K-stability and Moduli of Fano varieties II

Chenyang Xu (Princeton University)

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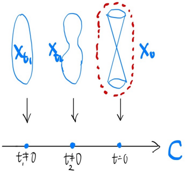
Degeneration of Fano varieties

- Started with a family of Fano varieties over a punctured disk $X^\circ \to C^\circ = C \setminus \{0\}$, we would like to investigate the question to find a optimal degeneration X_0.



- We can always get a klt Fano degeneration X_0. But often non-unique.

Assume X_t are K-semistable,

1. Does there always exist a K-semistable degeneration?
2. Assuming the existence, is it unique?

- Good moduli space
- Higher rank finite generation
- Properness

Definition (Filtrations)

- Let $R_\bullet := \bigoplus_m R_m = H^0(-mK_X)$.
- $\mathcal{F}^\lambda$ is a decreasing graded filtration indexed by $\lambda \in \mathbb{R}$.
- $\mathcal{F}^{\lambda-\varepsilon} R_m = \mathcal{F}^\lambda R_m$ for $0 < \varepsilon \ll 1$.
- $\mathcal{F}^\lambda R_m \cdot \mathcal{F}^{\lambda'} R_{m'} \subseteq \mathcal{F}^{\lambda+\lambda'} R_{m+m'}$.
- There exists $e_- < e_+$, $\mathcal{F}^{me_-} R_m = R_m$ and $\mathcal{F}^{me_+} R_m = 0$.

- $\mathrm{Gr}_{\mathcal{F}} R_\bullet$ is a (double) graded ring.
- Define $S_m(\mathcal{F}) := \frac{1}{mN_m} \sum_\lambda \lambda \dim \mathrm{Gr}^\lambda_{\mathcal{F}} R_m = \frac{1}{mN_m} \sum_i a_i$, where $N_m = \dim(R_m)$ and a_i are all jumping numbers of $\mathcal{F}$ on R_m.
- Example: $\mathcal{F}^\lambda_E R_m = \{s \in R_m \mid \mathrm{ord}_E(s) \geq \lambda\}$. So $S_m(E) = S_m(\mathcal{F}_E)$.

- Let $V^t_\bullet \subset R_\bullet$ given by $V^t_m := \mathcal{F}^{tm}R_m$.
- $\lim_m \frac{1}{N_m} \dim \mathcal{F}^{tm}R_m = \frac{\mathrm{vol}(V^t_\bullet)}{(-K_X)^n}$.
- Taking a derivative, $\lim_m \, \mathrm{d}v_m = \, \mathrm{d}v_{\mathrm{DH},\mathcal{F}}$ as distributions, where $\mathrm{d}v_m := \frac{-1}{N_m}\frac{\mathrm{d}\dim \mathcal{F}^{tm}R_m}{\mathrm{d}t} = \frac{1}{N_m}\sum_i \delta_{\frac{a_i}{m}}$ and $\mathrm{d}v_{\mathrm{DH},\mathcal{F}} := \frac{-1}{(-K_X)^n}\frac{\mathrm{dvol}(V^t_\bullet)}{\mathrm{d}t}$.
- So $\lim_m S_m(\mathcal{F}) = \lim_m \int t \, \mathrm{d}v_m \to S(\mathcal{F}) = \int t \, \mathrm{d}v_{\mathrm{DH},\mathcal{F}}$.
- (Lazarsfeld-Mustaţă 09, Kaveh-Khovanskii 09, Boucksom-Chen 10) The theory of Okounkov body provides a powerful tool to study $S(\mathcal{F})$.

- Let $I_{m,\lambda} = \mathrm{Bs}(\mathcal{F}^{\lambda} R_m \to R_m)$. So $I^t_\bullet := \{I_{m,mt}\}_m$ form a multiplicative sequence of graded ideals.
- Define $\mu(\mathcal{F}) = \sup\{t \mid \mathrm{lct}(X; I^t_\bullet) \geq 1\}$, where $\mathrm{lct}(X; I^t_\bullet) = \inf_E \frac{A_X(E)}{\mathrm{ord}_E(I^t_\bullet)}$ and $\mathrm{ord}_E(I^t_\bullet) = \lim_m \frac{1}{m}\mathrm{ord}_E(I_{m,mt})$.

Definition-Theorem (Fujita 15, X-Zhuang 20)

$\mathrm{D}(\mathcal{F}) := \mu(\mathcal{F}) - S(\mathcal{F})$. X is K-semistable iff $\mathrm{D}(\mathcal{F}) \geq 0$ for any $\mathcal{F}$.

