# K-stability and Moduli of Fano varieties III

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- Projectivity
- Local stability theory
- Explicit examples

# Projectivity of KSBA moduli

Fix a family $f\colon X\to B$ of KSB(A) stable varieties over a projective base $B$ of maximal variation, i.e. $B\to\mathcal{M}^{\mathrm{KSBA}}$ is generically finite.

- (Kollár 90) For $m\gg 0$, $\det(f_*\mathcal{O}_X(mK_{X/B}))$ is big if the vector bundle $f_*(mK_{X/B})$ is nef.
- (Fujino 12) For a KSBA family, $f_*\mathcal{O}_X(mK_{X/B})$ is nef.
- (Kovác-Patakfalvi 15) $\det(f_*\mathcal{O}_X(m(K_{X/B}+\Delta)))$ is big.
- (Patakfalvi-Xu 15) The CM line bundle $\lim_{m\to\infty}\frac{(n+1)!}{m^{n+1}}\det(f_*\mathcal{O}_X(mK_{X/B}))$ is big.

## Definition

Let $f: X \to B$ be a family of $n$-dimensional klt Fano varieties. We can write the Knudson-Mumford expansion

$$\det(f_*(\mathcal{O}(-mK_{X/B}))) = M_{n+1}^{\binom{n+1}{m}} \otimes \cdots \otimes M_0$$

for $\mathbb{Q}$-line bundles $M_{n+1}, \ldots, M_0$ on $B$. The CM ($\mathbb{Q}$-)line bundle $\lambda_{X/B}$ of $f$ is defined to be $M_{n+1}^{-1}$.
If $B$ is normal proper, then $\lambda_{X/B} = -f_*(-K_{X/B})^{n+1}$.

- The CM bundle can be defined on any finite type substack of $\mathfrak{X}^{\mathrm{Fano}}_{n,V}$. In particular, we get $\lambda_{\mathrm{CM}}$ on $\mathfrak{X}^{\mathrm{K}}_{n,V}$.
- Since $\deg(\lambda_{\mathcal{X}/\mathbb{P}^1}) = \mathrm{Fut}(\mathcal{X}) = 0$ if $X$ is K-polystable, $\mathrm{Aut}^{\circ}(X)$ on $\lambda_{\mathrm{CM}}^{\otimes N}$ is trivial. So replacing $N$ by a larger multiple, $\lambda_{\mathrm{CM}}^{\otimes N}$ descends to $\Lambda_{\mathrm{CM}}^{\otimes N}$ on $X^{\mathrm{K}}_{n,V}$ for a $\mathbb{Q}$-line bundle $\Lambda_{\mathrm{CM}}$, i.e. the CM line bundle.

## Theorem (Codogni-Patakfalvi 19, X.-Zhuang 20)

$\Lambda_{\mathrm{CM}}$ is ample on $X^{\mathrm{K}}_{n,V}$. In particular, $X^{\mathrm{K}}_{n,V}$ is projective.

- The positivity of CM line bundle was first observed by differential geometers, e.g. Tian, for families of smooth Kähler-Einstein manifolds.
- $\Lambda_{\mathrm{CM}}$ is not positive for families of Fano varieties without the assumption of K-stability.

- Let $f\colon X\to C$ be a family of Fano varieties over a smooth projective curve. Denote by the Harder-Narashimhan filtration

$$0\subset E_{1,m}\subset\cdots\subset E_{k,m}=A_m=f_*(-mK_{X/C}),$$

with slope of $E_{i,m}/E_{i-1,m}$ to be $\mu_i$. Define a filtration of $A$ with jumping numbers $\mu_1,\ldots,\mu_k$ and $\mathcal{F}^{\mu_i}_{\rm CM}A_m=E_{i,m}$,

- Let $t\in C$ be a general point with $R_m=A_m\otimes_{O_C}k(t)$ and

$$\mathcal{G}^{\lambda}_{\rm CM}R_m=\mathrm{Im}(\mathcal{F}^{\lambda}_{\rm CM}A_m\to A_m\to R_m).$$

- Let $F$ be a fiber of $f$. Define the filtration $\mathcal{G}_{gg}$ on $R_m$ to be

$$\mathcal{G}^{\lambda}_{gg}R_m:=\mathrm{Im}\left(H^0(-mK_{X/C}\otimes(-\lceil\lambda\rceil F))\to R_m\right).$$

We have $\mathcal{G}^{\lambda}_{gg}\subset\mathcal{G}^{\lambda}_{\rm CM}\subset\mathcal{G}^{\lambda-2g(C)}_{gg}$.

