Research statement
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My research interests are in graph theory, combinatorics, and algorithms. Due to wide applications to other disciplines in mathematics, computer science, biology, economics, etc, these topics attracted extensive attention in recent decades. My work includes well-quasi-ordering, graph structure, graph coloring, extremal graph theory, extremal combinatorics, combinatorial optimization, domination problems, and applications to algorithms.

In this article, I will briefly describe part of my recent work and discuss some open problems that I would like to explore in the near future related to the project funded by my NSF grant.

1 Previous work

This section addresses part of my results in the past few years.

1.1 Well-quasi-ordering and structure theorems

A quasi-ordering $\preceq$ on a set S is a reflexive and transitive relation. It is a *well-quasi-ordering* if for every infinite sequence $s_1, s_2, ...$ in S, there exist $i < j$ such that $s_i \preceq s_j$. A graph G is a *minor* of a graph H if G can be obtained from a subgraph of H by contracting edges. One of the prominent and well-known results in graph theory is Robertson and Seymour's Graph Minors project, which brought a giant breakthrough in graph theory with 23 papers and was awarded the George Pólya Prize and the Fulkerson Prize. In the series of papers, they confirm Wagner's conjecture (now known as the Graph Minor Theorem): finite graphs are well-quasi-ordered by the minor relation [41], and prove the existence of a polynomial time algorithm for minor testing, which settles one of Garey and Johnson's problems. In addition, in the (currently) last paper of the Graph Minors series, they prove the same for weak immersions [42]. Furthermore, they develop many structural theorems that are widely applied in graph theory and theoretical computer science.

The results about well-quasi-ordering graphs are very powerful. If a property $\mathcal{Q}$ is closed under a relation $\preceq$ that well-quasi-orders graphs, then there exist finitely many graphs $H_1, H_2, ..., H_n$ (only depending on $\mathcal{Q}$) such that every graph G satisfies $\mathcal{Q}$ if and only if $H_i \not\preceq G$ for every $1 \leq i \leq n$. In particular, every property that is closed under deleting vertices, edges, and contracting edges can be characterized by finitely many graphs, and hence can be decided in polynomial time. Similarly, Fellows and Langston [14] used the fact that weak immersion well-quasi-orders graphs to deduce polynomial time algorithms for many problems that were unknown to be polynomial time solvable, and gave more efficient algorithms for some other problems.

Unlike the relations of minor and weak immersion, the relation of topological minor does not well-quasi-order finite graphs in general. For every positive integer k, we say that a graph is a *Robertson chain* of length k if it can be obtained by doubling the edges of the path of length k, and we say that the ends of the Robertson chain are the ends of the original path. Let G_k be the graph obtained by adding four vertices of degree one to the Robertson chain of length k, where each of the ends is adjacent to two new vertices. Then there do not exist $i \neq j$ such that G_i contains G_j as a topological minor. However, Robertson in the 1980's conjectured that the Robertson chain is the only obstruction. More precisely, he conjectured that for every positive integer k, the topological minor relation well-quasi-orders the graphs that do not contain a topological minor isomorphic to the Robertson chain of length k.

Robertson's conjecture immediately implies all known results about well-quasi-ordering graphs by the topological minor relation, including two conjectures of Vázsonyi: trees and graphs with maximum degree at most three are well-quasi-ordered by the topological minor relation, where the former was independently proved by Kruskal [20] and Tarkowski [43], and the latter was proved by Robertson and Seymour [42].

I proved Robertson's conjecture in my thesis [22]. Prior to the announcement of our proof, Robertson's conjecture was considered difficult, and very few progress was made in the literature [8] over decades.

In addition, we [33] are able to characterize all well-quasi-ordered topological ideals. A *topological ideal* is a set of graphs such that any topological minor of a member of the set belongs to this set. Note that for every topological minor closed property $\mathcal{Q}$, the set of graphs satisfying $\mathcal{Q}$ is a topological ideal. The characterization involves a notion called "Robertson family", which consists of the graphs that can be obtained from a Robertson chain by taking 16 different types of small modifications. We omit the formal definition of the Robertson family.

Theorem 1 ([22, 33]) *If k is a positive integer, then graphs with no topological minor isomorphic to the Robertson chain of length k are well-quasi-ordered by the topological minor relation. Furthermore, a topological ideal is well-quasi-ordered by the topological minor relation if and only if it contains only finitely many members of the Robertson family.*

The proof of Theorem 1 is long and difficult. It will be split into a series of papers [30, 31, 32, 33] for publication.

