-y|^{d+1}}(g(y)-g(x))dy \tag{38}$$

and

$$\delta_h u_{out} = cP.V.\int_{\mathbb{R}^d}\delta_{-h}\left[\psi\left(\frac{|x-y|}{\ell}\right)\frac{(x-y)^\perp}{|x-y|^{d+1}}\right]\theta(y)dy \tag{39}$$

with the ψ we used before and with an ℓ we'll choose shortly. We used translation invariance and, in (38) we used the fact that $g = \delta_h\theta$ and the vanishing of the spherical averages of the kernel, while in (39) we moved the finite difference onto the kernel. We bound $\delta_h u_{in}$ using the expression (24):

$$|\delta_h u_{in}(x)| \leq C\sqrt{\ell D_2(g)} \tag{40}$$

and we bound $\delta_h u_{out}$ by

$$|\delta_h u_{out}(x)| \leq C\frac{|h|}{\ell}\|\theta\|_{L^\infty}. \tag{41}$$

These bounds are easily obtained using Schwartz inequalities in the first and the homogeneity and smoothness of the kernel in the second. The term $\alpha|h|^{-2\alpha-1}g^2|\delta_h u_{in}|$ in (35) can be hidden in $\frac{1}{4}|h|^{-2\alpha}D_2(g)$ (using Young's inequality) and the price is

$$C\alpha^2\ell|h|^{-2-2\alpha}g^4.$$

Let us choose

$$\ell = |h|g^{-1}\|\theta\|_{L^\infty} \tag{42}$$

and so the price is

$$C\alpha^2|h|^{-1-2\alpha}g^3\|\theta\|_{L^\infty}$$

i.e.,

$$\alpha|h|^{-2\alpha-1}g^2|\delta_h u_{in}| \leq \frac{1}{4}|h|^{-2\alpha}D_2(g) + C\alpha^2|h|^{-1-2\alpha}g^3\|\theta\|_{L^\infty} \tag{43}$$

The term $\alpha|h|^{-2\alpha-1}g^2|\delta_h u_{out}|$ in (35) is bounded with our choice (42) of ℓ by

$$\alpha|h|^{-2\alpha-1}g^2|\delta_h u_{out}| \leq C\alpha|h|^{-2\alpha-1}g^3. \tag{44}$$

Putting together the bounds (43) and (44) and using (36) in (34) we have

$$L(|h|^{-2\alpha}g) + C|h|^{-2\alpha-1}g^3\|\theta\|_{L^\infty}^{-1} \leq C\left(\alpha + \alpha^2\|\theta\|_{L^\infty}\right)g^3|h|^{-2\alpha-1} \tag{45}$$

The right-hand side and the dissipation have the same order of magnitude, $g^3|h|^{-2\alpha-1}$ as it befits a critical case. There are no adjustable parameters, except one: α itself. If this is chosen small enough

$$\alpha\|\theta\|_{L^\infty} \leq c \tag{46}$$

then we obtain

$$L(|h|^{-2\alpha-1}g) \leq 0 \tag{47}$$

and consequently,

$$\sup_{h\neq 0}\sup_{x,t}\frac{|\delta_h\theta(x,t)|}{|h|^{2\alpha}} \leq \sup_{h\neq 0}\sup_{x}\frac{|\delta_h\theta_0(x)|}{|h|^{2\alpha}}. \tag{48}$$

We proved thus ([**10**])

THEOREM 1. *Let $\theta_0 \in L^\infty(\mathbb{R}^d)$. There exists α depending only on $\|\theta_0\|_{L^\infty}$ such that, if $\theta(x,t)$ solves (10) then*

$$\|\theta(\cdot,t)\|_{C^\alpha} \le \|\theta_0\|_{C^\alpha} \tag{49}$$

holds for all $t \ge 0$.

In the case of bounded domains we can obtain global existence of weak solutions ([**7**]).

4. Higher regularity

The proof of higher regularity of solutions given in ([**12**]) is done using the Littlewood-Paley decomposition. Here we present a different proof based on a version of the nonlinear lower bound on the fractional Laplacian. In this section we consider the supercritical case

$$\partial_t\theta + u\cdot\nabla\theta + \Lambda^s\theta = 0, \quad u = R^\perp\theta, \tag{50}$$

with $0 < s \le 1$, and assume we are given a solution on a time interval $[0,T]$ and that the solution is bounded in C^α, with $\alpha > 1-s$

$$\sup_{t\in[0,T]}\|\theta\|_{C^\alpha} = \Gamma < \infty \tag{51}$$

We use now a version of the lower bound on $D_2(g)$

$$D_2(g)(x) = g(x)\Lambda^s g(x) - \frac{1}{2}(\Lambda^s(g^2))(x) = \frac{c}{2}\int_{\mathbb{R}^d}\frac{(g(x)-g(y))^2}{|x-y|^{d+s}}dy \tag{52}$$

suitable for the case $g = \partial_1 f$ with $f \in C^\alpha$ ([**11**]):

$$D_2(g)(x) \ge Cg^{2+\frac{s}{1-\alpha}}\|f\|_{C^\alpha}^{-\frac{s}{1-\alpha}} \tag{53}$$

The proof of this inequality is very similar to the proof of (27) and is left as an exercise for the dilligent reader. We differentiate (50), denote $g = \nabla\theta$ and multiply by $\nabla\theta$.

$$\frac{1}{2}\left(\partial_t + u\cdot\nabla + \Lambda^s\right)|g|^2 + D_2(g) = -g(\nabla u)g \tag{54}$$