- One direction is easy: $\mu(\mathcal{F}_v) \leq A(v)$, as $v(I^t_\bullet(\mathcal{F}_v)) \geq t$.
- (Fujita 15) We can approximate $\mathcal{F}$ by a sequence of finitely generated ones $\{\mathcal{F}_m\}_m$ from which we get a sequence of test configurations $\{\mathcal{X}_m\}_m$. Moreover, $\mathrm{D}(\mathcal{F}_m) \geq \mathrm{Ding}(\mathcal{X}_m)$ and $\lim_m \mathrm{D}(\mathcal{F}_m) = \mathrm{D}(\mathcal{F})$.
- (XZ20) Let v compute $\mathrm{lct}(X; I^{\mu(\mathcal{F})}_\bullet)$, $D(\mathcal{F}) \geq c(A_X(v) - S(v))$.

- Let $f\colon X \to \mathrm{Spec}(\mathbb{O})$ be a family of Fano varieties where $\mathbb{O}$ is a DVR with fractional field $\mathbb{K}$. Let E be a divisor over X.

Definition

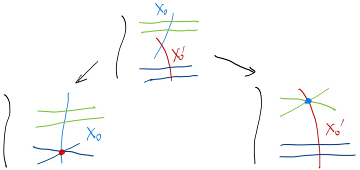
Let $A_m := f_*(mK_X), A_\bullet = \bigoplus_m A_m$ and $R_m = H^0(-mK_{X_0})$. Define

$$\mathcal{G}_E^\lambda R_m = \mathrm{Im}(\mathcal{F}_E^\lambda A_m \to A_m \to R_m)\,.$$

- For another family of Fano varieties $f'\colon X' \to \mathrm{Spec}(\mathbb{O})$ with $X_{\mathbb{K}} \cong X'_{\mathbb{K}}$, let $a = A_X(X'_0) - 1 = A_{X,X_0}(X'_0)$ and $a' = A_{X',X'_0}(X_0)$. Consider X'_0 over X to get $\mathcal{G}$ and X_0 over X' to get $\mathcal{G}'$,

  Key isomorphism: $\mathrm{Gr}^p_{\mathcal{G}} R_m \cong \mathrm{Gr}^{(a+a')m-p}_{\mathcal{G}'} R'_m$.

  In particular, $S_{X_0}(\mathcal{G}) + S_{X'_0}(\mathcal{G}') = a + a'$.

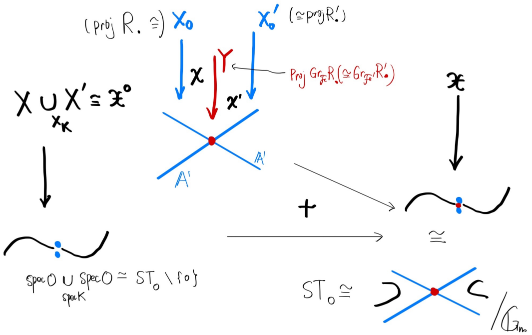


- Let $J_m = \mathrm{Bs}(\mathcal{F}^{am}_{X'_0} A_m \to A_m)$, so $\mathrm{lct}(X, X_0; J_m) \le \frac{1}{m}$. Thus $\mu(\mathcal{G}) \le a$ and similarly, $\mu(\mathcal{G}') \le a'$.
- If X_0 and X'_0 are K-semistable, then $a = \mu(\mathcal{G})(= S(\mathcal{G}))$, $\mathrm{lct}(X, X_0; \{J_m\}_m) = 1$, and X'_0 computes the log canonical threshold.
- This implies $\mathrm{Gr}_{\mathcal{F}_{X'_0}} A_\bullet$ is finitely generated, which implies $\mathrm{Gr}_{\mathcal{G}} R_\bullet$ is finitely generated.
- Let $Y = \mathrm{ProjGr}_{\mathcal{G}} R_\bullet$, then $X_0 \overset{\mathcal{X}}{\rightsquigarrow} Y \overset{\mathcal{X}'}{\leftsquigarrow} X'_0$ with weight opposite to each other up to a shift, which implies $\mathrm{Fut}(\mathcal{X}) = -\mathrm{Fut}(\mathcal{X}')$.
- So $\mathrm{Fut}(\mathcal{X}) = \mathrm{Fut}(\mathcal{X}') = 0$ and Y is K-semistable.

