K-stability and Moduli of Fano varieties I

Chenyang Xu (Princeton University)

2025 Summer Research Institute

July 16, 2025

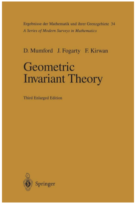
- Moduli of varieties
- Definition of K-stability
- K-stability via valuations

Moduli of varieties: history

- Goal: for a class of varieties $\{X\}$, construct a space $\mathcal{M}$, such that each object X in the class precisely corresponds to a point in $[X] \in \mathcal{M}$.
- Example: $\dim$ k linear subspace in $\mathbb{C}^n$, $\mathrm{Grass}(k, n)$.
- Example (Grothendieck): subschemes of $\mathbb{P}^n$ can be parametrized, by the Hilbert scheme.
- Parametrizing isomorphic classes of varieties needs a deeper theory.
- Need an embedding $|L|: X \to \mathbb{P}^N$ (up to $\mathrm{PGL}(N+1)$), i.e. a polarization L.
- There are natural line bundles $\omega_X^{\otimes m} = (\wedge^{\dim X} T_X)^{\otimes -m}$ ($m \in \mathbb{Z}$).

First phase: Moduli of curves

- There has been a long history for people trying to parametrize varieties, going back to Abel, Jacobi, Riemann, Weierstrass, Teichmuller etc.
- Moduli of curves C of higher genus g ($g \geq 2$), i.e. $\omega_C > 0$: $\mathcal{M}_g$, Mumford's geometric invariant theory (GIT) (65).
- Deligne-Mumford compactification (69): $\overline{\mathcal{M}}_g$, parametrizing compact (possibly reducible) curves C with nodal singularities, such that ω_C is ample.



Second phase: Moduli of varieties with $\omega_X > 0$

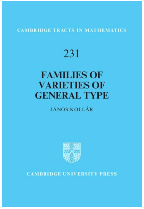
Moduli theory for X with ample K_X in higher dimensions:

- Include all smooth X with ample K_X.
- Difficulty: characterize the limit.
- Mumford's GIT doesn't work well in higher dimensions, as the GIT limit of $|\omega_X^{\otimes m}|\colon X \to \mathbb{P}^{N(m)}$ might be different for all large m.
- Kollár-Shepherd-Barron(-Alexeev) (KSB or KSBA) theory (88) proposes:

KSB(A) stability = ($\omega_X > 0$) + (mild singularities)

- The relative minimal model program yields a limit as such (similar to Deligne-Mumford's construction).

- By late 2010s, the construction was completed.
- It is intertwining with the progress of MMP.
- One foundational notion is local KSB(A) stability (Kollár): right definition of a family of higher dimensional varieties (or pairs) over a general base. It is (far) more subtle than flatness.



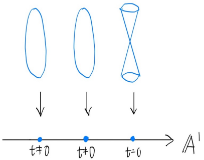
Third phase: how about Fano varieties?

- A (smooth or mildly singular) projective variety X is called Fano if $-K_X$ is ample. They belong to one of the trichotomy of classfying varieties by their sign of K_X.
- $\mathrm{dim}(X) = 2$, there are 10 families of smooth del Pezzo surfaces.
- (Iskovskikh, Mori-Mukai 82) $\mathrm{dim}(X) = 3$, there are 105 families of smooth Fano threefolds.
- (Kollár-Miyaoka-Mori 92) In any fixed dimension, there are only finitely many families of smooth Fano varieties.
- Explicit geometry has been studied for decades from many perspectives: rationality, birational automorphisms, Mori theory, cohomology, derived categories, etc..

For Fano varieties, before 2010 very few algebraic geometers thought general moduli theory of Fano varieties is possible!

- A simple example:

$$(x_0^2 + \cdots x_{n-1}^2 + t \cdot x_n^2 = 0) \subset \mathbb{P}^n \times \mathbb{A}^1_t \, .$$

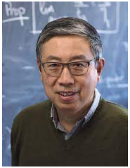


Kähler-Einstein Problem

- Kähler-Einstein problem (Kähler 1933, Calabi 1950s): find a Kähler metric ω such that $\mathrm{Ric}(\omega) = \lambda\omega$ for a constant λ.
- $[\mathrm{Ric}(\omega)] = -K_X$. For $\lambda = 0$ or -1, KE metric exists uniquely.
- Yau (80s) had deep reasons to believe the existence of KE metrics is equivalent to some algebraic 'stability' condition.
- Tian (97) proposed a condition called K-stability; and Donaldson (01) observed it was indeed algebraic.

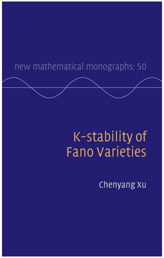
Theorem (Yau-Tian-Donaldson Conjecture)

A Fano variety X has a KE metric if and only if X is K-polystable.