Notice that our assumption $\alpha > 1-s$ makes the situation subcritical (in the sense of Goldilocks): the lower bound (52) is better than cubic,

$$D_2(g) \ge C\Gamma^{-\frac{s}{1-\alpha}}g^{3+\frac{s+\alpha-1}{1-\alpha}}. \tag{55}$$

Now in order to bound the right hand side of (54) we have again a situation in which $\nabla u \sim g$ in spirit but not in flesh. We split

$$\nabla u = \nabla u_{in} + \nabla u_{med} + \nabla u_{out} \tag{56}$$

where

$$\nabla u_{in}(x) = cP.V.\int_{\mathbb{R}^d}\chi_1(|x-y|)\frac{(x-y)^\perp}{|x-y|^{d+1}}(g(y)-g(x))dy, \tag{57}$$

$$\nabla u_{med}(x) = \int_{\mathbb{R}^d}\chi_2(|x-y|)\frac{(x-y)^\perp}{|x-y|^{d+1}}\nabla_y(\theta(y)-\theta(x))dy, \tag{58}$$

and

$$\nabla u_{out} = \int_{\mathbb{R}^d}\chi_3(|x-y|)\frac{(x-y)^\perp}{|x-y|^{d+1}}\nabla\theta(y)dy. \tag{59}$$

We employed here a radial partition of unity $\chi_1(r)+\chi_2(r)+\chi_3(r) = 1$, where χ_1 is supported on $[0,2\rho)$, χ_2 supported on $[\rho,2)$ and χ_3 supported on $(1,\infty)$. We choose χ_i so that $0 \le \chi_i(r) \le 1$ and $|\chi_2'(r)| \le C\rho^{-1}$, $|\chi_3'(r)| \le C$. (For example $\chi_1(r) = \phi(\frac{r}{\rho})$, $\chi_2(r) = -\phi(\frac{r}{\rho}) + \phi(r)$ and $\chi_3(r) = 1-\phi(r)$ with ϕ smooth, nonincreasing, $0 \le \phi(r) \le 1$, identically equal to 1 on $[0,1]$ and compactly supported in $[0,2)$). We'll choose $\rho < \frac{1}{2}$ below. We use (52) and a Schwartz inequality for ∇u_{in}

$$|\nabla u_{in}(x)| \le C\rho^{\frac{s}{2}}\sqrt{D_2(g)}. \tag{60}$$

For ∇u_{med} we integrate by parts and use the assumption on θ:
$$|\nabla u_{med}(x)| \le C\Gamma\rho^{-1+\alpha}. \tag{61}$$
For ∇u_{out} we just integrate by parts
$$|\nabla u_{out}(x)| \le C\|\theta\|_{L^\infty}. \tag{62}$$
We choose ρ to balance the first two terms:
$$\rho = C\left[\Gamma D_2(g)^{-\frac{1}{2}}\right]^{\frac{1}{1-\alpha+\frac{s}{2}}} \tag{63}$$
and we get therefore the upper bound
$$|g(\nabla u)g| \le C\left[\|\theta\|_{L^\infty} + \Gamma^{\frac{s}{s+2(1-\alpha)}} D_2(g)^{\frac{1-\alpha}{s+2(1-\alpha)}}\right] g^2. \tag{64}$$
Hiding the term involving $D_2(g)$ results in
$$C\Gamma^{\frac{s}{s+2(1-\alpha)}} D_2(g)^{\frac{1-\alpha}{s+2(1-\alpha)}} g^2 \le \frac{1}{2}D_2(g) + C\Gamma^{\frac{s}{1-\alpha+s}} g^{\frac{2(s+2(1-\alpha))}{1-\alpha+s}}$$
The beauty of this ugly calculation is that the exponent of g above, $\frac{2(s+2(1-\alpha))}{1-\alpha+s}$, is strictly smaller than the exponent of g in the lower bound (55), $3+\frac{s+\alpha-1}{1-\alpha}$, if, and only if $s+\alpha-1>0$, which is precisely our situation. This allows to hide again the right hand side,
$$C\Gamma^{\frac{s}{s+2(1-\alpha)}} D_2(g)^{\frac{1-\alpha}{s+2(1-\alpha)}} g^2 \le \frac{3}{4}D_2(g) + C\Gamma^{\frac{3s}{s+\alpha-1}}$$
Putting these considerations together results in the bound
$$\frac{1}{2}\left(\partial_t + u\cdot\nabla + \Lambda^s\right)|g|^2 + \frac{1}{4}D_2(g) \le C\left[\|\theta\|_{L^\infty} + \Gamma\right] g^2 + C\Gamma^{\frac{3s}{s+\alpha-1}}. \tag{65}$$
The Γ added in the term with g^2 in the righ hand side of (65) is not needed if the ρ defined in (63) obeys $\rho < \frac{1}{2}$. If this inequality fails, it fails because we have $D_2(g) \le C\Gamma^2$. In that case, using (60, 61, 62) with $\rho = \frac{1}{4}$ we obtain that
$$|\nabla u(x)| \le C[\|\theta\|_{L^\infty} + \Gamma],$$
and that makes (65) true in all cases. We obtain therefore

THEOREM 2. *Let θ be a solution of (50) obeying the bound (51) on $[0,T]$. Then there exists a constant C depending on Γ, $\|\theta_0\|_{L^\infty}$ and T such that*
$$\sup\|\nabla\theta\|_{L^\infty} \le C[\|\nabla\theta_0\|_{L^\infty} + 1]. \tag{66}$$

Passing now to $C^{1,\alpha}$ bounds is easy, and higher still regularity can be obtained by calculus inequalities.

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