ELECTROCONVECTION IN A MAGNETIC FIELD

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ABSTRACT. Electroconvection in a porous medium under a strong transversal magnetic field is described by an active scalar equation for the charge density. The equation has global weak solutions with L^∞ data. We show that for strong enough magnetic fields, L^∞-small solutions are smooth globally in time and they obey surface quasigeostrophic equations in the limit of infinite magnetic field strength.

1. INTRODUCTION

Electroconvection, the flow of charges in fluids, is characterized by the fact that the charge density and the fluid's velocity and pressure are directly coupled. The charges in the solvent exert forces on the fluid. The fluid responds to these forces while convecting the charges. This results in a nonlinearly coupled system of equations for the charge density and the fluid's velocity and pressure. The subject belongs to a large class of electrohydrodynamic problems of broad scientific interest. Experimental and theoretical studies [19–21, 34] on smectic films revealed complex dynamical behavior in rotating, two-dimensional conductive fluids subjected to three-dimensional electrostatic forces. In experimental studies, applied transversal magnetic fields significantly impact the dynamics of the electroconvection [4, 24, 27–29]. In particular, when a magnetic field is applied, the critical voltage required for instability increases, leading to enhanced system stability [4]. Magnetic fields applied to ordered electrically sensitive fluid, such as liquid crystals in porous media, have been documented to produce modifications in phase transitions, changes in orientational order, alterations in the elastic properties and in the behavior of the director field [18, 25, 38].

In this paper we are concerned with mathematical properties of electroconvection in a magnetic field. We show that in the limit of strong transversal magnetic field, the solutions of equations of electroconvection in porous media converge to solutions of the surface quasigeostrophic equation (SQG).

The charge density $q(x,t)$ of electroconvection in porous media is an active scalar. Active scalars [8, 35] are transported by incompressible velocities they create by means of a time-independent equation. This widely studied class of equations includes the two-dimensional incompressible vorticity equation, the surface quasigeostrophic equation SQG and generalized g-SQG equations interpolating between them. The SQG active scalar arose in geophysics [26] as a model of the large scale mid-latitude surface temperature evolution in quasigeostrophic flow. Electroconvection of charges in porous media under strong magnetic fields and atmosphere-ocean thermal dynamics are such disparate physical systems, and yet, mathematically, they turn out to be related. The inviscid SQG equation is studied in the context of singularity formation in fluids [8, 11, 15]. Global regularity from arbitrary smooth data is not known. The equation is ill posed in spaces of low regularity [16, 17]. Global weak solutions were obtained in [36] (see also [12] for bounded domains). Well-posedness and loss of regularity results have been obtained for g-SQG in the case the initial data is a patch (that is a step function) with smooth boundary in a half-space [23, 31, 32, 39] and fronts [5, 22]. Solutions that exist for all time have been constructed. They include steady radial solutions, time-periodic rotating solutions [7] and quasiperiodic solutions (see the monograph [37] and references therein). The critical dissipative SQG equation has global smooth solutions from arbitrary data [6, 30]. Critical dissipative SQG is L^∞ critical. Small L^∞ data lead to stability and global existence [9]; for large data, an initial data

Date: August 10, 2025.
Key words and phrases. electroconvection, imposed magnetic field, Darcy's law, surface quasigeostrophic equation.
MSC Classification: 35Q35, 35R11, 78A25.

dependent modulus of continuity [30] or a C^α norm [13] are non-increasing in time. These quantities depend on the size of the initial data in L^∞ and are highly sensitive to the structure of the equation: finding them relies on the linear nonlocal equation of state of the velocity in terms of the scalar. This is a source of difficulty in analyzing nonlinear nonlocal top-order perturbations of the critical SQG, such as we encounter in the present work. Once these quantities are finite, higher regularity follows [14].

We describe now the electroconvection setting we are discussing in this paper. We consider an incompressible fluid occupying a very thin region in space, represented as the two dimensional plane $\Omega = \mathbb{R}^2 = \{(x,y,z) \mid z = 0\} \subset \mathbb{R}^3$. We denote by q a charge density in Ω, denote by $\tilde{E} = (\tilde{E}_1, \tilde{E}_2, \tilde{E}_3)$ and $\tilde{B} = (\tilde{B}_1, \tilde{B}_2, \tilde{B}_3)$ three-dimensional electric and magnetic fields, and by ρ the total charge density $\rho = 2q\delta_\Omega$ as a distribution in the whole space. We also denote by $E = (E_1, E_2)$ the restriction of $(\tilde{E}_1, \tilde{E}_2)$ to Ω. The three-dimensional electric field $\tilde{E}$ obeys Gauss' law

$$\nabla_3 \cdot \tilde{E} = \frac{1}{\epsilon_0}\rho = \frac{1}{\epsilon_0} 2q\delta_\Omega, \tag{1.1}$$

where ϵ_0 is the vacuum permittivity and we denote by ∇_3 the three dimensional gradient. The electric field is given by an electrostatic potential Φ,

$$\tilde{E} = -\nabla_3 \Phi. \tag{1.2}$$

Gauss' law gives the Poisson equation obeyed by Φ,

$$-\Delta_3 \Phi = \frac{1}{\epsilon_0} 2q\delta_\Omega. \tag{1.3}$$

We discuss the case when q and Φ are 2π periodic functions of x and y and Φ vanishes as $|z| \to \infty$. The potential Φ is then given by

$$\Phi(x,y,z) = \begin{cases} \frac{1}{\epsilon_0} e^{-z\Lambda}\Lambda^{-1} q, & z > 0, \\ \frac{1}{\epsilon_0} e^{z\Lambda}\Lambda^{-1} q, & z < 0, \end{cases} \tag{1.4}$$

where $\Lambda = \sqrt{-\Delta}$ is the square root of the two dimensional Laplacian with 2π periodic boundary conditions. Therefore, $E := (\tilde{E}_1, \tilde{E}_2)|_{z=0}$ can be expressed as

$$E = -\frac{1}{\epsilon_0} \nabla \Lambda^{-1} q \tag{1.5}$$

where ∇ is the two dimensional gradient. We denote by R the Riesz transforms

$$R = \nabla \Lambda^{-1} \tag{1.6}$$

and thus the electric field restricted to the domain occupied by fluid is

$$E = -\frac{1}{\epsilon_0} Rq. \tag{1.7}$$

The magnetic field generated by the electric field is negligible. We consider an imposed external magnetic field $\tilde{B} = (0,0,B)$ where $B \geq 0$ is a constant independent of both time and space. Notice that this external magnetic field $\tilde{B}$ satisfies the divergence-free condition, has zero curl, and is time-independent.

The total current density is given by the sum of the advective current $(u_1, u_2, 0)q$ and the Ohmic conduction current $\sigma \tilde{E}'$, where $\sigma > 0$ is the conductivity of the medium and $\tilde{E}'$ is the electric field experienced by the fluid element in its rest frame [33]. When the fluid is moving with respect to the external magnetic field at the velocity u, using the Lorentz transformation we have

$$\tilde{E}' = \tilde{E} + (u_1, u_2, 0) \times \tilde{B}.$$

Therefore,

$$\tilde{j} = (u_1, u_2, 0)q + \sigma(\tilde{E} + (u_1, u_2, 0) \times \tilde{B}). \tag{1.8}$$

The restriction of $\tilde{j}$ to the surface is denoted by $j = (\tilde{j}_1, \tilde{j}_2)|_{z=0}$ and is therefore

$$j = uq + \sigma(E - Bu^\perp), \tag{1.9}$$

where $u^\perp = (-u_2, u_1)$. The surface charge density q obeys the conservation

$$\partial_t q + \nabla \cdot j = 0 \tag{1.10}$$

which is, in view of (1.9),

$$\partial_t q + u \cdot \nabla q + \sigma(\frac{1}{\epsilon_0}\Lambda q - B\nabla \cdot u^\perp) = 0. \tag{1.11}$$

The fluid's divergence-free velocity u obeys an equation forced by the electrostatic force qE plus the Lorentz force $(\tilde{j} \times \tilde{B})$ in the plane, which is computed as

$$(\tilde{j} \times \tilde{B})_{|\mathbb{R}^2} = \frac{\sigma}{\epsilon_0} BR^\perp q - Bqu^\perp - \sigma B^2 u. \tag{1.12}$$

To ease the notation we take $\sigma = \epsilon_0 = 1$. The total force exerted on the fluid is given by

$$F = BR^\perp q - qRq - Bqu^\perp - B^2 u. \tag{1.13}$$

There are two important velocity contributions in F due to the imposed external magnetic field. The first one, $-Bqu^\perp$, yields a rotation effect similar to a Coriolis force with frequency Bq. The second one $-B^2 u$, is a strong damping or friction effect. Electroconvection equations couple (1.11) to a momentum equation driven by F. For instance, the Navier-Stokes equations forced by F are

$$\partial_t u + u \cdot \nabla u - \nu \Delta u + B^2 u + Bqu^\perp + \nabla p = BR^\perp q - qRq, \quad \nabla \cdot u = 0, \tag{1.14}$$

where $\nu > 0$ is the kinematic viscosity. In the absence of an applied transversal magnetic field, the Navier-Stokes electroconvection system, i.e the system (1.11), (1.14) with $B = 0$, was shown to have global solutions in [10]. The long time dynamics were described in [1] in $\mathbb{T}^2$ and in [3] in bounded domains, both with, and without time independent body forces in the fluid.

If we replace viscous friction by Darcy's law and neglect inertial time dependence, we arrive at the equations

$$\mu u + \nabla p = F, \quad \nabla \cdot u = 0, \tag{1.15}$$

where $\mu > 0$ is porosity. This is the fluid equation we consider in this paper. Taking $\mu = 1$ and retaining B as the only (large) parameter, (1.15) becomes the time independent law

$$(1 + B^2)u + Bqu^\perp + \nabla p = BR^\perp q - qRq, \quad \nabla \cdot u = 0. \tag{1.16}$$

Together with the charge density equation (1.11), this gives rise to the following system,

$$\partial_t q + u \cdot \nabla q + \Lambda q - B\nabla \cdot u^\perp = 0, \tag{1.17}$$

$$(1 + B^2)u + Bqu^\perp + \nabla p = BR^\perp q - qRq, \tag{1.18}$$

$$\nabla \cdot u = 0. \tag{1.19}$$

The unknowns are q, u, p. Using $-B\nabla \cdot u^\perp = B\omega$, taking the two dimensional curl of (1.18), and replacing in (1.17), the latter becomes

$$\partial_t q + \frac{B}{1 + B^2}(R^\perp q \cdot \nabla q) + \frac{1}{1 + B^2} u \cdot \nabla q + \frac{1}{1 + B^2}\Lambda q = 0. \tag{1.20}$$

In the absence of an external magnetic field, i.e. when $B = 0$, the system (1.17)-(1.19) describes electroconvection in a porous medium. The global existence for large data of smooth solutions of this system is a challenging open problem. Global regularity for small initial data in Besov spaces smaller than L^∞ was obtained in [2].

We consider three time scales, $t = t_{lab}$ the laboratory time scale, t_1, the magnetic gyration time scale or first magnetic time scale, and t_2, the magnetic friction time scale or second magnetic time scale. They are related by

$$t_1 = \frac{B}{1 + B^2} t \quad \text{and} \quad t_2 = \frac{1}{B} t_1 = \frac{1}{1 + B^2} t. \tag{1.21}$$

In laboratory time scale, the equation for the unknown $q(x,t)$ is (1.20), that is

$$\partial_t q + \frac{B}{1+B^2}(R^\perp q \cdot \nabla q) + \frac{1}{1+B^2}\left(u[q]\cdot\nabla + \Lambda\right) q = 0, \tag{1.22}$$

with $u[q] = u$ the unique solution of

$$(1+B^2)\left(u + \frac{B}{1+B^2}\mathbb{P}(qu^\perp)\right) = BR^\perp q - \mathbb{P}(qRq) \tag{1.23}$$

where $\mathbb{P} = \mathbb{I} + \nabla(-\Delta)^{-1}\nabla\cdot = \mathbb{I} + R\otimes R$ is the Leray projector on divergence free vector fields. The time independent equation of state $u[q]$ and its properties are described in detail below in Section 2.