Robertson's conjecture is turned out to be harder than expected even for graphs of bounded tree-width. We needed a new technique to convert vertex-cuts of the tree-decomposition into edge-cuts [30], which serves the first essential step toward a proof of Theorem 1. Moreover, building on our earlier result about a general structural theorem for excluding a fixed graph as a topological minor [29] which improves results of Grohe, Marx [17], and Dvořák [9], we further developed other structural theorems for excluding the Robertson chain of a fixed length as a topological minor [31, 32]. These serve other essential steps toward the proof of Theorem 1 and prove another Robertson's conjecture (Theorem 2 mentioned below). Then we [32, 33] applied them to reduce the problem to graphs in which the topological minor relation is the same as the minor relation or the weak immersion relation, which reduces Theorem 1 to a result of Robertson and Seymour.

Theorem 2 ([31, 32]) *(Informal description) For every positive integer k, every graph with no topological minor isomorphic to the Robertson chain of length k is "nearly subcubic."*

The notion of being "nearly subcubic" mentioned in Theorem 2 needs a couple of definitions to be precisely stated, so we only give an informal description here.

1.2 Erdős-Pósa property

In addition to proving well-quasi-ordering results, our structure theorem for excluding a fixed graph as a topological minor [29] has an application to problems about the Erdős-Pósa property. A family $\mathcal{F}$ of graphs has the *Erdős-Pósa property* if there exists a function f such that for every integer k, every graph G either contains k pairwise disjoint subgraphs where each isomorphic to a member of $\mathcal{F}$, or contains a set of at most $f(k)$ vertices intersecting all subgraphs isomorphic to members of $\mathcal{F}$.

The Erdős-Pósa property is closely related to operation research. Several problems in graph theory can be formulated as one of the following two optimization problems: given a family $\mathcal{H}$ of graphs, what is the maximum number of disjoint subgraphs isomorphic to members of $\mathcal{H}$ in a given graph G, and what is the minimum size of a subset of $V(G)$ intersecting all such subgraphs?

One important and powerful idea in combinatorial optimization is the primal-dual method, which attempts to show that the solution of each of the above problems is bounded in terms of the other. This is exactly what the Erdős-Pósa property describes.

For a graph H, denote the sets of graphs containing H as a minor and a topological minor as $\mathcal{M}(H)$ and $\mathcal{T}(H)$, respectively. Robertson and Seymour [38] in 1986 proved that $\mathcal{M}(H)$ has the Erdős-Pósa property if and only if H is planar. In the same paper, they asked for the characterization for the graphs H in which $\mathcal{T}(H)$ has the Erdős-Pósa property.

Postle, Wollan and I [28] are able to answer Robertson and Seymour's question in full by providing a complete characterization described in Theorem 3. This characterization requires a number of definitions to be precisely stated, so only an informal description is included here. We also prove that testing whether $\mathcal{T}(H)$ has the Erdős-Pósa property for the input graph H is NP-hard, which implies that a clean characterization is unlikely to be existed.

Theorem 3 ([28]) *(Informal description.) Let H be a graph. Then $\mathcal{T}(H)$ has the Erdős-Pósa property if and only if the following hold.*

1. *H can be drawn in the plane such that all vertices of degree at least four are incident with the same face.*

2. *For every component L of H not properly contained in another component as a topological minor and for every "partition" of L, the partially ordered set on the parts ordered by the topological minor relation contains at most two maximal elements, and each part is "symmetrically" contained in some maximal part as a topological minor.*

3. *For every pair of components L_1, L_2 of H each not properly contained in another component as a topological minor and for every "partition" of $L_1 \cup L_2$, the partially ordered set on the parts ordered by the topological minor relation contains at most three maximal elements.*

Furthermore, testing whether $\mathcal{T}(H)$ has the Erdős-Pósa property for the input graph H is NP-hard.

As mentioned earlier, a family $\mathcal{F}$ satisfies the Erdős-Pósa property if and only if the integral solutions of the packing and covering problems with respect to $\mathcal{F}$ are bounded in term of each other. By the duality theorem of linear programmings, the solutions of the LP-relaxation of these two problems are always tied to each other. Hence, it was considered whether this still holds if we only allow the fractional solutions having small denominators.