Theorem (Blum-X. 18)

X_0 and X'_0 degenerates to the same K-semistable Fano variety Y.

- Let $\mathrm{ST}_{\mathbb{O}} = [\mathrm{Spec}\,(\mathbb{O}[s,t]/(st-\pi))\,/\mathbb{G}_m]$ $(\mu\colon\ t\to\mu t,\ s\to\mu^{-1}s)$.
- Let $o=[(0,0)/\mathbb{G}_m]$. Then $\mathrm{ST}_{\mathbb{O}}\setminus\{o\}=\mathrm{Spec}(\mathbb{O})\cup_{\mathbb{K}}\mathrm{Spec}(\mathbb{O})$.



- $\mathrm{ST}_{\mathbb{O}} = [\mathbb{A}^1_s/\mathbb{G}_m]\cup_{t=0}(\mathrm{ST}_{\mathbb{O}}\setminus\{o\})\cup_{s=0}[\mathbb{A}^1_t/\mathbb{G}_m]$.

Alper-Halper-Leistner-Heinloth's Theorem

- A stack $\mathfrak{M}$ is S-complete iff $\mathrm{ST}_{\mathbb{O}} \setminus \{o\} \to \mathfrak{M}$ can be uniquely extended to a morphism $\mathrm{ST}_{\mathbb{O}} \to \mathfrak{M}$. A stack $\mathfrak{M}$ is Θ-reductive iff $[\mathrm{Spec}\,\mathbb{O}[t] \setminus \{0_k\}/\mathbb{G}_m] \to \mathfrak{M}$ can be uniquely extended to a morphism $[\mathrm{Spec}\,\mathbb{O}[t]/\mathbb{G}_m] \to \mathfrak{M}$.
- (Alper) An Artin stack $\mathfrak{M} \to M$ admits a good moduli space (as an algebraic space) if there is an étale cover $\{U_\alpha\}_\alpha \to M$ such that $U_\alpha \times_M \mathfrak{M} \to U_\alpha$ is given by $[\mathrm{Spec}(A_\alpha)/G_\alpha] \to \mathrm{Spec}(A_\alpha^{G_\alpha})$ for a reductive group G_α acts on an affine scheme $\mathrm{Spec}(A_\alpha)$.

Theorem (A-HL-H 18)

If a finite type Artin stack $\mathfrak{M}$ with affine stabilizers and separated diagonal is S-complete and Θ-reductive, then $\mathfrak{M}$ admits a separated good moduli space $\mathfrak{M} \to M$.

Good moduli space

For an Artin stack, admitting a separated good moduli space presents very strong properties on its geometry.

Theorem (Alper-Blum-Halper-Leistner-X. 19)

The stack $\mathfrak{X}^{\mathrm{K}}_{n,V}$ is S-complete and Θ-reductive. As a corollary, it admits a separated good moduli space $\mathfrak{X}^{\mathrm{K}}_{n,V} \to X^{\mathrm{K}}_{n,V}$, whose points parametrizes K-polystable Fano varieties.

- Corollary: if X is K-polystable, then $\mathrm{Aut}(X)$ is reductive.
- (Zhuang 20) X is K-semistable (resp. K-polystable) over k if and only if it is K-semistable (resp. K-polystable) over $\bar{k}$.

- Good moduli space
- Higher rank finite generation
- Properness

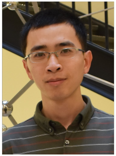
Theorem (Liu-X.-Zhuang 21, X.-Zhuang 22)

If X is a Fano variety with $\delta(X) < \frac{\dim X+1}{\dim X}$, then any quasi-monomial minimizer v with $\delta(X) = \frac{A_X(v)}{S_X(v)}$ satisfies $\mathrm{Gr}_v R_\bullet$ is finitely generated.

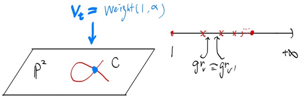
- If $\mathrm{Gr}_v R_\bullet$ is finitely generated, then in a neighborhood U of v in the minimal rational subspace in $\mathrm{QM}(Y, E)$, $A_X(\cdot)$ and $S_X(\cdot)$ are linear on U.

Corollary

There exists a divisor E, such that $\delta(X) = \frac{A_X(E)}{S_X(E)}$. So if X is K-stable, then there exists $\varepsilon > 0$ such that $A_X(E) > (1+\varepsilon)S_X(E)$.