In the first (gyration) magnetic time scale, the equation for $q_1(x,t_1) = q(x, \frac{1+B^2}{B}t_1)$ where q solves (1.22) becomes

$$\partial_{t_1} q_1 + R^\perp q_1 \cdot \nabla q_1 + \frac{1}{B}\left(u[q_1]\cdot\nabla + \Lambda\right) q_1 = 0. \tag{1.24}$$

The initial data are the same, $q(x,0) = q_1(x,0)$, and there is no rescaling of space variable x.

In the second magnetic time scale, the equation for the boosted field

$$Q(x,t_2) = Bq_1(x, Bt_2) = Bq(x,(1+B^2)t_2) \tag{1.25}$$

becomes

$$\partial_{t_2} Q + R^\perp Q \cdot \nabla Q + \left(u\left[\frac{Q}{B}\right]\cdot\nabla + \Lambda\right) Q = 0. \tag{1.26}$$

The initial data is $Q(x,0) = Bq(x,0)$ and there is no rescaling of space variable. Note that $Q = Bq$ is the frequency of rotation produced by the force F.

For fixed B, the three equations (1.22), (1.24), (1.26) are versions of the same equation, and they are entirely equivalent. We refer to these equations as Darcy law electroconvection in a magnetic field, DECM equations.

The main results of this work are as follows. We prove first that weak solutions of (1.20) with initial data in L^∞ exist globally (Theorem 3.1). As $B\to\infty$, in laboratory time scale, the equation converges to $\partial_t q = 0$. It is only in the first magnetic time scale that nontrivial dynamics arise. The solutions q_1 of (1.24) converge to solutions of inviscid SQG. We prove (Theorem 4.1)

Theorem 1.1. *Any family of weak solutions $q^B$of (1.20) has a subsequence such that $q^B(x, \frac{1+B^2}{B}t)$ converges weakly as $B\to\infty$ to a weak solution $q(x,t)$ of the inviscid SQG*

$$\partial_t q + (R^\perp q)\cdot\nabla q = 0,$$

in $L^2(0,T;L^2)$.

The dissipative nature of the DECM equation emerges in longer time scales, and the boosted field Q converges in the second magnetic time scale to solutions of critical dissipative SQG. We prove global regularity of the boosted field Q (Theorem 5.1) for small L^∞ initial data and large enough B. In laboratory time scale, the regularity result is (Remark 5.1)

Theorem 1.2. *There exist $c>0$ and $C>0$ such that, if q_0 obeys $B\|q_0\|_{L^\infty}\le c$ and $\|q_0\|_{H^3}\le C(1+B^2)$, then the solution of (1.20) with initial data q_0 exists globally, is unique, and satisfies*

$$\|q(\cdot,t)\|_{H^3} \le \|q_0\|_{H^3} e^{-\frac{t}{4(1+B^2)}}$$

for all $t\ge 0$.

The regularity in H^3 implies by parabolic estimates C^∞ regularity, a fact that follows by well known methods and is not pursued in this paper.

We prove convergence to dissipative critical SQG without assuming the condition of small initial L^∞ norm of q, but we do need to assume enough regularity to allow for absolute continuity of the L^2 norm of the DECM solution. Of course, this regularity is guaranteed for short time or for arbitrary time if the initial L^∞ norm is small. The result we prove is (Theorem 5.2 and Remark 5.2)

Theorem 1.3. *Let q be a strong solution of (1.20) on $[0,(1+B^2)T]$ with initial data $q_0 = \frac{1}{B}Q_0$, with fixed $Q_0 \in H^3$. There exists a constant C depending only on $\|Q_0\|_{H^3}$ and T such that*

$$\sup_{t\in[0,T]} \|Bq(\cdot,(1+B^2)t) - \overline{Q}(t)\|_{L^2} \le \frac{C}{1+B^2}$$

where $\overline{Q}$ is the global smooth solution of critical SQG

$$\partial_t Q + (R^\perp Q)\cdot\nabla Q + \Lambda Q = 0$$

with initial data Q_0.

The constant C above is in fact finite as soon as $Q_0 \in C^\alpha$, for any $\alpha > 0$. The result holds for fixed Q_0 while B varies, and not for fixed q_0. If we fix q_0 and consider the family of initial data Bq_0 for the critical SQG, then the constant C depends badly on B.

The paper is organized as follows. In Section 2 we discuss in detail the equation of state for the velocity. We show that solving for the velocity in terms of q yields a unique solution

$$u[q] = R^\perp\left(w[q] + \frac{Bq}{1+B^2}\right),$$

where w is small in H^3 compared to $\|q\|_{H^3}^2$ for large B (Proposition 2.2). This is a crucial ingredient in the proof of global regularity. In Section 3 we prove the global existence of weak solutions by introducing a judicious approximation. Section 4 is where we prove convergence to inviscid SQG equations in the first magnetic time scale. In Section 5 we prove the global regularity and dissipative SQG limit in the second magnetic time scale. We provide the proof of Proposition 2.2 in Appendix A and the proof of global regularity of the approximation in Appendix B.

2. Equation of State for the Velocity in Terms of the Charge Density

Taking the curl of (1.23) we obtain the time-independent equation for the vorticity,

$$\omega = \frac{1}{1+B^2}\left(R^\perp q\cdot\nabla q\right) - \frac{B}{1+B^2}\left(u\cdot\nabla q + \Lambda q\right), \tag{2.1}$$

where

$$\omega = \nabla^\perp\cdot u = \Delta\psi, \quad \nabla\cdot u = 0. \tag{2.2}$$

Writing $\omega = -\Lambda v$ where $v = \Lambda\psi$ with ψ the stream function, we have from (2.1) after applying $-\Lambda^{-1}$ to both sides

$$v - \frac{B}{1+B^2}R\cdot(qR^\perp v) = \frac{B}{1+B^2}\left(q - \frac{1}{B}R\cdot(qR^\perp q)\right), \tag{2.3}$$

and

$$u = R^\perp v. \tag{2.4}$$

We denote by L_q the operator

$$L_q(f) = R\cdot(qR^\perp f) \tag{2.5}$$

Both q and f are scalar. Because $R\cdot R^\perp = 0$, this is a commutator,

$$L_q(f) = [R,q]\cdot R^\perp f. \tag{2.6}$$

Now (2.3) can be written as

$$v - \frac{B}{1+B^2}L_q v = \frac{B}{1+B^2}\left(q - \frac{1}{B}L_q q\right). \tag{2.7}$$

The operator $\mathbb{I} - \frac{B}{1+B^2}L_q$ is bounded in L^2 (if $q \in L^\infty$) and unconditionally invertible. Because q is real, the operator L_q is bounded anti-selfadjoint in L^2 so its spectrum lies in a segment of $i\mathbb{R}$. The inverse of $\mathbb{I} - \frac{B}{1+B^2}L_q$ can be defined by familiar functional calculus, and has norm less than 1, no matter how large is the norm of $\|q\|_{L^\infty}$. We recall that under DECM equations this norm is not growing in time.

Introducing

$$w = v - \frac{B}{1+B^2}q \tag{2.8}$$

the equation (2.7) becomes

$$w - \frac{B}{1+B^2}L_q w = -\frac{1}{(1+B^2)^2}L_q q, \tag{2.9}$$

that is,

$$w[q] = -\frac{1}{(1+B^2)^2}\left(\mathbb{I} - \frac{BL_q}{1+B^2}\right)^{-1}(L_q q) \tag{2.10}$$

and (2.4) becomes

$$u[q] = R^{\perp}\left(w[q] + \frac{Bq}{1+B^2}\right). \tag{2.11}$$

The left hand side of (1.23) defines an operator T_q in H, the space of periodic divergence-free vector fields in L^2:

$$T_q(u) = (1+B^2)\left(u + \frac{B}{1+B^2}\mathbb{P}(qu^{\perp})\right). \tag{2.12}$$

Using (2.4) we express in terms of v,

$$T_q(u) = (1+B^2)\left(R^{\perp}v - \frac{B}{1+B^2}\mathbb{P}(qRv)\right). \tag{2.13}$$

Now, in view of

$$(\mathbb{P}f)_i = (\delta_{ij} + R_iR_j)\,f_j \tag{2.14}$$

$$L_q(v) = -R^{\perp}\cdot(qRv), \tag{2.15}$$

and

$$R_iR_i = -\mathbb{I}, \tag{2.16}$$

by applying from the left $R_i^{\perp} = \epsilon_{ji}R_j$ to (2.13) (where $\epsilon_{ji} = 0$ if $j = i$, and it is the signature of the permutation $(1,2)\mapsto(j,i)$ if $j \neq i$), we obtain

$$R^{\perp}\cdot T_q(u) = -(1+B^2)\left(\mathbb{I} - \frac{B}{1+B^2}L_q\right)(v). \tag{2.17}$$

This shows that u determined by inverting T_q is the same as $u[q]$. Indeed, using (2.7) to solve for v, we deduce

$$R^{\perp}\cdot T_q(u) = -(1+B^2)\frac{B}{1+B^2}\left(q - \frac{1}{B}L_q q\right) = -Bq + L_q q. \tag{2.18}$$

Now, because $T_q(u)$ is divergence-free, we have

$$T_q(u) = -R^{\perp}(R^{\perp}\cdot T_q(u)), \tag{2.19}$$

and using (2.18) we deduce

$$T_q(u) = BR^{\perp}q - R^{\perp}(L_q q). \tag{2.20}$$

Because $\mathbb{P}(qRq)$ is divergence free we have that

$$\mathbb{P}(qRq) = -R^{\perp}\left(R^{\perp}\cdot\mathbb{P}(qRq)\right), \tag{2.21}$$

and using (2.15) we see that

$$\mathbb{P}(qRq) = R^{\perp}(L_q q) \tag{2.22}$$

and thus, from (2.20) that

$$T_q(u) = BR^{\perp}q - \mathbb{P}(qRq). \tag{2.23}$$

This shows that $u = R^{\perp}v$ solves (2.23). The solution is unique, and this establishes the converse implication as well. We have thus,

Proposition 2.1. *Let $q \in L^\infty$. The unique solution in H of (1.23),*

$$u[q] = T_q^{-1}\left(BR^\perp q - \mathbb{P}(qRq)\right) \tag{2.24}$$

is given by (2.11) with $w[q]$ solving (2.9) given by (2.10).