Solution

K-(semi,polystable)stable Fano varieties provides a subclass of Fano varieties with a good moduli theory, similar to stable vector bundles.

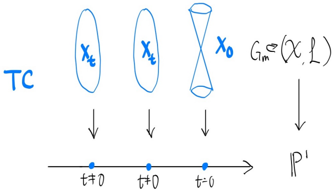


- Moduli of varieties
- Definition of K-stability
- K-stability via valuations

- Fix a klt Fano variety X. Test configuration (TC): $(\mathcal{X}, \mathcal{L})$ is $\mathbb{G}_m$-equivariant over $\mathbb{P}^1$, with $\mathcal{X}$ normal and $\mathcal{L}$ a $\mathbb{Q}$-line bundle, such that

$$(\mathcal{X}, \mathcal{L})|_{\mathbb{P}^1 \setminus \{0\}} \cong_{\mathbb{G}_m} (X, -K_X) \times (\mathbb{P}^1 \setminus \{0\})$$

with $\mathbb{G}_m$-trivially acting on the first factor.



- $\text{Fut}(\mathcal{X}, \mathcal{L}) := \frac{1}{(n+1)(-K_X)^n}(n\mathcal{L}^{n+1} + (n+1)\mathcal{L}^n \cdot K_{\mathcal{X}/\mathbb{P}^1})$.

Definition (Tian 97, Donaldson 01)

- X is *K-semistable* iff $\mathrm{Fut}(\mathcal{X}, \mathcal{L}) \geq 0$ for any TC $\mathcal{X}$.
- X is *K-stable* iff $\mathrm{Fut}(\mathcal{X}, \mathcal{L}) \geq 0$ and "$=$" holds only if $\mathcal{X} \cong_{\mathbb{G}_m} X \times \mathbb{P}^1$ with $\mathbb{G}_m$-acts on the first factor trivially.
- X is *K-polystable* iff $\mathrm{Fut}(\mathcal{X}, \mathcal{L}) \geq 0$ and "$=$" holds only if $\mathcal{X}|_{\mathbb{A}^1} \cong_{\mathbb{G}_m} X \times \mathbb{A}^1$ for a $\mathbb{G}_m$-action on X.

Compare to GIT

- Fix X, $\mathcal{X}$ are unbounded families.
- There exists a line bundle λ (the CM-line bundle) on the Hilbert scheme, such that $\mathrm{Fut}(\mathcal{X}, \mathcal{L}) = \mathrm{wt}(0, \lambda)$, but λ is not ample on Hilbert scheme.

MMP Perspective

Theorem (C. Li-X. 12)

For any TC $\mathcal{X}$, we can use MMP techniques to give a sequence of operations, to get a TC $\mathcal{X} \dashrightarrow \mathcal{X}^{\mathrm{s}}$ such that

1. $\mathcal{X}^{\mathrm{s}}$ is special, i.e. $K_{\mathcal{X}^{\mathrm{s}}}$ is $\mathbb{Q}$-Cartier, and $\mathcal{X}^{\mathrm{s}}_0$ is klt.
2. there exists a positive $d \in \mathbb{Z}$, such that $\mathrm{Fut}(\mathcal{X}^{\mathrm{s}}) \leq d \cdot \mathrm{Fut}(\mathcal{X})$.

- This implies to test K-(semi,poly)stability it suffices to only consider special TCs.
- For a special TC $\mathcal{X}$, $\mathrm{Fut}(\mathcal{X}) = \frac{-1}{(n+1)(-K_X)^n}(-K_{\mathcal{X}/\mathbb{P}^1})^{n+1}$.
- It is still hard to determine all special TCs for X when $\dim(X) \geq 2$, e.g. $X \cong \mathbb{P}^2$.
- This represents an early evidence that MMP plays a deep role in K-stability theory.
- It is first among a series of results showing testing K-stability are equivalent on different class of objects, which is indispensable in our understanding of the moduli problem.

Ding stability

- (Berman 12) Let $K_{\mathcal{X}/\mathbb{P}^1} + \mathcal{L} = D$ supporting on $\mathcal{X}_0$. $\mathrm{Ding}(\mathcal{X}) := \frac{-1}{(n+1)(-K_X)^n}\mathcal{L}^{n+1} + \mathrm{lct}(\mathcal{X}, D; \mathcal{X}_0) - 1$.
- We can define Ding stability notions, replacing $\mathrm{Fut}(\mathcal{X})$ by $\mathrm{Ding}(\mathcal{X})$.
- For a special TC, $\mathrm{Ding}(\mathcal{X}) = \mathrm{Fut}(\mathcal{X})$.
- (Fujita 15) To test Ding (semi-poly)stability, it only suffices to test on special TCs.