- Any minimizer v is an lc place of a $\mathbb{Q}$-complement D, but unlike the divisorial case, this is not enough to guarantee $\mathrm{Gr}_v R_\bullet$ is finitely generated.
- Example: consider $(\mathbb{P}^2, C)$ where C is the nodal cubic. Let v_t be the valuation given by weight $(1, t)$ $t \in (0, +\infty)$ with respect to the two analytic branches. Then $\mathrm{Gr}_{v_t} k[x_0, x_1, x_2]$ is finitely generated if and only if $t \in \mathbb{Q}$ or $t \in (\frac{7-5\sqrt{3}}{2}, \frac{7+5\sqrt{3}}{2})$.



- (J. Peng) Similar pictures hold for all del Pezzo surfaces with irreducible nodal 1-complement.
- We first need to sort out a finer condition that a minimizer satisfies to guarantee the finite generation.

Step 1: Let $(Y,\Gamma)\to X$ be a log resol. such that $v=v_\alpha\in \mathrm{QM}(Y,\Gamma)$. $G\geq 0$ on X is a birational transform of an ample divisor on Y in general position. Then v is an lc place of (X,D) with $D\geq \varepsilon G$.

Proof.

- Let D_m be an m-basis type divisor compatible with both $\mathcal{F}_v$ and $\mathcal{F}_G$. So $D_m\geq a_m G$ with $\lim_{m\to\infty} a_m\to a>0$.
- By linear Diophantine approximation, there exist integral approximations of $\alpha_1,\dots,\alpha_r$ such that $\|\alpha_i - p_i\cdot\alpha\|\leq\varepsilon$ for some integers p_i, and α contained in the convex cone generated by α_i. Each α_i corresponds to a divisor $\Gamma_i=v_{p_i\alpha_i}$.
- So for any $\varepsilon>0$, there is a $\mathbb{Q}$-complement D'_m with $D'_m\geq \frac{a\delta}{2}G$ such that $A_{X,D'_m}(\Gamma_i)<\varepsilon$.
- By global ACC (HMX 12), this implies, there exists a $\mathbb{Q}$-complement $D\geq\frac{a\delta}{2}G$, such that Γ_i are lc places of (X,D).

$\square$

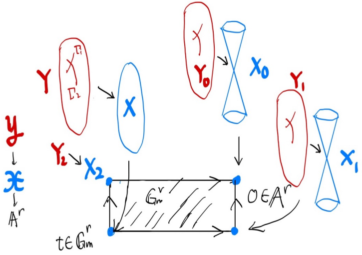
- Step 2: Using minimal model program, we can replace $(Y, \Gamma + \Theta)$ by a dlt pair over X such that $-K_Y - \Gamma - \Theta$ is ample, $\lfloor \Gamma + \Theta \rfloor = \Gamma$ and $v \in \mathrm{QM}(Y, \Gamma)$.
- Step 3: $\mathrm{QM}(Y, \Gamma)$ is a simplex and

Theorem (X.-Zhuang 22)

If v and v' belong to the same interior facet of $\mathrm{QM}(Y, \Gamma)$, $\mathrm{Gr}_v R_\bullet \cong \mathrm{Gr}_{v'} R_\bullet$.

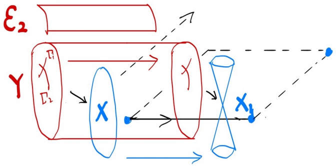
- Let $\Gamma = \sum_{i=1}^r \Gamma_i$. For $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r_{\geq 0}$, we define a filtration
$$\mathcal{F}^\lambda_\alpha R_m = \mathrm{Span}\Big\{ s \in R_m \mid \textstyle\sum_{i=1}^r \alpha_i \cdot \mathrm{ord}_{\Gamma_i}(s) \geq \lambda \Big\}.$$
Then $\mathcal{F}^\lambda_\alpha R_m \subseteq \mathcal{F}^\lambda_{v_\alpha} R_m$.
- Moreover, if $\mathrm{Gr}_{\mathcal{F}_\alpha} R_\bullet$ is integral, then $\mathcal{F}^\lambda_\alpha = \mathcal{F}^\lambda_{v_\alpha}$.