Note that $L_q(f)$ is bilinear, in particular $BL_q = L_{Bq}$. In view of (2.10) and (2.11) we have

$$u[q] = \frac{B}{1+B^2}R^\perp\left(q - \frac{1}{B^2}\sum_{n=0}^\infty\left(\frac{BL_q}{1+B^2}\right)^{n+1}q\right). \tag{2.25}$$

For $Q = Bq$ we have

$$U[Q] = R^\perp\left(\frac{1}{1+B^2}Q + W[Q]\right) \tag{2.26}$$

with

$$W[Q] = -\frac{1}{B^2(1+B^2)}\sum_{n=0}^\infty\left(\frac{L_Q}{(1+B^2)}\right)^{n+1}Q. \tag{2.27}$$

The scalar w and velocity u are unchanged,

$$W[Q] = w\left[\frac{Q}{B}\right], \tag{2.28}$$

$$U[Q] = u\left[\frac{Q}{B}\right]. \tag{2.29}$$

Let us consider a Banach algebra $\mathcal{B}$, smaller than L^∞ and where the Riesz transforms are bounded. Two canonical examples are C^α and H^s, $s > 1$. Then, if $Q \in \mathcal{B}$ we have that L_Q is bounded in $\mathcal{B}$ with

$$\|L_Q\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} \le \gamma\|Q\|_{\mathcal{B}}. \tag{2.30}$$

The expressions (2.25), (2.27) are explicit analytic expansions in $\mathcal{B}$, and we have

$$\|W[Q]\|_{\mathcal{B}} \le \frac{\gamma}{B^2(1+B^2)^2}\frac{\|Q\|_{\mathcal{B}}^2}{\left(1 - \frac{\gamma}{1+B^2}\|Q\|_{\mathcal{B}}\right)}. \tag{2.31}$$

If

$$\|Q\|_{\mathcal{B}} \le \frac{1+B^2}{2\gamma} \tag{2.32}$$

then

$$\|W(Q)\|_{\mathcal{B}} \le \frac{2\gamma}{B^2(1+B^2)^2}\|Q\|_{\mathcal{B}}^2. \tag{2.33}$$

These considerations can be greatly improved by removing the quantitative condition (2.32) in H^s spaces. This is done by taking advantage of the unconditional invertibility in L^2, provided $Q \in L^\infty$, and then using the same cancellation and commutator estimates for higher derivatives. Because we are interested in the DECM transport equations, we work in a Sobolev space H^s that is smaller than $W^{1,\infty}$ to guarantee Lipschitz velocities, which means that we need $s > 2$. For simplicity we take s an integer, $s = 3$. We have

Proposition 2.2. *For any $Q \in H^3$, the equation*

$$w - \frac{1}{1+B^2}R\cdot(QR^\perp w) = -\frac{1}{B^2(1+B^2)^2}R\cdot(QR^\perp Q) \tag{2.34}$$

has a unique solution $w = W[Q]$. There exists an absolute constant C such that w obeys

$$\|w\|_{L^2}^2 \le \frac{C}{B^4(1+B^2)^4}\|Q\|_{L^4}^4, \tag{2.35}$$

and

$$\|\nabla\Delta w\|_{L^2}^2 \le \frac{C(\|Q\|_{L^\infty}^5 + \|Q\|_{L^\infty}^3 + \|Q\|_{L^\infty}^2)}{B^4(1+B^2)^4}\|Q\|_{H^3}^4 + \frac{C\|Q\|_{L^\infty}^2}{B^4(1+B^2)^4}\|Q\|_{H^3}^2. \tag{2.36}$$

We expressed w solving (2.9) as a function of $Q = Bq$, and thus rewrote (2.9) in terms of Q as (2.34). The function w is the same, $W[Q] = w\left[\frac{Q}{B}\right]$. Note that the leading order bound on $\|\nabla\Delta w\|_{L^2}$ of (2.36) relative to the square of the H^3 norm is $B^{-2}(1+B^2)^{-2}$, the same as in the abstract bound (2.33), but it holds without the condition (2.32). The proof of Proposition 2.2 is found in Appendix A.

We conclude this section by showing some continuity properties of T_q^{-1}.

Proposition 2.3. *Let $q \in L^\infty$, and let the linear operator $T_q : H \to H$ be defined in (2.12)*

$$T_q u = (1 + B^2)u + B\mathbb{P}\left(qu^\perp\right). \tag{2.37}$$

(1) The operator T_q is invertible and its inverse T_q^{-1} obeys

$$\|T_q^{-1}u\|_{L^2} \le \frac{1}{1+B^2}\|u\|_H \tag{2.38}$$

for all $u \in H$.

(2) If $q_1, q_2 \in L^\infty$, then, for every $u \in H$, we have

$$\|T_{q_1}^{-1}u - T_{q_2}^{-1}u\|_H \le \frac{B}{(1+B^2)^2}\|q_1 - q_2\|_{L^\infty}\|u\|_H. \tag{2.39}$$

(3) Let $\{q_n\}_{n\in\mathbb{N}}$ be a sequence of functions such that $\{q_n\}_{n\in\mathbb{N}}$ is uniformly bounded in L^∞ and $\{q_n\}_{n\in\mathbb{N}}$ converges pointwise to $q \in L^\infty$ almost everywhere. Then, for every $u \in H$, $\left\{T_{q_n}^{-1}u\right\}_{n\in N}$ converges strongly in H to $T_q^{-1}u$.

(4) Let $\{q_n\}_{n\in\mathbb{N}}$ be a sequence of functions such that $\{q_n\}_{n\in\mathbb{N}}$ is uniformly bounded in L^∞ and $\{q_n\}_{n\in\mathbb{N}}$ converges strongly in L^p to $q \in L^\infty$ for some $p \in (1,\infty)$. Then, for every $u \in H$, $\left\{T_{q_n}^{-1}u\right\}_{n\in N}$ has a subsequence that converges strongly in H to $T_q^{-1}u$.

Proof. We prove now (1)–(4).

(1) The linear bounded operator $J_q(u) = \mathbb{P}(qu^\perp)$ is anti-selfadjoint in H, and therefore $\mathbb{I} + \frac{B}{1+B^2}J_q$ is invertible with inverse of norm less than 1. Taking the scalar product of the equation $T_q(u) = f$ with u and using the anti-symmetry of J_q we have

$$(1+B^2)\|u\|_H^2 = (u,f)_H \le \|u\|_H\|f\|_H \tag{2.40}$$

and thus $(1+B^2)\|u\|_H \le \|f\|_H$. Because $u = T_q^{-1}f$ we have

$$\|T_q^{-1}f\|_H \le \frac{1}{1+B^2}\|f\|_H. \tag{2.41}$$

(2) Let $q_1, q_2 \in L^\infty$. We have

$$\|T_{q_1}^{-1}u - T_{q_2}^{-1}u\|_H = \|T_{q_1}^{-1}(T_{q_2} - T_{q_1})T_{q_2}^{-1}u\|_H \le \frac{1}{1+B^2}\|(T_{q_2} - T_{q_1})T_{q_2}^{-1}u\|_H. \tag{2.42}$$

But, for $f = T_{q_2}^{-1}u$,

$$\|T_{q_2}f - T_{q_1}f\|_H = B\|\mathbb{P}((q_2 - q_1)f^\perp)\|_H \le B\|q_2 - q_1\|_{L^\infty}\|f\|_H. \tag{2.43}$$

It follows that

$$\|T_{q_1}^{-1}u - T_{q_2}^{-1}u\|_H \le \frac{B}{1+B^2}\|q_2 - q_1\|_{L^\infty}\|T_{q_2}^{-1}u\|_{L^2} \le \frac{B}{(1+B^2)^2}\|q_1 - q_2\|_{L^\infty}\|u\|_H \tag{2.44}$$

for any $u \in H$.

(3) Let $u \in H$. Then we have

$$\|T_{q_n}^{-1}u - T_q^{-1}u\|_H = \|T_{q_n}^{-1}(T_q - T_{q_n})T_q^{-1}u\|_H = \|T_{q_n}^{-1}\left(B\mathbb{P}((q - q_n)(T_q^{-1}u)^\perp)\right)\|_H. \tag{2.45}$$

Using the uniform boundedness of the operators $T_{q_n}^{-1}$ and the Leray projector $\mathbb{P}$ on L^2, we bound the latter as follows,

$$\|T_{q_n}^{-1}u - T_q^{-1}u\|_H \le \frac{B}{1+B^2}\|(q-q_n)(T_q^{-1}u)^{\perp}\|_{L^2}. \tag{2.46}$$

Since $|(q-q_n)(T_q^{-1}u)^{\perp}| \le \left(\|q\|_{L^\infty} + \sup_{n\in\mathbb{N}}\|q_n\|_{L^\infty}\right)|(T_q^{-1}u)^{\perp}|$, $(T_q^{-1}u)^{\perp}$ is bounded in L^2 by a constant multiple of $\|u\|_{L^2}$, and q_n converges pointwise to q a.e., we deduce that

$$\lim_{n\to\infty}\|(q-q_n)(T_q^{-1}u)^{\perp}\|_{L^2} = 0 \tag{2.47}$$

by the Lebesgue Dominated Convergence Theorem, and consequently

$$\lim_{n\to\infty}\|T_{q_n}^{-1}u - T_q^{-1}u\|_H = 0. \tag{2.48}$$

(4) This follows from (3) and the fact that $\{q_n\}_{n\in\mathbb{N}}$ has a subsequence that converges to x for a.e. $x\in\mathbb{T}^2$.

□

3. Existence of Global Weak Solutions

In this section, we prove the existence of global weak solutions of (1.22) with (1.23), for L^2 initial charge density on $\mathbb{T}^2$.

For each $\epsilon\in(0,1)$, we let J_ϵ be a standard mollifier, and we consider the ϵ-approximate system

$$\partial_t q^\epsilon + \frac{1}{1+B^2}u^\epsilon\cdot\nabla q^\epsilon + \frac{B}{(1+B^2)}J_\epsilon R^{\perp}q^\epsilon\cdot\nabla q^\epsilon + \frac{1}{(1+B^2)}\Lambda q^\epsilon - \epsilon\Delta q^\epsilon = 0, \tag{3.1}$$

$$u^\epsilon = J_\epsilon T_{J_\epsilon q^\epsilon}^{-1}\left[-\mathbb{P}\left(J_\epsilon q^\epsilon R J_\epsilon q^\epsilon\right) + BR^{\perp}J_\epsilon q^\epsilon\right], \tag{3.2}$$

$$q^\epsilon(0) = J_\epsilon q_0, \tag{3.3}$$

in $\mathbb{T}^2$ with periodic boundary conditions.

Proposition 3.1. *Let $\epsilon\in(0,1)$. Let $q_0\in L^2$. Then the ϵ-approximate system (3.1)-(3.3) has a unique global smooth solution.*

The proof of this proposition is found in Appendix B.

Proposition 3.2. *Let $\epsilon\in(0,1)$ and $T>0$. Suppose $q_0\in L^\infty$. Then the family $\{q^\epsilon\}$ of solutions of (3.1)-(3.3) is uniformly bounded in $L^\infty(0,T;L^\infty)$ and $L^2(0,T;H^{\frac12})$. Moreover, the family $\{u^\epsilon\}$ is uniformly bounded in $L^\infty(0,T;L^2)$.*

Proof. The time evolution of the L^2 norm of q^ϵ is described by the energy balance

$$\frac{1}{2}\frac{d}{dt}\|q^\epsilon\|_{L^2}^2 + \frac{1}{(1+B^2)}\|\Lambda^{\frac12}q^\epsilon\|_{L^2}^2 + \epsilon\|\nabla q^\epsilon\|_{L^2}^2 = 0. \tag{3.4}$$

Here we used the fact that u^ϵ and $\mathcal{J}_\epsilon R^{\perp}q^\epsilon$ are divergence-free. Applying the Grönwall inequality, we obtain

$$\|q^\epsilon(t)\|_{L^2}^2 + \frac{2}{(1+B^2)}\int_0^t\|\Lambda^{\frac12}q^\epsilon(s)\|_{L^2}^2ds + 2\epsilon\int_0^t\|\nabla q^\epsilon(s)\|_{L^2}^2ds = \|q^\epsilon(0)\|_{L^2}^2 \le \|q(0)\|_{L^2}^2. \tag{3.5}$$

for all $t\ge0$. Now we address the L^p evolution of q^ϵ for even integers $p>2$. To this end, we multiply the equation obeyed by q^ϵ by $(q^\epsilon)^{p-1}$, integrate over $\mathbb{T}^2$, make use of the divergence-free property obeyed by the vector fields u^ϵ and $R^{\perp}q^\epsilon$, apply the Córdoba–Córdoba inequality, and obtain

$$\frac{1}{p}\frac{d}{dt}\|q^\epsilon\|_{L^p}^p \le 0. \tag{3.6}$$

Integrating in time from 0 to t, we infer that

$$\|q^\epsilon(t)\|_{L^p} \le \|q^\epsilon(0)\|_{L^p} \le \|q(0)\|_{L^p} \tag{3.7}$$

for any $t \ge 0$. Letting $p \to \infty$, we obtain the L^∞ bound

$$\|q^\epsilon(t)\|_{L^\infty} \le \|q(0)\|_{L^\infty} \tag{3.8}$$

for all positive times $t \ge 0$. As a consequence of the above estimates, it follows that $\{q^\epsilon\}$ is uniformly bounded in $L^\infty(0,T;L^\infty)$ and $L^2(0,T;H^{\frac{1}{2}})$. Finally, we turn our attention to the regularity of the ϵ-regularized velocities. Due to the boundedness of the operators $\mathcal{J}_\epsilon$, $T^{-1}_{\mathcal{J}_\epsilon q^\epsilon}$, $\mathbb{P}$, and R on L^2, we have

$$\begin{aligned}\|u^\epsilon\|_{L^2} =& \left\|J_\epsilon T^{-1}_{J_\epsilon q^\epsilon}\left[-\mathbb{P}\left(J_\epsilon q^\epsilon R J_\epsilon q^\epsilon\right) + BR^\perp J_\epsilon q^\epsilon\right]\right\|_{L^2} \\ \le& C\left\|J_\epsilon q^\epsilon R J_\epsilon q^\epsilon\right\|_{L^2} + C\|q^\epsilon\|_{L^2} \\ \le& C(\|q^\epsilon\|_{L^\infty} + 1)\|q^\epsilon\|_{L^2}\end{aligned} \tag{3.9}$$

for some positive constant C depending on B and some universal constants. Therefore, the family $\{u^\epsilon\}$ is uniformly bounded in $L^\infty(0,T;L^2)$.