In particular, Thomas (see [18]) conjecture that the planarity is no longer required to tie the solutions of the packing problem and the covering problem if we pack minors half-integrally: for every graph H, $\mathcal{M}(H)$ satisfies the corresponding Erdős-Pósa property if we allow the solutions of the packing problem having denominators 1 or 2.

I [24] prove that it is true for $\mathcal{T}(H)$, which easily implies Thomas' conjecture. One application of this result is providing a polynomial time approximation scheme for maximizing any graph parameter that does not increase under taking vertex-deletion on the class of graphs that can only half-integrally pack a bounded number of apex-minors, generalizing a result of Demaine and Hajiaghayi [5]. A graph is *apex* if it can be made planar by deleting at most one vertex.

Theorem 4 ([24]) *For every graph H, there exists a function f such that for every graph G and positive integer k, either there exist k H topological minors in G such that each vertex of G is contained in at most two of them, or there exists a set of at most $f(k)$ vertices intersecting all H topological minors in G.*

For every apex-graph H, integer k, and graph parameter α that does not increase under taking vertex-deletion, there exists a polynomial q and a constant c such that for every $\epsilon > 0$, there exists an algorithm for finding α with approximation ratio $1 - \epsilon$ running in time $O(c^{1/\epsilon} q(|V(G)|))$ on graphs G that do not half-integrally pack k H-minors.

Based on Theorem 3, Theorem 4 shows that the gap between the Erdős-Pósa properties with respect to packing and half-integral packing topological minors is huge in the sense that testing the former is NP-hard but testing the latter is constant time solvable.

The Erdős-Pósa property with respect to the weak immersion containment were also considered. Due to the nature of packing edge-disjoint paths in immersions, it is more reasonable to consider an edge-variant of the Erdős-Pósa property. I [23] observe that the characterization for the graphs H in which the set of graphs containing H as a weak immersion has the edge-variant of the Erdős-Pósa property can be as complicated as the topological minor case, and prove that the characterization is much cleaner if we restrict the host graphs to be 4-edge-connected graphs. The 4-edge-connectivity cannot be replaced by the 3-edge-connectivity.

Theorem 5 ([23]) *For every graph H, there exists a function f such that for every positive integer k, every 4-edge-connected graph either contains k pairwise edge-disjoint subgraphs where each contains H as a weak immersion, or contains a set of at most $f(k)$ edges intersecting all such subgraphs.*

Theorem 5 shows that the edge-variant of the Erdős-Pósa property for weak immersions can be ensured by simply assuming the host graphs has edge-connectivity at least 4. However, there is no constant c to ensure the Erdős-Pósa property for topological minors even if we assume that host graphs are c-connected [28].

1.3 Cycle lengths and minimum degree

The study of the relationship between the minimum degree of graphs and the lengths of their cycles is a classical research direction in graph theory. For example, Dirac in the 1950's proved that every graph on n vertices with minimum degree at least $n/2$ contains a cycle with length n.

Thomassen [45] in 1983 proposed the following two conjectures: every graph with minimum degree at least $k + 1$ contains cycles of all even lengths modulo k; every 2-connected non-bipartite graph with minimum degree at least $k + 1$ contains cycles of all lengths modulo k. The complete graph on $k + 1$ vertices shows that the minimum degree conditions in these conjectures are best possible. Motivated by these conjectures, Ma and I [25] proved several theorems about the minimum degree conditions for forcing the existence of cycles with certain lengths.

First, we prove the following theorem, which confirms both conjectures of Thomassen mentioned above in a stronger sense when k is even.

Theorem 6 ([25]) *If G is a graph with minimum degree at least $k + 1$, then G contains $\lfloor k/2 \rfloor$ cycles with consecutive even lengths. Furthermore, if G is 2-connected and non-bipartite, then G contains $\lfloor k/2 \rfloor$ cycles with consecutive odd lengths.*

If we increase the connectivity, we can obtain cycles with consecutive lengths. The following theorem improves a result of Fan [13], which was originally asked by Bondy and Vince [3].

Theorem 7 ([25]) *If G is a 3-connected non-bipartite graph with minimum degree at least $k + 1$, then G contains $2\lfloor \frac{k-1}{2} \rfloor$ cycles with consecutive lengths.*

The 3-connectivity in Theorem 7 cannot be dropped since there are infinitely many 2-connected non-bipartite graphs with arbitrarily large minimum degree but containing no two cycles with consecutive lengths. Though dropping the 3-connectivity prevents us from obtaining cycles of consecutive lengths, we can still obtain cycles whose lengths form a long arithmetic progress of common difference one or two.