- We construct a locally (KSBA) stable $\mathbb{G}_m^r$-equivariant family $\mathcal{X} \to \mathbb{A}^r$ such that $\mathcal{X}_t \cong X$ for $t \in \mathbb{G}_m^r$, and $X_0 \cong \mathrm{Proj}(\mathrm{Gr}_{\mathcal{F}_\alpha} R_\bullet)$ $(\forall \alpha \in \mathbb{R}^r_{>0}) \cong \mathrm{Proj}(\mathrm{Gr}_{\Gamma_r} \cdots \mathrm{Gr}_{\Gamma_1} R_\bullet)$ (which doesn't depend on the order of Γ_i).



- Aim to show there exist $\mathbb{G}_m$-models $\phi \colon (\mathcal{Y}, \Gamma_{\mathcal{Y}} + \Theta_{\mathcal{Y}}) \to \mathcal{X} \to \mathbb{A}^r$ such that ϕ has dlt Fano fibers, with $\Gamma_{\mathcal{Y}}$ birational to $\Gamma \times \mathbb{A}^r$.
- For any i, over $(t_1, \ldots, t_r) \in \mathbb{G}_m^{r-1} \times \mathbb{A}^1$ with $t_1 \cdots t_{i-1} t_{i+1} \cdots t_r \neq 0$, the family is determined by the degeneration induced by $\mathbb{G}_m^{r-1} \times \Gamma_i$. We need to extend the family over the codimension ≥ 2 strata. It is unique for a flat family of polarized varieties.

- Assume $\mathcal{Y}_{r-1} \to \mathcal{X}_{r-1} \to \mathbb{A}^{r-1}$ exists, with $\mathcal{E}_r \subset \mathcal{Y}_{r-1}$ birational to $\Gamma_r \times \mathbb{A}^{r-1}$. BCHM implies $\mathcal{E}_r$ yields a family $\mathcal{X} \to \mathbb{A}^r = (\mathbb{A}^{r-1} \times \mathbb{A}^1)$ extending $\mathcal{X}_{r-1} \times \mathbb{G}_m \to \mathbb{A}^{r-1} \times \mathbb{G}_m$.
- As $\mathcal{E}_r$ is an lc place of the locally stable family $(\mathcal{X}_{r-1}, \mathcal{D}_{r-1})/\mathbb{A}^{r-1}$, $(\mathcal{X}, \mathcal{D})$ is birationally crepant equivalent to $(\mathcal{X}_{r-1}, \mathcal{D}_{r-1}) \times \mathbb{A}^1$, thus $(\mathcal{X}, \mathcal{D}) \to \mathbb{A}^r$ is a locally stable family.
- We extract $\Gamma \times \mathbb{A}^r$ to get $\mathcal{Y}$, and there exists $\Theta_{\mathcal{Y}} \geq \Theta_{r-1} \times \mathbb{A}^1$ with $\mu^*(K_{\mathcal{X}} + \mathcal{D}) \geq K_{\mathcal{Y}} + \Gamma_{\mathcal{Y}} + \Theta_{\mathcal{Y}}$, and $(\mathcal{Y}, \Gamma_{\mathcal{Y}} + \Theta_{\mathcal{Y}})$ is dlt and Fano over $\mathbb{A}^r$.



- Observation: For any $0 \leq G \sim_{\mathbb{Q}} -K_Y - \Gamma - \Theta$ such that G does not contain the intersection Z of Γ_i, the degeneration of $(Y, \Gamma + \Theta + \varepsilon G)$ still yields a locally stable family.

  We apply this twice:

  1. Let $Z_0 = \cap_{i=1}^r \Gamma_{\mathcal{Y}_i} \bigcap \phi^{-1}(\mathbf{0})$ with $\mathbf{0} = (0, \ldots, 0) \in \mathbb{A}^r$. For any proper closed subset $W \subsetneq Z_0$, pick G such that $Z \not\subset G$, but W is in the degeneration of G. The above implies Z_0 is the unique minimal lc center of $(\mathcal{Y}_r, \Gamma_r + \Theta_r + \sum_{i=1}^r (t_i = 0))$.
  2. Pick $G = \frac{1}{2m}(G_+ + G_-)$ for general $G_+ \in |mL + \Gamma_i|$ and $G_- \in |mL - \Gamma_i|$ for $L = -m(K_Y - \Gamma - \Theta)$ and sufficiently divisible m. This implies that every component of $\Gamma_{\mathcal{Y}}$ is Cartier around $\eta(Z_0)$.

  So $(\mathcal{Y}, \Gamma_{\mathcal{Y}} + \Theta_{\mathcal{Y}}) \to \mathbb{A}^r$ has dlt fibers.