□

Proposition 3.3. *Let $\epsilon \in (0,1)$ and $T > 0$. Let $q_0 \in L^\infty$. The family $\{\partial_t q^\epsilon\}$ of solutions of (3.1)-(3.3) is uniformly bounded in $L^2(0,T;H^{-\frac{3}{2}})$.*

Proof. We take an arbitrary test function $\phi \in H^{\frac{3}{2}}$ and consider the L^2 inner product of ϕ with each term in (3.1). For the linear terms, we have

$$(\epsilon\Delta q^\epsilon, \phi)_{L^2} \le \epsilon\|\Lambda^{\frac{1}{2}} q^\epsilon\|_{L^2}\|\phi\|_{H^{\frac{3}{2}}} \tag{3.10}$$

and

$$(\Lambda q^\epsilon, \phi)_{L^2} \le \|q^\epsilon\|_{L^2}\|\phi\|_{H^1}. \tag{3.11}$$

For the nonlinear terms, we integrate by parts, apply the Hölder and Sobolev inequalities, and obtain

$$(u^\epsilon \cdot \nabla q^\epsilon, \phi)_{L^2} = -(u^\epsilon \cdot \nabla\phi, q^\epsilon)_{L^2} \le \|u^\epsilon\|_{L^2}\|\nabla\phi\|_{L^4}\|q^\epsilon\|_{L^4} \le C\|u^\epsilon\|_{L^2}\|\Lambda^{\frac{1}{2}} q^\epsilon\|_{L^2}\|\phi\|_{H^{\frac{3}{2}}} \tag{3.12}$$

and

$$(J_\epsilon R^\perp q^\epsilon \cdot \nabla q^\epsilon, \phi)_{L^2} = -(J_\epsilon R^\perp q^\epsilon \cdot \nabla\phi, q^\epsilon)_{L^2} \le C\|q^\epsilon\|_{L^2}\|\Lambda^{\frac{1}{2}} q^\epsilon\|_{L^2}\|\phi\|_{H^{\frac{3}{2}}}. \tag{3.13}$$

Putting the above estimates together and using the uniform bounds derived in Proposition 3.2, we conclude that $\{\partial_t q^\epsilon\}$ is uniformly bounded in $L^2(0,T;H^{-\frac{3}{2}})$.

□

Theorem 3.1. *Let $T > 0$, let $q_0 \in L^\infty$. Then there exists a global weak solution q of the DECM equation (1.20), satisfying*

$$q \in C_{w^*}([0,T];L^\infty) \cap L^2(0,T;H^{\frac{1}{2}}). \tag{3.14}$$

Proof. As $\{q^\epsilon\}_{\epsilon\in(0,1)}$ is uniformly bounded in $L^\infty(0,T;L^\infty) \cap L^2(0,T;H^{\frac{1}{2}})$ and $\{\partial_t q^\epsilon\}_{\epsilon\in(0,1)}$ is uniformly bounded in $L^2(0,T;H^{-\frac{3}{2}})$, there exists a subsequence, also denoted by $\{q^\epsilon\}_{\epsilon\in(0,1)}$, and a limit q, such that

$$q^\epsilon \overset{*}{\rightharpoonup} q \text{ in } L^\infty(0,T;L^\infty), \quad q^\epsilon \rightharpoonup q \text{ in } L^2(0,T;H^{\frac{1}{2}}), \quad \partial_t q^\epsilon \rightharpoonup \partial_t q \text{ in } L^2(0,T;H^{-\frac{3}{2}}) \tag{3.15}$$

by the Banach-Alaoglu theorem. In addition, by the Aubin-Lions and Lions-Magenes lemmas, we have

$$q^\epsilon \to q \text{ in } L^2(0,T;L^2) \cap C([0,T);H^{-\frac{1}{2}}). \tag{3.16}$$

Since $q \in L^\infty(0,T;L^\infty)$, it follows that $q \in C_{w^*}([0,T);L^\infty)$.

We denote by

$$u := T_q^{-1}\left[-\mathbb{P}\left(qRq\right) + BR^\perp q\right]. \tag{3.17}$$

and show that $u^\epsilon \to u$ strongly in $L^2(0,T;L^2)$. Denoting

$$v^\epsilon := -\mathbb{P}\left(J_\epsilon q^\epsilon R J_\epsilon q^\epsilon\right) + BR^\perp J_\epsilon q^\epsilon, \qquad v := -\mathbb{P}\left(qRq\right) + BR^\perp q,$$

and estimating as in (3.9), we have $v^\epsilon, v \in L^\infty(0,T;L^2)$. We show that $v^\epsilon \to v$ strongly in $L^2(0,T;L^2)$. The convergence of the linear terms follows directly from the boundedness of the Riesz transform on L^p spaces, the convergence property of the mollifiers $\mathcal{J}_\epsilon$, and the strong convergence of q^ϵ obtained in (3.16). As for the nonlinear terms, we have

$$\begin{aligned}\|\mathcal{J}_\epsilon q^\epsilon R\mathcal{J}_\epsilon q^\epsilon - qRq\|_{L^2} &\le \|\mathcal{J}_\epsilon q^\epsilon - q\|_{L^4}\|R\mathcal{J}_\epsilon q^\epsilon\|_{L^4} + \|q\|_{L^\infty}\|R(\mathcal{J}_\epsilon q^\epsilon - q)\|_{L^2} \\ &\le C\|\mathcal{J}_\epsilon q^\epsilon - q\|_{L^2}^{\frac14}(\|q^\epsilon\|_{L^6} + \|q\|_{L^6})^{\frac34}\|q^\epsilon\|_{L^4} + C\|q\|_{L^\infty}\|\mathcal{J}_\epsilon q^\epsilon - q\|_{L^2}.\end{aligned} \tag{3.18}$$

As $q, q^\epsilon \in L^\infty(0,T;L^\infty)$ and $q^\epsilon \to q$ in $L^2(0,T;L^2)$, we deduce that $v^\epsilon \to v$ in $L^2(0,T;L^2)$. We write

$$\begin{aligned}u^\epsilon - u &= \mathcal{J}_\epsilon T^{-1}_{\mathcal{J}_\epsilon q^\epsilon} v^\epsilon - T_q^{-1} v \\ &= \left(\mathcal{J}_\epsilon T^{-1}_{\mathcal{J}_\epsilon q^\epsilon}(v^\epsilon - v)\right) + \left(\mathcal{J}_\epsilon (T^{-1}_{\mathcal{J}_\epsilon q^\epsilon} v - T_q^{-1} v)\right) + \left(\mathcal{J}_\epsilon T_q^{-1} v - T_q^{-1} v\right) := I_1 + I_2 + I_3.\end{aligned}$$

In view of the uniform boundedness of $T^{-1}_{\mathcal{J}_\epsilon q^\epsilon}$ in L^2 and the convergence $v^\epsilon \to v$ in $L^2(0,T;L^2)$, we conclude that $I_1 \to 0$ in $L^2(0,T;L^2)$. Since $\mathcal{J}_\epsilon q^\epsilon \to q$ in $L^2(0,T;L^2)$ and $v \in L^\infty(0,T;L^2)$, an application of Proposition 2.3 yields the convergence of I_2 to 0 in $L^2(0,T;L^2)$. Due to the convergence properties of the mollifiers $\mathcal{J}_\epsilon$, we have $I_3 \to 0$ in $L^2(0,T;L^2)$. Therefore, we conclude that $u^\epsilon \to u$ strongly in $L^2(0,T;L^2)$.

Finally, let ϕ be a test function in $C^\infty([0,T]\times\mathbb{T}^2)$. By virtue of (3.15), (3.16), and the regularity of $u, u^\epsilon, q, q^\epsilon$, we have

$$\langle \partial_t q^\epsilon, \phi\rangle \to \langle \partial_t q, \phi\rangle, \quad \langle \Lambda q^\epsilon, \phi\rangle \to \langle \Lambda q, \phi\rangle, \quad \epsilon\langle \Delta q^\epsilon, \phi\rangle \to 0,$$

and

$$\langle u^\epsilon \cdot \nabla q^\epsilon, \phi\rangle \to \langle u \cdot \nabla q, \phi\rangle, \quad \langle \mathcal{J}_\epsilon R^\perp q^\epsilon \cdot \nabla q^\epsilon, \phi\rangle \to \langle R^\perp q \cdot \nabla q, \phi\rangle.$$

By a density argument, it follows that

$$\partial_t q + \frac{1}{1+B^2} u \cdot \nabla q + \frac{B}{(1+B^2)} R^\perp q \cdot \nabla q + \frac{1}{(1+B^2)}\Lambda q = 0 \quad \text{in} \quad L^2(0,T;H^{-\frac32}),$$

where u obeys (3.17). The initial condition $q(0) = q_0$ holds due to the weak continuity in time of q.

□

4. Convergence to Inviscid SQG

As mentioned in the introduction, solutions $q(x,\cdot)$ of (1.20) obey (1.24) on the first magnetic time scale, that is $q^B(x,t) = q(x, \frac{1+B^2}{B}t)$ obeys

$$\partial_t q^B + R^\perp q^B \cdot \nabla q^B + \frac{1}{B} u^B \cdot \nabla q^B + \frac{1}{B}\Lambda q^B = 0, \tag{4.1}$$

with

$$u^B = -\frac{1}{1+B^2}\mathbb{P}\left(q^B R q^B + B q^B (u^B)^\perp - B R^\perp q^B\right), \tag{4.2}$$

in $\mathbb{T}^2$, with fixed initial data $q^B(0) = q_0$.

In this section, we prove that any family $\{q^B\}_{B\ge1}$ of weak solutions has a subsequence that converges as $B \to \infty$ to a weak solution of the inviscid SQG equation

$$\partial_t q + (R^\perp q)\cdot \nabla q = 0 \tag{4.3}$$

in $\mathbb{T}^2$.