Theorem 8 ([25]) *If G is a 2-connected non-bipartite graph with minimum degree at least $k+3$, then G contains k cycles whose lengths form an arithmetic progression with common difference one or two.*

Theorem 8 shows that minimum degree $k+3$ suffices (in a stronger sense) for the second Thomassen's conjecture mentioned above when k is odd. By dropping the 2-connectivity and the non-bipartiteness, we can prove the following theorem, which shows that minimum degree $k+4$ is sufficient (in a stronger sense) for the first Thomassen's conjecture mentioned above when k is odd.

Theorem 9 ([25]) *If G is a graph with minimum degree at least $k+4$, then G contains k cycles whose lengths form an arithmetic progression with common difference one or two.*

The minimum degree conditions in Theorems 8 and 9 are currently best known in the literature and just small additive constants away from the ones in Thomassen's conjectures.

1.4 Graph coloring

Graph coloring is one of the most important and broadly studied topics in graph theory. It has been widely applied to scheduling and partitioning problems.

1.4.1 Variations of Hadwiger's conjecture

Hadwiger conjectured that every graph with no K_{t+1}-minor can be partitioned into t parts such that every component of each subgraph induced by a part contains at most one vertex. It is widely considered one of the deepest conjectures in graph theory and remains open for the case $t \geq 6$. The celebrated Four Color Theorem is implied by (in fact, equivalent with) the case $t = 4$.

Variations of Hadwiger's conjecture for seeking a partition with no large components in each subgraph induced by a part have been extensively considered. In particular, Alon, Ding, Oporowski and Vertigan [2] proved that for every integer d, there exists a number c such that every K_t-minor free graph of maximum degree at most d can be partitioned into four parts such that every component of each subgraph induced by a part has at most c vertices. Esperet and Joret [11] improved a special case of the mentioned result of Alon et al. by proving that only three parts are required for graphs embeddable in a fixed surface and asked whether it can be generalized to every proper minor-closed family of graphs.

Oum and I [26] provided a positive answer of Esperet and Joret's question and hence improved the mentioned result of Alon et al. In fact, our result proves a stronger statement. We show that the condition of the lack of minors can be replaced by the lack of minors with special properties.

Theorem 10 ([26]) *For every graph H and positive integer d, there exists a number c such that if a graph G of maximum degree at most d has no odd H-minor, then $V(G)$ can be partitioned into three parts X_1, X_2, X_3 such that every component of $G[X_i]$ contains at most c vertices for every $i = 1, 2, 3$.*

Theorem 10 is best possible in the sense that the number three cannot be reduced. Furthermore, the condition on the maximum degree cannot be dropped if the partition only allows a constant number of parts.

Another application of Theorem 10 is that as long as graphs of a proper minor closed family can be partitioned into k subgraphs with bounded maximum degree, such graphs can be partitioned into $3k$ subgraphs with bounded component size. It shows that for any proper minor closed family, the ratio between the minimum number of required parts for partitioning graphs into graphs with bounded component size and the minimum number of required parts for partitioning graphs into graphs with bounded maximum degree is at most three, which serves a tight link between two variants of the chromatic number that attract wide attentions.

1.4.2 Edge-density of minimal non-3-colorable graphs

Intuitively, graphs with fewer edges can be properly colored by a smaller number of colors. Kostochka and Yancy [19] confirmed this intuition by proving that every non-3-colorable graph contains a subgraph H with $|E(H)| \geq \frac{5|V(H)|-2}{3}$. Postle and I [27] improved this bound when the graphs have no cycle of length less than five.

Theorem 11 ([27]) *There exist positive numbers ϵ, c such that every non-3-colorable graph with no cycle of length less than five contains a subgraph H with $|E(H)| \geq (\frac{5}{3}+\epsilon)|V(H)|+c$.*

Together with Euler's formula, Theorem 11 immediately implies a result of Thomassen [46] and a result of Thomas and Walls [44]: every graph with no cycle of length less than five embeddable in the torus or the Klein bottle is properly 3-colorable. Theorem 11 not only provides a unified proof of the results of Thomassen, Thomas and Walls, but also shows that the 3-colorability of those graphs are indeed due to sparsity instead of topological properties.