- (Work in progress by Z. Chen) There exists a birational model $Z' \to Z$ and an r-dimensional polyhedral convex cone $\mathcal{C} \subset \mathrm{Div}(Z')$, such that $\mathrm{Gr}_{\mathcal{F}_\alpha} R_\bullet \cong \bigoplus_{L \in \mathcal{C}} H^0(Z', L)$.

- Good moduli space
- Higher rank finite generation
- Properness

Theorem (Blum-Halpern-Leistner-Liu-X. 20, B-L-X.-Zhuang 25)

$X^{\mathrm{K}}_{n,V}$ is proper.

- Both proofs fundamentally depend on the HRFG Theorem.
- Halpern-Leistner's Θ-stratification: if any unstable object has a unique optimal destabilizing (up to reparametrization) satisfying some natural properties, then the good moduli space of the semistable objects is proper.
- The optimal destabilization arises from minimizing functions defined on all special TC $\mathcal{X}$.
- If a special TC $\mathcal{X}$ yields a divisorial valuation E, then $\frac{\mathrm{Fut}(\mathcal{X})}{\|\mathcal{X}\|_{\mathrm{m}}} = \delta(E) - 1$ ($\|\cdot\|_{\mathrm{m}}$ is the minimum norm defined by Dervan).
- However, the minimizer of E is not unique. B-HL-L-X constructs a second term to make an alphabetic order $(\frac{\mathrm{Fut}(\mathcal{X})}{\|\mathcal{X}\|_{\mathrm{m}}}, \frac{\mathrm{Fut}(\mathcal{X})}{\|\mathcal{X}\|_{2}})$, which yields a Θ-stratification on $\mathfrak{X}^{\mathrm{Fano}}_{n,V}$.

Relative stability theory (Blum-Liu-X.-Zhuang 25)

Definition

Let $f\colon X \to \mathrm{Spec}(\mathbb{O})$ be a family of klt Fano varieties with $\delta(X_0) < \min\{\delta(X_{\overline{K}}), 1\}$. For any divisor E over X, we define

$$\delta_t(E) := \frac{A_{X,(1-t)X_0}(E)}{S(\mathcal{G}_E)} \ \text{ and } \ \delta_t(X) := \inf_E \delta_t(E)\,.$$

- If $t = 0$, then $\delta_0(X) = \delta(X_0)$.
- For $t \in [0,1]$, there exists a quasi-monomial v computing $\delta_t(X)$.
- v is over the special fiber and satisfies $\mathrm{Gr}_v A_\bullet$ is finitely generated. In particular, there is a divisorial minimizer E.
- If $0 < t \ll 1$, after a base change of $\mathrm{Spec}(\mathbb{O})$, we can get a family of Fano varieties $X' \to \mathrm{Spec}(\mathbb{O})$ with $X_K = X'_K$, such that X'_0 is the birational transform of E.
- We have $\delta(X_0) \le \delta_0(E) < \delta_t(E) = \delta_t(X) \le \delta(X'_0)$.

- If E_m computes $\min_{D_m} \mathrm{lct}(X, (1-t)X_0; D_m) := \delta_m$ for all m-basis type divisors D_m. Let $X' \to \mathrm{Spec}(\mathbb{O})$ with an integral fiber X'_0 birational to E_m.
- Any D_m compatible with ord_{E_m}, its birational base change yields an m-basis type divisor D'_m compatible with ord_{X_0}, and vice versa.
- If $\delta_m \leq 1$, then $(X, (1-t)X_0 + \delta_m D_m)$ is lc implies $(X', X'_0 + \delta_m D'_m)$ is lc (by two-divisor game).
- So $\min_{D'_m} \mathrm{lct}(X', X'_0; D'_m)$
  $$= \min_{D'_m}\{\mathrm{lct}(X', X'_0; D'_m) \mid D'_m \text{ is compatible with } \mathrm{ord}_{X_0}\} \geq \delta_m$$
- In general, we extract E for $g\colon Y \to X$. Let $\lambda_m := \inf_{D_m} \mathrm{lct}(Y, (1-t)g_*^{-1}X_0 + E; g_*^{-1}D_m)$. The above argument implies for $m \gg 0$, $\delta_m(X'_0) \geq \min\{\lambda_m, \min_{D_m} \mathrm{lct}(X, (1-t)X_0; D_m)\}$.
- We show $\lim_m \lambda_m \geq \delta_t(X)$. Taking a limit $m \to \infty$, $\delta(X'_0) \geq \delta_t(X)$.

Thank you very much!