Theorem 4.1. *Let $T > 0$ and $\{q^B\}_{B\ge1}$ be a family of weak solutions of the DECM equation (4.1) on $[0,T]$. Then the family has a subsequence that converges weakly as $B \to \infty$ in $L^2(0,T;L^2)$ to a weak solution of the inviscid SQG equation (4.3).*

Remark 4.1. *In the laboratory time scale, any family q^B of weak solutions of (1.20) has a subsequence such that $q^B(x, \frac{1+B^2}{B}t)$ converges weakly as $B \to \infty$ to a weak solution $q(x,t)$ of the inviscid SQG equation (4.3) in $L^2(0,T;L^2)$.*

Proof. Since $\left\{q^B\right\}_{B\geq 1}$ is uniformly bounded in $L^2(0,T;L^2)$, there is an unbounded increasing sequence $\{B_n\}_{n\in\mathbb{N}}$ and a subsequence of solutions, denoted by $\left\{q^{B_n}\right\}_{n\in\mathbb{N}}$, converging weakly in $L^2(0,T;L^2)$ to some scalar function $q \in L^2(0,T;L^2)$. Due to the uniform boundedness of $\left\{q^{B_n}\right\}_{n\in\mathbb{N}}$ in $L^\infty(0,T;L^\infty)$, the divergence-free property obeyed by u^{B_n} and $R^\perp q^{B_n}$, the boundedness of the Riesz transform on L^2, and the fact that $B_n \geq 1$, we have

$$\begin{aligned}\|\partial_t \Lambda^{-1} q^{B_n}\|_{L^2} &\leq \frac{1}{B_n}\|q^{B_n}\|_{L^2} + \frac{1}{B_n}\|\Lambda^{-1}(u^{B_n}\cdot\nabla q^{B_n})\|_{L^2} + \|\Lambda^{-1}(R^\perp q^{B_n}\cdot\nabla q^{B_n})\|_{L^2} \\ &\leq \|q^{B_n}\|_{L^2} + \|u^{B_n}\|_{L^2}\|q^{B_n}\|_{L^\infty} + \|R^\perp q^{B_n}\|_{L^2}\|q^{B_n}\|_{L^\infty} \\ &\leq C\left(\|q_0\|_{L^\infty}^3 + \|q_0\|_{L^\infty}^2 + \|q_0\|_{L^\infty}\right),\end{aligned} \tag{4.4}$$

and consequently, we deduce that $\left\{\partial_t\Lambda^{-1}q^{B_n}\right\}_{n\geq 0}$ is uniformly bounded in $L^2(0,T;L^2)$. As $\left\{\Lambda^{-1}q^{B_n}\right\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^2(0,T;H^1)$, there exists a subsequence, also denoted by $\left\{\Lambda^{-1}q^{B_n}\right\}$, that converges strongly in $L^2(0,T;H^{\frac12})$ to some scalar function Q. But $\left\{q^{B_n}\right\}_{n\in\mathbb{N}}$ converges weakly to q in $L^2(0,T;L^2)$, and so $\left\{\Lambda^{-1}q^{B_n}\right\}_{n\in\mathbb{N}}$ converges weakly to $\Lambda^{-1}q$ in $L^2(0,T;H^1)$. Therefore, $Q = \Lambda^{-1}q$. Now we show that q obeys

$$\int_{\mathbb{T}^2}(q(t)-q_0)\phi(x)dx = -\int_0^t\int_{\mathbb{T}^2}(R^\perp q\cdot\nabla q)\phi(x)dx \tag{4.5}$$

for a.e. $t\in[0,T]$ and for all $\phi\in H^3(\mathbb{T}^2)$. To this end, we rewrite (4.1) as

$$q^{B_n}(t) - q_0 = -\int_0^t R^\perp q^{B_n}\cdot\nabla q^{B_n}ds - \frac{1}{B_n}\int_0^t u^{B_n}\cdot\nabla q^{B_n}dx - \frac{1}{B_n}\int_0^t \Lambda q^{B_n}ds. \tag{4.6}$$

As $\left\{\Lambda^{-1}q^{B_n}\right\}_{n\in\mathbb{N}}$ converges strongly in $L^2(0,T;H^{\frac12})$, it has a subsequence, still denoted $\left\{\Lambda^{-1}q^{B_n}\right\}_{n\in\mathbb{N}}$, which converges strongly in L^2 to $\Lambda^{-1}q$ for a.e. $t\in[0,T]$, and hence,

$$\int_{\mathbb{T}^2}(q^{B_n}-q)\phi(x)dx = \int_{\mathbb{T}^2}\Lambda^{-\frac12}(q^{B_n}-q)\Lambda^{\frac12}\phi dx \to 0 \tag{4.7}$$

for a.e. $t\in[0,T]$ and for all $\phi\in H^{\frac12}$. In view of the uniform bound $\|q^{B_n}\|_{L^2}\leq\|q_0\|_{L^2}$ that holds for all $t\in[0,T]$, we have

$$\frac{1}{B_n}\left|\int_{\mathbb{T}^2}\int_0^t\Lambda q^{B_n}\phi dxds\right| \leq \frac{1}{B_n}\|q_0\|_{L^2}\|\Lambda\phi\|_{L^2}T \to 0, \tag{4.8}$$

and thus,

$$\frac{1}{B_n}\int_0^t\int_{\mathbb{T}^2}\Lambda q^{B_n}\phi dxds \to 0 \tag{4.9}$$

for all $t\in[0,T]$ and $\phi\in H^1$. By making use of the uniform velocity bound $\|u^{B_n}\|_{L^2}\leq C(\|q_0\|_{L^\infty}^2+\|q_0\|_{L^\infty})$, we estimate

$$\begin{aligned}\left|\int_0^t\int_{\mathbb{T}^2}u^{B_n}\cdot\nabla q^{B_n}\phi dxds\right| &\leq C\int_0^t\|u^{B_n}\|_{L^2}\|q^{B_n}\|_{L^\infty}\|\nabla\phi\|_{L^2}dx \\ &\leq C\|q_0\|_{L^\infty}^2(\|q_0\|_{L^\infty}+1)\|\nabla\phi\|_{L^2}T,\end{aligned} \tag{4.10}$$

and hence,

$$\frac{1}{B_n}\int_0^t\int_{\mathbb{T}^2}u^{B_n}\cdot\nabla q^{B_n}\phi dxds \to 0 \tag{4.11}$$

for all $t\in[0,T]$ and $\phi\in H^1$. Finally, we consider the difference

$$\int_0^t\int_{\mathbb{T}^2}R^\perp q^{B_n}\cdot\nabla q^{B_n}\phi\, dxds - \int_0^t\int_{\mathbb{T}^2}R^\perp q\cdot\nabla q\phi\, dxds, \tag{4.12}$$

and we decompose it into the sum of two terms $\mathcal{I}_n$ and $\mathcal{O}_n$ where

$$\mathcal{I}_n = \int_0^t \int_{\mathbb{T}^2} \left(R^\perp q \cdot \nabla(q^{B_n} - q) + R^\perp(q^{B_n} - q) \cdot \nabla q\right) \phi dxds \tag{4.13}$$

and

$$\mathcal{O}_n = \int_0^t \int_{\mathbb{T}^2} R^\perp(q^{B_n} - q) \cdot \nabla(q^{B_n} - q)\phi dxds. \tag{4.14}$$

Since $q \in L^\infty(0,T;L^\infty)$, it follows that both quantities $R^\perp q \cdot \nabla\phi$ and $R^\perp \cdot (q\nabla\phi)$ lie in $L^2(0,T;L^2)$ for any $\phi \in H^2$, and thus $\mathcal{I}_n \to 0$ by the weak convergence of q^{B_n} to q in $L^2(0,T;L^2)$. As for the term $\mathcal{O}_n$, we have

$$\mathcal{O}_n = -\int_0^t \int_{\mathbb{T}^2} R^\perp(q^{B_n} - q)(q^{B_n} - q) \cdot \nabla\phi dxds \tag{4.15}$$

via integration by parts, and thus

$$\begin{aligned}
\mathcal{O}_n &= -\int_0^t \int_{\mathbb{T}^2} \nabla^\perp\Lambda^{-1}(q^{B_n} - q)(q^{B_n} - q) \cdot \nabla\phi dxds \\
&= \int_0^t \int_{\mathbb{T}^2} \Lambda^{-1}(q^{B_n} - q)\nabla^\perp(q^{B_n} - q) \cdot \nabla\phi dxds \\
&= \int_0^t \int_{\mathbb{T}^2} \Lambda^{-1}(q^{B_n} - q)\Lambda\nabla^\perp\Lambda^{-1}(q^{B_n} - q) \cdot \nabla\phi dxds \\
&= \int_0^t \int_{\mathbb{T}^2} R^\perp(q^{B_n} - q) \cdot \Lambda\left(\Lambda^{-1}(q^{B_n} - q)\nabla\phi\right) dxds \\
&= \int_0^t \int_{\mathbb{T}^2} R^\perp(q^{B_n} - q) \cdot \left[\Lambda(\Lambda^{-1}(q^{B_n} - q)\nabla\phi) - \nabla\phi\Lambda(\Lambda^{-1}(q^{B_n} - q)\right] dxds - \mathcal{O}_n.
\end{aligned} \tag{4.16}$$

The latter yields the identity

$$\mathcal{O}_n = \frac{1}{2}\int_0^t \int_{\mathbb{T}^2} R^\perp(q^{B_n} - q)[\Lambda, \nabla\phi]\Lambda^{-1}(q^{B_n} - q)dxds. \tag{4.17}$$

An application of the Cauchy-Schwarz inequality in the spatial variable gives rise to

$$|\mathcal{O}_n| \le C\int_0^t \left(\|q^{B_n} - q\|_{L^2}\|[\Lambda, \nabla\phi]\Lambda^{-1}(q^{B_n} - q)\|_{L^2}\right) ds. \tag{4.18}$$

Using the periodic commutator estimate

$$\|[\Lambda, f]g\|_{L^2} \le C\|\nabla f\|_{L^4}\|g\|_{L^4} \tag{4.19}$$

that holds for any $f \in W^{1,4}$ and $g \in L^4$, and standard continuous Sobolev embeddings, we bound

$$\|[\Lambda, \nabla\phi]\Lambda^{-1}(q^{B_n} - q)\|_{L^2} \le C\|\phi\|_{H^3}\|\Lambda^{-\frac{1}{2}}(q^{B_n} - q)\|_{L^2}, \tag{4.20}$$

and consequently, we obtain

$$|\mathcal{O}_n| \le C\|\phi\|_{H^3}\|q_0\|_{L^2}\int_0^t \|\Lambda^{-\frac{1}{2}}(q^{B_n} - q)\|_{L^2} ds \to 0 \tag{4.21}$$

due to the strong convergence of $\Lambda^{-1}q^{B_n}$ to $\Lambda^{-1}q$ in $L^2(0,T;H^{\frac{1}{2}})$. Therefore, q is a weak solution of the inviscid SQG equation. □

5. Global Regularity and Convergence to Dissipative Critical SQG

We consider here the DECM equation for the boosted field Q. As mentioned in the introduction, if $q(x,\cdot)$ is a solution of (1.20) in laboratory time scale, then

$$Q(x,t) = Bq(x,(1+B^2)t) \tag{5.1}$$

solves (1.26), which we write as

$$\partial_t Q + R^\perp\left(\left(1 + \frac{1}{1+B^2}\right)Q + w\right) \cdot \nabla Q + \Lambda Q = 0 \tag{5.2}$$

with

$$w = W[Q], \tag{5.3}$$

which is the unique solution of (2.34) discussed in Proposition 2.2. We recall that Proposition 2.2 shows that w is small relative to $\|Q\|^2$ in H^3 when B is large. When B is of order 1, this term is difficult to handle and global existence for large data of solutions of the equation is not known, just as in the case of electroconvection in porous media. We prove here

Theorem 5.1. *There exist constants $c > 0$, $C > 0$ such that if $Q_0 \in H^3(\mathbb{T}^2)$ obeys*

$$\|Q_0\|_{L^\infty} \le c \tag{5.4}$$

and

$$\|Q_0\|_{H^3} \le CB(1 + B^2), \tag{5.5}$$

then the solution of (5.2), (5.3) with initial data Q_0 is unique, exists for all time and obeys

$$\|Q(t)\|_{H^3} \le \|Q_0\|_{H^3} e^{-\frac{t}{4}} \tag{5.6}$$

for all $t \ge 0$.