- $S_m(\mathcal{G}_{\mathrm{CM}}) = \frac{1}{m \dim R_m} \deg(A_m)$. Let $m \to \infty$, $S(\mathcal{G}_{\mathrm{CM}}) = \frac{-\deg(\lambda_f)}{(n+1)(-K_{X_t})^n}$.
- $\mu(\mathcal{G}_{\mathrm{CM}}) \leq 0$ for any family of Fano varieties: if $\mu(\mathcal{G}_{\mathrm{CM}}) > 0$, then there exists a section $s \in A_m \otimes \mathcal{O}_C(-P)$, yielding a divisor
$$\Gamma \in H^0(X, \mathcal{O}_X(-mK_{X/C}(-F)))$$
such that $(X, \frac{1}{m}\Gamma)$ has lc general fibers. Contradiction.
- If $X_t$ is K-semistable, $\mathrm{D}(X_t, \mathcal{G}_{\mathrm{CM}}) \geq 0$ which implies $\deg(\lambda_f) \geq 0$.
- For ampleness, we want $f_*(-mK_{X/C})$ to be a nef vector bundle.
- If $X_t$ is uniformly K-stable, $\mu(\delta, \mathcal{F}) \geq S(\mathcal{F})$ for any $\mathcal{F}$, where $\delta = \delta(X)$ and $\mu(\delta, \mathcal{F}) := \sup_\lambda \{\mathrm{lct}(X_t; I^\lambda_\bullet) \geq \delta\}$.
- Applying to $\mathcal{G}_{\mathrm{CM}}$, this implies there exists $D \sim_{\mathbb{Q}} -K_{X/C} - S(\mathcal{G}_{\mathrm{CM}})F$ such that $(X, \delta D)$ is lc along $X_t$.
- $K_{X/C} + \delta D \sim -(\delta - 1)K_{X/C} + \frac{\delta \deg(\lambda_f)}{(n+1)(-K_X)^n} F$, so $f_*(-mK_{X/C}) \otimes \lambda_f^{\otimes C(m)}$ is nef by Fujita-Kawamata type semipositivity of pushforward.

- In general $f_*(-mK_{X/C}) \otimes \lambda_f^{\otimes C(m)}$ is not nef for K-polystable $X_t$, e.g. $X = \mathrm{Bl}_x\mathbb{P}^2 \to C = \mathbb{P}^1$. After twisting it to $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, the pushforward becomes nef.
- If $X$ admits a torus $\mathbb{T}$-action. Then $R_\bullet = \oplus_\alpha R_\alpha$ for $\alpha \in M = \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$. A twist $\mathcal{F}_\xi$ of a filtration $\mathcal{F}$ by an element $\xi \in N \otimes \mathbb{R} = \mathrm{Hom}(M, \mathbb{R})$ is defined as $\mathcal{F}_\xi^\lambda R_\alpha := \mathcal{F}^{\lambda - \langle \xi, \alpha \rangle} R_\alpha$.
- For $\xi \in N$, we define $A_{\bullet,\xi} = \bigoplus_\alpha A_\alpha \otimes \mathcal{O}_C(\langle \xi, \alpha \rangle \cdot P)$.
- $f_\xi \colon X_\xi := \mathrm{Proj}(A_{\bullet,\xi}) \to C$ satisfies that the induced CM filtration is $\mathcal{G}_{\mathrm{CM},\xi}$, and $\lambda_f = \lambda_{f_\xi}$ as $\mathrm{Fut}(X_t, \xi) = 0$.

**Theorem (Liu-X.-Zhuang 21)**

If $X$ is K-polystable, then there exists $\delta > 1$, such that for any filtration $\mathcal{F}$, we can find $\xi$ which satisfies $\mu(\delta, \mathcal{F}_\xi) \geq S(\mathcal{F}_\xi)$.