2 Future work

I would like to address problems related to three cognate graph containment relations (graph minors, topological minors and immersions) in the next few years. Conjectures in this research area have been proposed and attracted wide attention for more than 70 years. Though successes were reported over recent decades, such as results in the Graph Minors series and some of our earlier results mentioned in the previous section, several such conjectures remain open.

2.1 Nash-Williams' strong immersion conjecture

Nash-Williams conjectured in the 1960's that the relations of weak immersion and strong immersion are well-quasi-orderings of finite graphs. In other words, given an infinite sequence $G_1, G_2, ...$ of finite graphs, there exist $i < j$ such that G_j admits a weak (and strong, respectively) G_i-immersion. The conjecture on weak immersion was proved by Robertson and Seymour in the (currently) last paper of the Graph Minors series via very complicated arguments [42], but the conjecture on strong immersion, which implies the weak immersion conjecture, is still open. Furthermore, the conjecture on strong immersion implies that every strong immersion-closed property can be characterized by finitely many graphs. Note that the class of strong immersion-closed properties strictly contains the class of weak immersion-closed properties.

Conjecture 12 ([35]) *Graphs are well-quasi-ordered by the strong immersion relation.*

The strategy to solve the strong immersion conjecture is similar as the idea that we developed for attacking Robertson's conjecture. We shall develop structure theorem for excluding a fixed graph as a strong immersion. If there exists a sequence of graphs such that no graph contains another as a strong immersion, then every graph in the sequence does not strongly immerse the first graph, and hence we can apply our structure theorem. Based on the structure theorem (elaborated in the next subsection), we expect that such graphs have some tree structure such that each small piece either has maximum degree smaller than the original graph or can be drawn on a surface with smaller genus. And then the conjecture follows by induction on the maximum degree and genus.

2.2 Structure theorems for excluding strong immersions and its applications

One important achievement of Robertson and Seymour's Graph Minors series is developing the structure theorem for excluding a fixed graph as a minor [40]. Roughly speaking, every graph that does not contain a fixed graph as a minor has a certain tree structure. This theorem leads to many algorithmic applications on minor-closed families of graphs, such as the family of planar graphs, and it is an important step for proving the minor relation well-quasi-orders graphs. In comparison to the fact that structural theorems about graph minors were widely explored, only few about graph immersions could be found in the literature. As structural theorems about graph minors lead to great successes in deriving many algorithmic results, it is reasonable to expect the same for graph immersions. Based on the our proofs of structure theorems for excluding weak immersions [34] and topological minors [29], we expect that we can obtain a theorem for excluding a fixed graph as a strong immersion.

Besides proving well-quasi-ordering problems, the other motivation for investigating structure theorems for excluding strong immersions is in the topic of algorithms. It is known that minor testing [39], topological minor testing and weak immersion testing [16] are fixed-parameter tractable. That is, given a fixed graph H, testing whether an input graph on n vertices contains H as a minor, topological minor, or weak immersion can be done in time $O(f(H)n^c)$, where c does not depend on H or n. In consequence, due to the results that the minor, topological minor (with some extra requirement), and weak immersion relations well-quasi-order graphs, every property that is closed under minors, topological minor (with some extra requirement), or weak immersions can be decided in polynomial time. However, the existence of polynomial time algorithms for testing whether a given graph contains a fixed graph as a strong immersion is not known, but was positively conjectured by Grohe, Kawarabayashi, Marx and Wollan [16]. The information obtained by the structure of graphs that do not contain a fixed graph as a strong immersion seems essential for developing such a polynomial time algorithm.

Conjecture 13 ([16]) *There exist a function f and a number c such that for every graph H, whether a graph on n vertices admits a strong H-immersion can be decided in time $O(f(H)n^c)$.*

2.3 Graph coloring, minimum degree, and immersions

Lescure and Meyniel [21] and Abu-Khzam and Langston [1] independently conjectured a variant of Hadwiger's conjecture with respect to graph immersions.

Conjecture 14 ([1, 21]) *For every positive integer k, every graph that does not contain K_k as a weak immersion admits a proper $(k-1)$-coloring.*

The case $k \leq 7$ of Conjecture 14 were confirmed by DeVos, Kawarabayashi, Mohar and Okamura [7], but the case $k \geq 8$ remains open. Motivated by Conjecture 14, a question of minimum degree conditions for forcing a K_k-immersion was considered.