Remark 5.1. *In laboratory time scale, the result says that if q_0 obeys $B\|q_0\|_{L^\infty} \le c$ and $\|q_0\|_{H^3} \le C(1+B^2)$, then*

$$\|q(\cdot,t)\|_{H^3} \le \|q_0\|_{H^3} e^{-\frac{t}{4(1+B^2)}}. \tag{5.7}$$

Proof. We start by observing that, without loss of generality we may assume that

$$\int_{\mathbb{T}^2} Q(x,t)dx = 0, \tag{5.8}$$

because this average is time independent. Secondly, we note that the L^∞ norm of Q is nonincreasing in time. We denote by M

$$\|Q_0\|_{L^\infty} = M \tag{5.9}$$

and we have

$$\|Q(\cdot,t)\|_{L^\infty} \le M \tag{5.10}$$

a priori. Next, we use the fact that

$$\|Q\|_{H^3}^2 \sim \sum_{|\alpha|=3} \int_{\mathbb{T}^2} |\partial^\alpha Q(x,t)|^2 dx \tag{5.11}$$

because Q has mean zero. Above $\alpha = (\alpha_1, \alpha_2)$ is a multi-index, with $\alpha_j \in \mathbb{N}$. We recall the notation $\alpha! = \alpha_1!\alpha_2!$ and $|\alpha| = \alpha_1 + \alpha_2$. We compute the evolution

$$\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|=3}\int_{\mathbb{T}^2} |\partial^\alpha Q(x,t)|^2 dx + D^2 = R_1 + R_2, \tag{5.12}$$

where

$$D^2 = \sum_{|\alpha|=3} \|\partial^\alpha Q\|^2_{H^{\frac{1}{2}}}, \tag{5.13}$$

and

$$R_1 = (1 + \frac{1}{1+B^2}) \sum_{|\alpha|=3} \sum_{\beta+\gamma=\alpha, |\beta|>0} \frac{\alpha!}{\beta!\gamma!} \int_{\mathbb{T}^2} \left(\partial^\beta(R^\perp Q)\cdot\nabla\partial^\gamma Q\right)\partial^\alpha Q dx, \tag{5.14}$$

and

$$R_2 = \sum_{|\alpha|=3} \sum_{\beta+\gamma=\alpha, |\beta|>0} \frac{\alpha!}{\beta!\gamma!} \int_{\mathbb{T}^2} \left(\partial^\beta(R^\perp w)\cdot\nabla\partial^\gamma Q\right)\partial^\alpha Q dx. \tag{5.15}$$

The terms with $\beta = 0$ vanish because $R^\perp Q$ and $R^\perp w$ are divergence-free. Therefore $|\gamma| \le 2$. We estimate the two contributions R_1 and R_2 differently. The first one, like in critical SQG [9], is going to be absorbed in D^2 if M is small,

$$|R_1| \le CMD^2. \tag{5.16}$$

The second contribution is small compared to $\|Q\|_{H^3}^4$,

$$|R_2| \le C\frac{1}{B^2(1+B^2)^2}\|Q\|_{H^3}^4. \tag{5.17}$$

We note that

$$D^2 \ge C\|Q\|_{H^{3.5}}^2 \tag{5.18}$$

because Q has mean zero.

For R_1 we use L^3 bounds.

$$\|\partial^\beta Q\|_{L^3} \le CM^a D^{1-a} \tag{5.19}$$

where $a = \frac{19-6|\beta|}{15}$,

$$\|\nabla\partial^\gamma Q\|_{L^3} \le CM^b D^{1-b} \tag{5.20}$$

where $b = \frac{19-6(|\gamma|+1)}{15}$ and

$$\|\partial^\alpha Q\|_{L^3} \le CM^c D^{1-c} \tag{5.21}$$

where $c = \frac{19-18}{15}$. These are all the same Gagliardo-Nirenberg inequality

$$\|\partial^s Q\|_{L^3} \le C\|Q\|_{L^\infty}^{\frac{19-6|s|}{15}}\|Q\|_{H^{3.5}}^{\frac{6|s|-4}{15}} \tag{5.22}$$

for $1 \le |s| \le 3$. Noting that $a+b+c = 1$ because $|\beta|+|\gamma| = 3$, we proved (5.16).

For the term R_2, if $|\beta| = 1$ then we estimate $L^\infty - L^2 - L^2$.

$$\left|\int_{\mathbb{T}^2}(\partial^\beta R^\perp w\cdot\nabla\partial^\gamma Q)\partial^\alpha Q dx\right| \le C\|\nabla R^\perp w\|_{L^\infty}\|Q\|_{H^3}^2 \le C\|w\|_{H^3}\|Q\|_{H^3}^2. \tag{5.23}$$

If $|\beta| = 2$ then we estimate $L^4 - L^4 - L^2$. We use the inequality

$$\|\partial^\beta R^\perp w\|_{L^4} \le C\|w\|_{L^2}^{\frac{1}{6}}\|w\|_{H^3}^{\frac{5}{6}} \tag{5.24}$$

$$\|\nabla\partial^\gamma Q\|_{L^4} \le C\|Q\|_{L^2}^{\frac{1}{6}}\|Q\|_{H^3}^{\frac{5}{6}}. \tag{5.25}$$

These are two instances of the inequality

$$\|\partial^s Q\|_{L^4} \le C\|Q\|_{L^2}^{\frac{1}{6}}\|Q\|_{H^3}^{\frac{5}{6}} \tag{5.26}$$

for $|s| = 2$, so we have for $|\beta| = 2$,

$$\left|\int_{\mathbb{T}^2}(\partial^\beta R^\perp w\cdot\nabla\partial^\gamma Q)\partial^\alpha Q dx\right| \le C\|w\|_{L^2}^{\frac{1}{6}}\|w\|_{H^3}^{\frac{5}{6}}\|Q\|_{H^3}^2 \le C\|w\|_{H^3}\|Q\|_{H^3}^2. \tag{5.27}$$

Finally, for $|\beta| = 3$ we estimate using $L^2 - L^\infty - L^2$ to obtain

$$\left|\int_{\mathbb{T}^2}(\partial^\beta R^\perp w\cdot\nabla\partial^\gamma Q)\partial^\alpha Q dx\right| \le C\|w\|_{H^3}\|\nabla Q\|_{L^\infty}\|Q\|_{H^3} \le C\|w\|_{H^3}\|Q\|_{H^3}^2. \tag{5.28}$$

Putting these together we have

$$|R_2| \le C\|w\|_{H^3}\|Q\|_{H^3}^2. \tag{5.29}$$

Now we use the bound (2.36) of Proposition 2.2. Assuming without loss of generality that $M \le 1$ we obtain from (2.36)

$$\|w\|_{H^3} \le C\frac{M}{B^2(1+B^2)^2}\|Q\|_{H^3}(1+\|Q\|_{H^3}) \tag{5.30}$$

and using it in (5.29) we obtain (5.17).

Considering now

$$y(t) = \sum_{|\alpha|=3}\int_{\mathbb{T}^2}|\partial^\alpha Q(x,t)|^2 dx \tag{5.31}$$

and taking $CM \le \frac{1}{2}$ to absorb the term R_1 in $\frac{1}{2}D^2$ in view of (5.16), we have

$$\frac{dy}{dt} + y \le C\frac{1}{B^2(1+B^2)^2}y^2 \tag{5.32}$$

with some constant $C > 0$, where we used a Poincaré inequality $D^2 \ge y$. Thus, if

$$y(0) \le \frac{1}{3C}B^2(1+B^2)^2 \tag{5.33}$$

then, as long as

$$y(t) \le \frac{1}{2C}B^2(1+B^2)^2, \tag{5.34}$$

we have that y is decreasing in time and obeys

$$\frac{dy}{dt} + \frac{1}{2}y \le 0, \tag{5.35}$$

which implies

$$y(t) \le y(0)e^{-\frac{t}{2}}. \tag{5.36}$$

But, because y is decreasing, $y(t) \le y(0)$ and because of (5.33), the condition (5.34) is never violated, so (5.36) holds for all time. The condition (5.5) is the square root of (5.33) (with another name for the constant). The bound (5.6) is obtained by taking square roots of both sides of (5.36).

□

Theorem 5.2. *Let $T > 0$ and let $\overline{Q}$ be a solution on $[0,T]$ of the dissipative critical SQG equation*

$$\partial_t\overline{Q} + (R^\perp\overline{Q})\cdot\nabla\overline{Q} + \Lambda\overline{Q} = 0 \tag{5.37}$$

with initial data $\overline{Q}_0 \in H^3$. We consider a strong solution $Q \in L^\infty(0,T;H^{\frac{1}{2}}\cap L^\infty)\cap L^2(0,T;H^1)$ of (5.2), (5.3) with initial data $\overline{Q}_0$. Then, setting $\overline{C} = T\exp\{\int_0^T \|\nabla\overline{Q}\|_{L^\infty}dt\}$ we have

$$\sup_{t\in[0,T]}\|Q(t)-\overline{Q}(t)\|_{L^2} \le \overline{C}\left(\frac{1}{1+B^2}\|\overline{Q}_0\|_{L^2} + C\frac{1}{B^2(1+B^2)^2}\|\overline{Q}_0\|^2_{L^4}\right). \tag{5.38}$$

Remark 5.2. *In laboratory time scale, this result says that strong solutions q of DECM equations (1.20) with initial data $q_0 = \frac{\overline{Q}_0}{B} \in H^3$ obey*

$$\sup_{t\in[0,T]}\|Bq(\cdot,(1+B^2)t)-\overline{Q}(t)\|_{L^2} \le \overline{C}\left(\frac{1}{1+B^2}\|\overline{Q}_0\|_{L^2} + C\frac{1}{B^2(1+B^2)^2}\|\overline{Q}_0\|^2_{L^4}\right) = O\left(\frac{1}{1+B^2}\right) \tag{5.39}$$

where $\overline{Q}$ is the global smooth solution of critical SQG (5.37) with initial data $\overline{Q}_0$.

Proof. Let $\delta = Q - \overline{Q}$. Writing

$$V = \left(1+\frac{1}{1+B^2}\right)Q + w, \tag{5.40}$$

we have

$$\partial_t\delta + \Lambda\delta + (R^\perp V)\cdot\nabla\delta + R^\perp(V-\overline{Q})\cdot\nabla\overline{Q} = 0. \tag{5.41}$$

Because Q is a strong solution we have that $\partial_t Q \in L^{\frac{4}{3}}$, $Q \in L^\infty(0,T;L^4)$, and because $\overline{Q}$ is smooth, we can compute the evolution of δ in L^2, and after cancellations we have

$$\frac{1}{2}\frac{d}{dt}\|\delta\|^2_{L^2} + \|\delta\|^2_{H^{\frac{1}{2}}} \le \|\nabla\overline{Q}\|_{L^\infty}\|\delta\|_{L^2}\left((1+\frac{1}{1+B^2})\|\delta\|_{L^2} + \frac{1}{1+B^2}\|\overline{Q}\|_{L^2} + \|w\|_{L^2}\right). \tag{5.42}$$

Using (2.35) we have

$$\frac{d}{dt}\|\delta\|_{L^2} \le \|\nabla\overline{Q}\|_{L^\infty}\left((1+\frac{1}{1+B^2})\|\delta\|_{L^2} + \frac{1}{1+B^2}\|\overline{Q}_0\|_{L^2} + C\frac{1}{B^2(1+B^2)^2}\|\overline{Q}_0\|^2_{L^4}\right), \tag{5.43}$$

and thus

$$\sup_{t\in[0,T]}\|\delta(t)\|_{L^2}\le\left(\frac{1}{1+B^2}\|\overline{Q}_0\|_{L^2}+C\frac{1}{B^2(1+B^2)^2}\|\overline{Q}_0\|_{L^4}^2\right)T\exp\{\int_0^T\|\nabla\overline{Q}\|_{L^\infty}dt\}. \tag{5.44}$$

□

Appendix A. Proof of Proposition 2.2

Step 1. L^2 **bounds for** w**.** We multiply the equation (2.34) by w and integrate over $\mathbb{T}^2$. Since $R^\perp w\cdot Rw=0$, the nonlinear term in w vanishes, and we obtain the identity

$$\|w\|_{L^2}^2=\frac{1}{B^2(1+B^2)^2}\int_{\mathbb{T}^2}QR^\perp Q\cdot Rwdx. \tag{A.1}$$

Applications of the Cauchy-Schwarz and Young inequalities yield the bound

$$\|w\|_{L^2}^2\le\frac{1}{2}\|w\|_{L^2}^2+\frac{C}{B^4(1+B^2)^4}\|QR^\perp Q\|_{L^2}^2, \tag{A.2}$$

from which we deduce that

$$\|w\|_{L^2}^2\le\frac{C}{B^4(1+B^2)^4}\|Q\|_{L^4}^4, \tag{A.3}$$

after making use of the boundedness of the Riesz transform on L^4.