- Twisting $X$ by $\xi$ to get $X_\xi$, we can assume $\mu(\delta, \mathcal{G}_{\mathrm{CM}}) \geq S(\mathcal{G}_{\mathrm{CM}})$.

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# Local stability theory

- Fix $x \in X = \mathrm{Spec}(R)$ where $(R, \mathfrak{m}_x)$ is an $n$-dimensional klt local domain. Let $\mathrm{Val}_{X,x}$ denote all valuations $v$ of $K(R)$ centered on $x$.
- (Ein-Lazarsfeld-Smith 03): For $v \in \mathrm{Val}_{X,x}$, the volume $\mathrm{vol}(v) = \lim_{\lambda \to \infty} \frac{\dim(R/\mathfrak{a}_\lambda)}{\lambda^n/n!}$, where $\mathfrak{a}_\lambda := \{f \mid v(f) \geq \lambda\}$.

**Definition (Chi Li 15)**

Define the normalized volume of $v$ to be

$$\widehat{\mathrm{vol}}(v) = \begin{cases} A^n(v) \cdot \mathrm{vol}(v) & \text{if } A(v) < +\infty, \\ +\infty & \text{if } A(v) = +\infty, \end{cases}$$

and $\widehat{\mathrm{vol}}(x \in X) = \inf_{v \in \mathrm{Val}_{X,x}} \widehat{\mathrm{vol}}(v)$.

## Theorem (Stable Degen. Thm, conjectured by Li 15, Li-X. 16)

Let $x \in X$ be any klt singularity.

- (Blum 17, X. 19) $\widehat{\mathrm{vol}}$ on $\mathrm{Val}_{X,x}$ has a minimizer.
- (X. 19) A minimizer is quasi-monomial.
- (X.-Zhuang 20, Blum-Liu-Qi 22) The minimizer $v$ is unique up to rescaling.
- (X.-Zhuang 22) $\mathrm{Gr}_v R := \bigoplus_{\lambda \in \mathbb{R}_{\geq 0}} \mathfrak{a}_\lambda / \mathfrak{a}_{>\lambda}$ is finitely generated, where $\mathfrak{a}_{>\lambda} := \{f \in R \mid v(f) > \lambda\}$ .
- (Li-X. 17) $X_0 = \mathrm{Spec}(\mathrm{Gr}_v R)$ is K-semistable (with $\xi_v$).

- For uniqueness, aim to show (strict) convexity of $\widehat{\mathrm{vol}}(\cdot)$. Two valuations can not be connected.
- One embeds $\mathrm{Val}_{X,x} \hookrightarrow \mathrm{Fil}_{R,\mathfrak{m}}$ the space of filtered ideals $\{\mathfrak{a}_\lambda\}_{\lambda>0}$, with a strictly convex function $\widehat{\mathrm{mult}}(\cdot)\colon \mathrm{Fil}_{R,\mathfrak{m}} \to \mathbb{R}_{>0}$ such that it coincides with $\widehat{\mathrm{vol}}(\cdot)$ on minimizers.

## Distribution of volumes

What is the set $\mathrm{Vol}_n = \{\widehat{\mathrm{vol}}(x \in X) \mid x \in X$ is an $n$-dim klt sing.$\}$ ?

- $\mathrm{Vol}_2 = \{\frac{4}{m} \mid m \in \mathbb{N}\}$.

## Theorem (X.-Zhuang 24)

The only accumulation point of $\mathrm{Vol}_n$ is $\{0\}$.

- The degeneration of $(x \in X)$ to $(X_0, \xi)$ preserves the volume.
- For any fixed $\delta > 0$,
  $\{(X_0, \xi) \mid \widehat{\mathrm{vol}}(X_0, \xi) \geq \delta, (X_0, \xi)$ is K-semistable$\}$ is bounded.
- In the proof we need Birkar's theorem on boundedness of complements.

**Theorem (Fujita 15, Liu 16)**

If $X$ is K-semistable, for any $x \in X$, $(-K_X)^n \le \frac{(n+1)^n}{n^n}\widehat{\mathrm{vol}}(x \in X)$.