Question 15 *For each positive integer k, what is the minimum $f(k)$ such that every simple graph with minimum degree at least $f(k)$ contains K_k as a weak immersion.*

Since every graph that does not admit a proper $(k-1)$-coloring has a subgraph with minimum degree at least $k-1$, Devos, Kawarabayashi, Mohar and Okamura [7] conjectured that $f(k) \leq k-1$. This conjecture is true for $k \leq 7$ [7] but false for all $k \geq 8$ [4, 6]. Bear this in mind, Dvořák and Yepremyan [10] asked whether the following conjecture, which still implies Conjecture 14, is true.

Conjecture 16 *For every positive integer k, $f(k) \leq k$.*

Besides applications in graph coloring, Question 15 is of its own interest and attracts attention recently. Devos, Dvořák, Fox, McDonald, Mohar and Scheide [6] proved that $f(k) \leq 200k$; Dvořák and Yepremyan [10] improved their result to $f(k) \leq 11k+7$; Gauthier, Le and Wollan [15] further improved it to $f(k) \leq 7k+7$. My another objective is to prove Conjecture 16 or at least improve the known results on Question 15 by decreasing the leading coefficient of the bound of $f(k)$.

2.4 Graph partitioning and improper coloring

For a family of graphs $\mathcal{F}$ closed under taking subgraphs, the *defective chromatic number* (and *clustered chromatic number*, respectively) of $\mathcal{F}$ is the minimum t such that there exists a number N such that for every graph G in $\mathcal{F}$, there exists a partition of $V(G)$ into t parts such that the subgraph induced by each part has maximum degree at most N (and has no component containing more than $N+1$ vertices, respectively). These two notions are relaxations of the traditional coloring, as the chromatic number requires the number N to be zero. Motivated by Hadwiger's conjecture, they attract wide attention recently. For example, Edwards, Kang, Kim, Oum and Seymour [12] proved that the set of K_t-minor-free graphs has defective chromatic number $t-1$, and Oum and I [26] proved that for every graph H, the set of H-minor-free graphs with bounded maximum degree has clustered chromatic number at most three.

All known lower bounds of the defective chromatic number and clustered chromatic number of families of graphs come from few types of obstructions. In particular, the graph $G_{h,N}$ that is obtained from an $(N+1)$-nary tree with height h by adding edges between all ancestor-descendant pairs cannot be partitioned into $h-1$ parts where each part induces a graph with maximum degree at most N. Hence the defective chromatic number of $\{G_{h,N} : N \geq 1\}$ is at least h. This family witnesses a lower bound for defective chromatic number and can be viewed as an analog of complete graphs which witness a lower bound for chromatic number. Note that K_h is the maximal graph with h vertices, and $\{G_{h,N} : N \geq 1\}$ consists of the maximal graphs with connected tree-depth h. Ossona de Mendez, Oum and Wood [37] proposed the following analog of Hadwiger's conjecture.

Conjecture 17 ([37]) *For every graph H, the defective chromatic number of the family of graphs with no H-minor equals the connected tree-depth of H minus 1.*

Norin, Scott, Seymour and Wood [36] constructed a family with clustered chromatic number larger than the connected tree-depth of H, and conjectured that the clustered chromatic number of this family is the worst among all H minor free graphs.

Conjecture 18 ([36]) *For every graph H, the clustered chromatic number of the family of graphs with no H-minor is at most twice of the connected tree-depth of H minus 2.*

In addition, recall that Hadwiger's conjecture is true if we allow defective coloring [12], and the number of colors can be reduced to three for clustered coloring if we further assume graphs having bounded maximum degree, no matter how large the complete graph we exclude is [25]. As Hajós' conjecture is also true if we allow defective coloring [12], one might consider whether there exists a constant C such that every K_t-topological minor free graph with bounded maximum degree has clustered chromatic number at most C. However, it is not true, as the clustered chromatic number of the family of graphs with maximum degree at most $t-2$ is $\Omega(t)$. As graphs with maximum degree at most $t-2$ are the most obvious examples of graphs with no K_t-topological minor, it is natural to conjecture the following.

Conjecture 19 *For every sufficiently large integer t, if $f(t)$ is the clustered chromatic number of the family of graphs with maximum degree at most $t-2$, then for every integer k, the family of K_t-topological minor free graphs with maximum degree at most k has clustered chromatic number at most $f(t)$.*

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