Step 2. H^1 **bounds for** w**.** We take the L^2 inner product of the equation (2.34) obeyed by w with $-\Delta w$. Since ∇ and $R^\perp$ commutes and $R^\perp\nabla w\cdot R\nabla w=0$, we have

$$\int_{\mathbb{T}^2}\nabla R\cdot(QR^\perp w)\cdot\nabla wdx=-\int_{\mathbb{T}^2}\nabla QR^\perp w\cdot R\nabla wdx. \tag{A.4}$$

Using the interpolation inequality

$$\|\nabla Q\|_{L^\infty}\le C\|Q\|_{L^\infty}^{\frac{1}{2}}\|Q\|_{H^3}^{\frac{1}{2}}, \tag{A.5}$$

we estimate

$$\begin{aligned}\|\nabla w\|_{L^2}^2&=-\frac{1}{1+B^2}\int_{\mathbb{T}^2}\nabla QR^\perp w\cdot R\nabla wdx+\frac{1}{B^2(1+B^2)^2}\int_{\mathbb{T}^2}\nabla R\cdot(QR^\perp Q)\cdot\nabla wdx\\&\le\frac{1}{2}\|\nabla w\|_{L^2}^2+\frac{C}{(1+B^2)^2}\|\nabla QR^\perp w\|_{L^2}^2+\frac{C}{B^4(1+B^2)^4}\|\nabla R\cdot(QR^\perp Q)\|_{L^2}^2\\&\le\frac{1}{2}\|\nabla w\|_{L^2}^2+\frac{C}{(1+B^2)^2}\|Q\|_{L^\infty}\|Q\|_{H^3}\|w\|_{L^2}^2+\frac{C}{B^4(1+B^2)^4}\|Q\|_{L^2}^2\|Q\|_{L^\infty}\|Q\|_{H^3}.\end{aligned} \tag{A.6}$$

Due to the boundedness of w in L^2 obtained in (A.3), we infer that

$$\|\nabla w\|_{L^2}^2\le\frac{C(\|Q\|_{L^\infty}^5+\|Q\|_{L^\infty}^3)}{B^4(1+B^2)^4}\|Q\|_{H^3}. \tag{A.7}$$

Step 3. H^2 **bounds for** w**.** We apply Δ to the w-equation (2.34) and take the L^2 inner product with Δw. Using the cancellation law $R^\perp\Delta w\cdot R\Delta w=0$, we have

$$\int_{\mathbb{T}^2}\Delta R\cdot(QR^\perp w)\cdot\Delta wdx=-\int_{\mathbb{T}^2}\left(\Delta QR^\perp w+\nabla Q\cdot\nabla R^\perp w\right)\cdot R\Delta wdx. \tag{A.8}$$

Consequently, it holds that

$$\begin{aligned}\|\Delta w\|_{L^2}^2 &\le \frac{1}{2}\|\Delta w\|_{L^2}^2 + \frac{C}{(1+B^2)^2}\left(\|\Delta Q\|_{L^4}^2\|R^\perp w\|_{L^4}^2 + \|\nabla Q\|_{L^\infty}^2\|\nabla R^\perp w\|_{L^2}^2\right) + \frac{C}{B^4(1+B^2)^4}\|QR^\perp Q\|_{H^2}^2\\ &\le \frac{1}{2}\|\Delta w\|_{L^2}^2 + \frac{C}{(1+B^2)^2}\left(\|Q\|_{L^\infty}^{\frac{1}{2}}\|Q\|_{H^3}^{\frac{3}{2}}\|w\|_{L^2}\|\nabla w\|_{L^2} + \|Q\|_{L^\infty}\|Q\|_{H^3}\|\nabla w\|_{L^2}^2\right)\\ &\quad + \frac{C}{B^4(1+B^2)^4}\|Q\|_{L^\infty}^2\|Q\|_{H^3}^2\\ &\le \frac{1}{2}\|\Delta w\|_{L^2}^2 + \frac{C(\|Q\|_{L^\infty}^5 + \|Q\|_{L^\infty}^3 + \|Q\|_{L^\infty}^2)}{B^4(1+B^2)^4}\|Q\|_{H^3}^2.\end{aligned} \tag{A.9}$$

where the last two inequalities follow from standard continuous Sobolev embeddings, the boundedness of the Riesz transform on Sobolev and L^p spaces, the fact that H^2 is a Banach Algebra, Gagliardo-Nirenberg interpolation inequalities, and application of the bound (A.7). Thus, we deduce that

$$\|\Delta w\|_{L^2}^2 \le \frac{C(\|Q\|_{L^\infty}^5 + \|Q\|_{L^\infty}^3 + \|Q\|_{L^\infty}^2)}{B^4(1+B^2)^4}\|Q\|_{H^3}^2. \tag{A.10}$$

Step 4. H^3 bounds for w. The cancellation law $R^\perp\nabla\Delta w \cdot R\nabla\Delta w = 0$ gives rise to

$$\begin{aligned}\|\nabla\Delta w\|_{L^2}^2 = &-\frac{1}{1+B^2}\int_{\mathbb{T}^2}\left[\nabla\Delta(QR^\perp w) - QR^\perp\nabla\Delta w\right]\cdot R\nabla\Delta w dx\\ &+ \frac{1}{B^2(1+B^2)^2}\int_{\mathbb{T}^2}\nabla\Delta(QR^\perp Q)\cdot R\nabla\Delta w dx.\end{aligned} \tag{A.11}$$

By expanding and simplifying the commutator $[\nabla\Delta, QR^\perp]w$, and estimating using Hölder's inequality and continuous Sobolev embeddings, we obtain

$$\|\nabla\Delta(QR^\perp w) - QR^\perp\nabla\Delta w\|_{L^2}^2 \le C\|Q\|_{H^3}^2\|\Delta w\|_{L^2}^2. \tag{A.12}$$

Hence, we have

$$\|\nabla\Delta w\|_{L^2}^2 \le \frac{1}{2}\|\nabla\Delta w\|_{L^2}^2 + \frac{C}{(1+B^2)^2}\|Q\|_{H^3}^2\|\Delta w\|_{L^2}^2 + \frac{1}{B^4(1+B^2)^4}\|Q\|_{L^\infty}^2\|Q\|_{H^3}^2, \tag{A.13}$$

and so

$$\|\nabla\Delta w\|_{L^2}^2 \le \frac{C(\|Q\|_{L^\infty}^5 + \|Q\|_{L^\infty}^3 + \|Q\|_{L^\infty}^2)}{B^4(1+B^2)^4}\|Q\|_{H^3}^4 + \frac{C\|Q\|_{L^\infty}^2}{B^4(1+B^2)^4}\|Q\|_{H^3}^2. \tag{A.14}$$

due to (A.10).

APPENDIX B. PROOF OF PROPOSITION 3.1

Proof. For each integer $n \ge 1$, we consider the Galerkin approximants

$$\mathbb{P}_n\theta = \sum_{j=1}^n (\theta, \omega_j)_{L^2}\omega_j \tag{B.1}$$

where ω_j are the eigenfunctions of the negative Laplacian operator with periodic boundary conditions. For fixed $\epsilon > 0$ and $n \ge 1$, we consider the Galerkin approximate system

$$\partial_t q_n^\epsilon + \frac{1}{1+B^2}\mathbb{P}_n(u_n^\epsilon\cdot\nabla q_n^\epsilon) + \frac{B}{(1+B^2)}\mathbb{P}_n(J_\epsilon R^\perp q_n^\epsilon\cdot\nabla q_n^\epsilon) + \frac{1}{(1+B^2)}\Lambda q_n^\epsilon - \epsilon\Delta q_n^\epsilon = 0, \tag{B.2}$$

$$u_n^\epsilon = \mathbb{P}_n\left(J_\epsilon T_{J_\epsilon q_n^\epsilon}^{-1}\left[-\mathbb{P}\left(J_\epsilon q_n^\epsilon RJ_\epsilon q_n^\epsilon\right) + BR^\perp J_\epsilon q_n^\epsilon\right]\right), \tag{B.3}$$

with initial data $q_n^\epsilon(0) = \mathbb{P}_n q^\epsilon(0)$ and periodic boundary conditions. The latter is a finite-dimensional system of autonomous ODEs and has a unique smooth solution on a maximal time interval. Next, we derive *a priori* bounds.

Step 1. L^2 **bounds.** We take the L^2 inner product of the equation (B.2) obeyed by q_n^ϵ with q_n^ϵ. Using the self-adjointness of the Galerkin projectors $\mathbb{P}_n$, the identity $\mathbb{P}_n q_n^\epsilon = q_n^\epsilon$, and the divergence-free condition obeyed by both u_n^ϵ and $R^\perp q_n^\epsilon$, the nonlinear terms vanish, namely

$$(\mathbb{P}_n(u_n^\epsilon \cdot \nabla q_n^\epsilon), q_n^\epsilon)_{L^2} = (\mathbb{P}_n(J_\epsilon R^\perp q_n^\epsilon \cdot \nabla q_n^\epsilon), q_n^\epsilon)_{L^2} = 0. \tag{B.4}$$

This gives rise to the following energy balance

$$\frac{1}{2}\frac{d}{dt}\|q_n^\epsilon\|_{L^2}^2 + \frac{1}{(1+B^2)}\|\Lambda^{\frac{1}{2}} q_n^\epsilon\|_{L^2}^2 + \epsilon\|\nabla q_n^\epsilon\|_{L^2}^2 = 0. \tag{B.5}$$

Integrating in time from 0 to t and using the uniform boundedness of $\mathbb{P}_n$ on L^2, we infer that

$$\|q_n^\epsilon(t)\|_{L^2}^2 + \frac{2}{1+B^2}\int_0^t \|\Lambda^{\frac{1}{2}} q_n^\epsilon(s)\|_{L^2}^2 ds + 2\epsilon\int_0^t \|\nabla q_n^\epsilon(s)\|_{L^2}^2 ds = \|q_n^\epsilon(0)\|_{L^2}^2 \le \|q^\epsilon(0)\|_{L^2}^2. \tag{B.6}$$

for all $t \ge 0$.