- (Liu-X. 17) $\mathrm{Vol}(x \in X) \le n^n$ and '=' holds only for a smooth point.
- Gap Conjecture: The second largest volume is given by $(x_1^2 + \cdots + x_{n+1}^2 = 0)$ (which is $2(n-1)^n$).
- (Liu-X. 17) This is known when $n \le 3$.

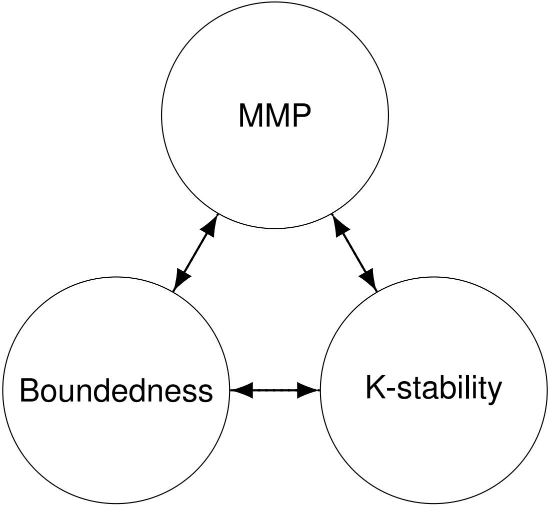
# Application to MMP by Han-Liu-Qi-Zhuang

- Let $X$ be klt projective general type.
- $\alpha_X(K_X) := \inf_{D\in|mK_X|} \mathrm{lct}(X; \frac{1}{m}D) > 0$. It satisfies that for any $x$ on $X$, $\widehat{\mathrm{vol}}(x \in X) \geq \alpha_X^n(K_X) \cdot \mathrm{vol}(K_X)$.
- For an MMP step $f\colon X \dashrightarrow X'$, $\alpha_{X'}(K_{X'}) \geq \alpha_X(K_X)$. So for any $x' \in X'$, $\widehat{\mathrm{vol}}(x' \in X')$ is bounded from below.
- (X.-Zhuang 20) $\pi_1^{\mathrm{loc}}(x' \in X') \leq \frac{n^n}{\widehat{\mathrm{vol}}(x' \in X')}$. So $K_{X'}$ has bounded Cartier index.
- (X.-Zhuang 24) $(x' \in X')$ has bounded minimal log discrepancy.

**Question**

Can we prove termination of flips (at least when $K_X$ is big)?

# Three-body



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## Chen-Donaldson-Sun, 2013

criterion for the existence of a Kähler-Einstein metric. On the other hand, we should point out that as things stand at present the result is of very limited use in concrete cases, so that there is no manifold $X$ known to us, not covered by other existence results and where we can deduce that $X$ has a Kähler-Einstein metric. This is because it seems a very difficult matter to test K-stability by a direct study of all possible degenerations. However, we are optimistic that this situation will change in the future, with a deeper analysis of the stability condition. As Yau [47] pointed

Which Fano varieties are K-(semi,poly)stable?

1. Show a general member of a family is K-(semi)stable.
2. Show a specific member is K-(semi)stable.
3. What are the limiting K-semistable/polystable Fano varieties?

## Question

1. Conjecture: all smooth degree $d$ Fano hypersurfaces $X \subset \mathbb{P}^{n+1}$ with $3 \leq d \leq n+1$ are K-stable.
2. What is the compactification?

1. A general hypersurface is K-stable.
2. We know a lot more than 10 years ago, but still far from complete.
3. We will use cubic threefolds as an example.

- (Zhuang 20) To check $X$ is K-(semi)polystable, we only need to look at $G$-invariant $E$ for reductive group $G \subset \mathrm{Aut}(X)$, i.e
  K-(semi)polystability=Equivariant K-(semi)polystability
- Example: The $n$-dimensional Fermat hypersurface of degree $d$ is a $\mu_d$-cover of $(\mathbb{P}^n, (1-\frac{1}{d})(\sum_{i=0}^{n}(z_i=0)+(\sum_{i=0}^{n} z_i=0))$.
- By interpolation, the latter pair is K-stable ($d \geq 3$), and therefore $X$ is K-stable.

(Liu-X. 17) The K-moduli containing a general smooth cubic threefold is the same as the GIT moduli.

- Aim to show all K-semistable limits are in $\mathbb{P}^4$, as a hypersurface is K-stable, then it's GIT stable (Paul-Tian 04).
- (T. Fujita 90) A normal limit $X$ of cubic hypersurfaces is a hypersurface in $\mathbb{P}^{n+1}$ if the degeneration $L$ of $\mathcal{O}(1)$ is Cartier.
- Fujita-Liu's inequality implies for K-semistable $X$, any $(x \in X)$ satisfies that $\widehat{\mathrm{vol}}(x \in X) \geq \frac{3(n-1)^n n^n}{(n+1)^n}$.
- If $\pi_1(x \in X) \neq \{e\}$, its local simply connected covering satisfies

$$\widehat{\mathrm{vol}}(y \in Y) \geq \frac{6(n-1)^n n^n}{(n+1)^n} > 2(n-1)^n.$$

It has to be smooth if the gap conjecture holds. This can NOT happen.

The strategy has also been applied to other cases:

- $\dim = 2$: (Mabuchi-Mukai 94) $(-K_S)^2 = 4$, (Odaka-Spotti-Sun 12) $(-K_S)^2 = 1, 2, 3$.
- (Ascher-DeVleming-Liu 18-21) K-moduli whose general members are $(\mathbb{P}^2, c \cdot C)$ ($\deg(C) = 4, 5, 6$), $(\mathbb{P}^1 \times \mathbb{P}^1, c \cdot C)$ ($C \in |\mathcal{O}(4, 4)|$) and $(\mathbb{P}^3, c \cdot S)$ ($\deg(S) = 4$).
- A-DV-L as well as C. Zhou develop a general framework of wall-crossing when varying the coefficient $c$.
- (Liu 20) Cubic fourfolds.

## Theorem (Abban-Zhuang)

*All smooth hypersurfaces* $X \subset \mathbb{P}^{n+1}$ *of degree* $d$ *with* $n + 2 - n^{1/3} \leq d \leq n + 1$ *are K-stable.*

- A key recipe is called the Abban-Zhuang method.
- To prove $\liminf_m \inf_{D_m} \mathrm{lct}_x(X; D_m) \geq \delta_0$, it suffices to show that there is a model $f\colon Y \to X$ with a divisor $E$ such that $x \in f(E)$, $\frac{A(E)}{S(E)} \geq \delta_0$, and $\liminf_m \inf_{D_m}(Y, E; f_*^{-1}D_m) \geq \delta_0$ is lc along $E$.
- By inversion of adjunction, the last inequality is implied by $\liminf_m \inf_{G_m} \mathrm{lct}(E, \Delta_E; G_m) \geq \delta_0$, where $(K_Y + E)|_E = K_E + \Delta_E$ and $f_*^{-1}D_m|_E = G_m$.
- For a hypersurface $X$, one can repeatedly choose $E = X \cap H$ for $H \in \mathcal{O}(1)$, and consider multi-graded linear series on $X \cap H \cap \cdots \cap H$ which is a curve or a surface.

# Fano threefolds

- The Abban-Zhuang method is particularly powerful when $X$ is a threefold, as $\dim(E) = 2$.
- There is a classification by Iskovskikh $\rho(X) = 1$ and Mori-Mukai $2 \leq \rho(X) \leq 10$. Total 105 families.
- (Abban-Zhuang) Except the family 1.9 and 1.10, i.e. $-K_X = H$ and $H^3 = 18$ and 22, all smooth Fano threefolds with Picard number 1 are K-stable/polystable.
- Conjecture: Every smooth member in 1.9 and 1.10 is K-semistable.
- The family 1.10 (or $V_{22}$) is of historic interests.
- (A-C-C-F-K-MG-S-S-V) There are precisely 79 families whose general member is K-polystable.
- Cheltsov and many others are actively working on Fano threefolds $X$ with $\rho(X) > 1$.

# Other Fano manifolds

- Conjecture: Let $C$ be a smooth projective curve with $g(C) \geq 2$. The moduli space $M_{r,L}$ of rank $r$ stable vector bundle with the determinant $L$ of degree $d$ $(\gcd(d,r) = 1)$ on $C$ is K-stable.
- Conjecture (LeBrun): All contact Fano manifolds/varieties are K-polystable.

*Thank you very much!*