Step 2. H^m **bounds for** $m \ge 1$**.** We multiply the equation (B.2) obeyed by q_n^ϵ by $\Lambda^{2m} q_n^\epsilon$ and we integrate over $\mathbb{T}^2$. We obtain the evolution equation

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Lambda^m q_n^\epsilon\|_{L^2}^2 + \frac{1}{(1+B^2)}\|\Lambda^{m+\frac{1}{2}} q_n^\epsilon\|_{L^2}^2 + \epsilon\|\Lambda^{m+1} q_n^\epsilon\|_{L^2}^2 \\
&= -\frac{1}{1+B^2}\int_{\mathbb{T}^2} \Lambda^{m-1}(u_n^\epsilon \cdot \nabla q_n^\epsilon)\Lambda^{m+1} q_n^\epsilon dx - \frac{B}{(1+B^2)}\int_{\mathbb{T}^2} \Lambda^{m-1}(J_\epsilon R^\perp q_n^\epsilon \cdot \nabla q_n^\epsilon)\Lambda^{m+1} q_n^\epsilon dx.
\end{aligned} \tag{B.7}$$

Applications of the Cauchy-Schwarz and Young inequalities give rise to the differential inequality

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Lambda^m q_n^\epsilon\|_{L^2}^2 + \frac{1}{(1+B^2)}\|\Lambda^{m+\frac{1}{2}} q_n^\epsilon\|_{L^2}^2 + \frac{\epsilon}{2}\|\Lambda^{m+1} q_n^\epsilon\|_{L^2}^2 \\
&\qquad \le \frac{1}{(1+B^2)^2}\|\Lambda^{m-1}(u_n^\epsilon \cdot \nabla q_n^\epsilon)\|_{L^2}^2 + \frac{B^2}{(1+B^2)^2}\|\Lambda^{m-1}(J_\epsilon R^\perp q_n^\epsilon \cdot \nabla q_n^\epsilon)\|_{L^2}^2.
\end{aligned} \tag{B.8}$$

By making use of periodic fractional product estimates and standard continuous Sobolev embeddings, we have

$$\begin{aligned}
\|\Lambda^{m-1}(u_n^\epsilon \cdot \nabla q_n^\epsilon)\|_{L^2}^2 &\le C\|\Lambda^{m-1} u_n^\epsilon\|_{L^4}^2 \|\nabla q_n^\epsilon\|_{L^4}^2 + C\|u_n^\epsilon\|_{L^\infty}^2 \|\Lambda^{m-1}\nabla q_n^\epsilon\|_{L^2}^2 \\
&\le C\|u_n^\epsilon\|_{H^{m+1}}^2 \|\Lambda^m q_n^\epsilon\|_{L^2}^2
\end{aligned} \tag{B.9}$$

provided that $m \ge 2$. We point out that the latter estimate trivially holds when $m = 1$ by the Hölder and Sobolev inequalities. By making use of the explicit relation (B.3), the boundedness of $\mathbb{P}_n$ on Sobolev spaces, and the boundedness of the mollifier J_ϵ from L^2 to H^{m+1}, and the boundedness of $T_{J_\epsilon q_n^\epsilon}^{-1}$ on L^2, we estimate

$$\begin{aligned}
\|u_n^\epsilon\|_{H^{m+1}}^2 &\le \epsilon^{-2m-2}\left\|T_{J_\epsilon q_n^\epsilon}^{-1}\left[-\mathbb{P}\left(J_\epsilon q_n^\epsilon R J_\epsilon q_n^\epsilon\right) + BR^\perp J_\epsilon q_n^\epsilon\right]\right\|_{L^2}^2 \\
&\le \epsilon^{-2m-2}(1+B^2)^{-2}\left\|-\mathbb{P}\left(J_\epsilon q_n^\epsilon R J_\epsilon q_n^\epsilon\right) + BR^\perp J_\epsilon q_n^\epsilon\right\|_{L^2}^2 \\
&\le \epsilon^{-2m-2}(1+B^2)^{-2}\left(\|J_\epsilon q_n^\epsilon\|_{L^4}^2 \|R J_\epsilon q_n^\epsilon\|_{L^4}^2 + B^2\|R^\perp J_\epsilon q_n^\epsilon\|_{L^2}^2\right).
\end{aligned} \tag{B.10}$$

As J_ϵ and R are bounded operators on L^4 and L^2, and due to the Ladyzhenskaya interpolation inequality, the latter yields

$$\begin{aligned}
\|u_n^\epsilon\|_{H^{m+1}}^2 &\le C\epsilon^{-2m-2}(1+B^2)^{-2}\left(\|q_n^\epsilon\|_{L^4}^4 + B^2\|q_n^\epsilon\|_{L^2}^2\right) \\
&\le C\epsilon^{-2m-2}(1+B^2)^{-2}\left(\|q_n^\epsilon\|_{L^2}^2\|\nabla q_n^\epsilon\|_{L^2}^2 + B^2\|q_n^\epsilon\|_{L^2}^2\right).
\end{aligned} \tag{B.11}$$

Consequently, we obtain the bound

$$\|\Lambda^{m-1}(u_n^\epsilon \cdot \nabla q_n^\epsilon)\|_{L^2}^2 \le C\epsilon^{-2m-2}(1+B^2)^{-2}\left(\|\nabla q_n^\epsilon\|_{L^2}^2 + B^2\right)\|q_n^\epsilon\|_{L^2}^2 \|\Lambda^m q_n^\epsilon\|_{L^2}^2. \tag{B.12}$$

Similarly, we have

$$\begin{aligned}
\|\Lambda^{m-1}(J_\epsilon R^\perp q_n^\epsilon \cdot \nabla q_n^\epsilon)\|_{L^2}^2 &\le C\|J_\epsilon R^\perp q_n^\epsilon\|_{H^{m+1}}^2 \|\Lambda^m q_n^\epsilon\|_{L^2}^2 \\
&\le C\epsilon^{-2m-2}\|q_n^\epsilon\|_{L^2}^2 \|\Lambda^m q_n^\epsilon\|_{L^2}^2.
\end{aligned} \tag{B.13}$$

Putting all these estimates together, the energy inequality (B.8) boils down to

$$\begin{aligned}\frac{d}{dt}\|\Lambda^m q_n^\epsilon\|_{L^2}^2 + \frac{2}{(1+B^2)}\|\Lambda^{m+\frac{1}{2}} q_n^\epsilon\|_{L^2}^2 + \epsilon\|\Lambda^{m+1} q_n^\epsilon\|_{L^2}^2 \\ \le \frac{C}{\epsilon^{2m+2}(1+B^2)^2}\left[\frac{1}{(1+B^2)^2}\left(\|\nabla q_n^\epsilon\|_{L^2}^2 + B^2\right) + B^2\right]\|q_n^\epsilon\|_{L^2}^2\|\Lambda^m q_n^\epsilon\|_{L^2}^2.\end{aligned} \tag{B.14}$$

Finally, we integrate in time from 0 to t, use the uniform bounds (B.6) derived in Step 1, and deduce that for any $T > 0$,

$$q_n^\epsilon \in L^\infty(0,T;H^m(\mathbb{T}^2)) \cap L^2(0,T;H^{m+1}(\mathbb{T}^2)). \tag{B.15}$$

Step 3. Convergence. The convergence follows from the uniform-in-n boundedness of solutions in the Lebesgue spaces $L^\infty(0,T;H^m(\mathbb{T}^2))$ and $L^2(0,T;H^{m+1}(\mathbb{T}^2))$ for all $m \in \mathbb{N}$ and the Aubin-Lions lemma. We point out that the u_n^ϵ converges to u^ϵ due to the uniform boundedness of mollifiers on Sobolev spaces and the Lipschitz estimates (2.39). The proof is standard and we omit the details.

Step 4. Uniqueness. Suppose there are two smooth solutions q_1^ϵ and q_2^ϵ with same initial data and with u_1^ϵ and u_2^ϵ determined by q_1^ϵ and q_2^ϵ respectively. Denoting the differences by $q^\epsilon = q_1^\epsilon - q_2^\epsilon$ and $u^\epsilon = u_1^\epsilon - u_2^\epsilon$, we have

$$\begin{aligned}\partial_t q^\epsilon + \frac{1}{1+B^2}(u_1^\epsilon\cdot\nabla q^\epsilon + u^\epsilon\cdot\nabla q_2^\epsilon) + \frac{B}{(1+B^2)}(J_\epsilon R^\perp q_1^\epsilon\cdot\nabla q^\epsilon + J_\epsilon R^\perp q^\epsilon\cdot\nabla q_2^\epsilon) \\ + \frac{1}{(1+B^2)}\Lambda q^\epsilon - \epsilon\Delta q^\epsilon = 0.\end{aligned} \tag{B.16}$$

Multiplying the latter by q^ϵ and integrating over $\mathbb{T}^2$ give

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|q^\epsilon\|_{L^2}^2 + \frac{1}{(1+B^2)}\|\Lambda^{\frac{1}{2}} q^\epsilon\|_{L^2}^2 + \epsilon\|\nabla q^\epsilon\|_{L^2}^2 + \frac{1}{1+B^2}\int_{\mathbb{T}^2} u^\epsilon\cdot\nabla q_2^\epsilon q^\epsilon dx \\ + \frac{B}{(1+B^2)}\int_{\mathbb{T}^2} J_\epsilon R^\perp q^\epsilon\cdot\nabla q_2^\epsilon q^\epsilon dx = 0.\end{aligned} \tag{B.17}$$

Using the boundedness of the operator $T_{J_\epsilon q_2^\epsilon}^{-1}$ on L^2, the Lipschitz estimate (2.39), and continuous Sobolev embeddings, we estimate $\|u^\epsilon\|_{L^2}$ as follows,

$$\begin{aligned}&\|u^\epsilon\|_{L^2} = \|u_1^\epsilon - u_2^\epsilon\|_{L^2} \\ &\le \left\|J_\epsilon(T_{J_\epsilon q_1^\epsilon}^{-1} - T_{J_\epsilon q_2^\epsilon}^{-1})\left[-\mathbb{P}\left(J_\epsilon q_1^\epsilon R J_\epsilon q_1^\epsilon\right) + BR^\perp J_\epsilon q_1^\epsilon\right]\right\|_{L^2} \\ &\qquad + \left\|J_\epsilon T_{J_\epsilon q_2^\epsilon}^{-1}\left[-\mathbb{P}\left(J_\epsilon q_1^\epsilon R J_\epsilon q^\epsilon + J_\epsilon q^\epsilon R J_\epsilon q_2^\epsilon\right) + BR^\perp J_\epsilon q^\epsilon\right]\right\|_{L^2} \\ &\le C\|J_\epsilon q^\epsilon\|_{L^\infty}\left\|-\mathbb{P}\left(J_\epsilon q_1^\epsilon R J_\epsilon q_1^\epsilon\right) + BR^\perp J_\epsilon q_1^\epsilon\right\|_{L^2} + C\|J_\epsilon q_1^\epsilon R J_\epsilon q^\epsilon + J_\epsilon q^\epsilon R J_\epsilon q_2^\epsilon\|_{L^2} + C\|q^\epsilon\|_{L^2} \\ &\le \frac{C}{\epsilon^2}\|q^\epsilon\|_{L^2}\left\|-\mathbb{P}\left(J_\epsilon q_1^\epsilon R J_\epsilon q_1^\epsilon\right) + BR^\perp J_\epsilon q_1^\epsilon\right\|_{L^2} + C(\|q_1^\epsilon\|_{H^2} + \|q_2^\epsilon\|_{H^2} + 1)\|q^\epsilon\|_{L^2} \\ &\le \frac{C}{\epsilon^2}(\|q_1^\epsilon\|_{L^4}^2 + \|q_1^\epsilon\|_{L^2})\|q^\epsilon\|_{L^2} + C(\|q_1^\epsilon\|_{H^2} + \|q_2^\epsilon\|_{H^2} + 1)\|q^\epsilon\|_{L^2}.\end{aligned} \tag{B.18}$$

Therefore, we obtain the bound

$$\int_{\mathbb{T}^2} u^\epsilon\cdot\nabla q_2^\epsilon q^\epsilon dx \le C_\epsilon(1 + \|q_1^\epsilon\|_{H^2}^4 + \|q_2^\epsilon\|_{H^2}^2)\|q^\epsilon\|_{L^2}^2. \tag{B.19}$$

By making use of the Hölder and Sobolev inequalities, we have

$$\int_{\mathbb{T}^2} J_\epsilon R^\perp q^\epsilon\cdot\nabla q_2^\epsilon q^\epsilon dx \le C\|q_2^\epsilon\|_{H^2}\|q^\epsilon\|_{L^2}^2. \tag{B.20}$$

Combining the above estimates gives rise to the energy inequality

$$\frac{d}{dt}\|q^\epsilon\|_{L^2}^2 \le C_\epsilon(1 + \|q_1^\epsilon\|_{H^2}^4 + \|q_2^\epsilon\|_{H^2}^2)\|q^\epsilon\|_{L^2}^2. \tag{B.21}$$

Consequently, the uniqueness of solutions follows from the Gronwall inequality.

□

ACKNOWLEDGEMENT

The work of PC was partially supported by NSF grant DMS-2106528 and by a Simons Collaboration Grant 601960. The work of M.I. was partially supported by NSF grant DMS 2204614. Q.L. was partially supported by an AMS-Simons travel grant.

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