Ding stability=K-stability

- Moduli of varieties
- Definition of K-stability
- K-stability via valuations

- Let $\mathcal{X}/\mathbb{A}^1$ be a TC. For any component $Y \subset \mathcal{X}_0$ which is not $X \times \{0\}$, then $\mathrm{ord}_Y|_{K(X)}$ is given by $c \cdot \mathrm{ord}_E$ for some $c \in \mathbb{Z}_{>0}$ and E a divisor over X, i.e. E is a divisor on a birational model $\mu\colon Y \to X$.
- $A_X(E) := \mathrm{mult}_E(\omega_{Y/X}) + 1 > 0$ since X it klt.
- $S_X(E)$ is the expected vanishing order, i.e.,

$$S_X(E) := \frac{1}{(-K_X)^n} \int_0^\infty \mathrm{vol}(\mu^*(-K_X) - tE)dt.$$

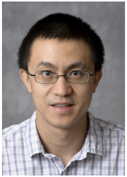
- If $\mathcal{X}$ is (nontrivial) special, with $\mathrm{ord}_{\mathcal{X}_0}|_{K(X)} = c \cdot \mathrm{ord}_E$, then

$$\mathrm{Fut}(\mathcal{X}) = c(A_X(E) - S_X(E))$$

Theorem (Fujita-Li Criterion 15)

$$\begin{aligned} \text{K-semistability} &\iff A_X(E) \geq S_X(E) \text{ for all } E; \\ \text{K-stability} &\iff A_X(E) > S_X(E) \text{ for all } E; \\ \text{Uniform K-stability} &\iff \exists \varepsilon > 0,\ A_X(E) > (1+\varepsilon) S_X(E) \text{ for all } E. \end{aligned}$$

- This gives a more computable characterization of K-stability.
- To prove "$\Longrightarrow$", we will show that it suffices to look at a sequence of divisors E arising from TCs.



- D_m is an *m-basis-type* divisor, if $D_m = \frac{1}{mN_m}\sum_{i=1}^{N_m}\mathrm{div}(s_i)$, where $\{s_1,\dots,s_{N_m}\}$ is a basis of $R_m = H^0(-mK_X)$.
- The vanishing order along ord_E yields a filtration on R_m. Define

$$S_m(E) = \max_{D_m}\mathrm{ord}_E(D_m) = \frac{1}{mN_m}\sum_{\lambda}\lambda\dim \mathrm{Gr}_E^{\lambda}R_m\,.$$

Definition-Theorem (Fujita-Odaka 16, Blum-Jonsson 17)

1. Define $\delta_m(X) = \inf_E \frac{A_X(E)}{S_m(E)}$. Then $\delta_m(X) = \min_{D_m}\mathrm{lct}(X, D_m)$, where $\mathrm{lct}(X, D_m) = \sup_t\{t \mid (X, tD_m) \text{ is lc}\}$.
2. $\lim_m \delta_m(X) = \delta(X) := \inf_E \frac{A_X(E)}{S_X(E)}$.

- So $\delta(X) \geq$ (resp. $>$) 1 iff X is K-semistable (resp. uniformly K-stable). In general, $\delta(X)$, called the stability threshold, gives a quantitative way to measure how 'stable' X is.

- An effective $\mathbb{Q}$-divisor is an *m-complement* if $mD \in |-mK_X|$, i.e. $D \sim_{\mathbb{Q}} -K_X$ and (X, D) is lc. D is a *$\mathbb{Q}$-complement* if it is an m-complement for some positive m.

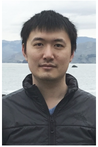
Theorem (Birkar 16)

There exists an $N = N(\dim X)$ such that any klt Fano variety X admits an N-complement.

- For an lc pair (X, D), E is an *lc place* of (X, D) if $A_X(E) = \mathrm{ord}_E(D)$, i.e. $A_{X,D}(E) = 0$.

Theorem (Blum-Y. Liu-X. 19)

1. (Lc places E of $\mathbb{Q}$-complements) $\Longleftrightarrow$ (Nontrivial TCs $\mathcal{X}$ such that $(\mathcal{X}, \mathcal{X}_0)$ is lc with irreducible $\mathcal{X}_0$).
2. When $\delta(X) < \frac{\dim(X)+1}{\dim(X)}$, $\delta_m(X) = \frac{A_X(E_m)}{S_m(E_m)}$ where E_m is the an lc place of a $\mathbb{Q}$-complement D_m for $m \gg 0$.
3. There exists $N = N(\dim(X))$ such that an lc place E of a $\mathbb{Q}$-complement is an lc place an N-complement.
4. In (2), $\delta(X) = \lim_m \frac{A_X(E_m)}{S(E_m)}$ where E_m are lc places of a fixed N-complement D.
5. (Openness of K-semistability) $(b \in B) \to \min\{1, \delta(X_{\bar{b}})\}$ is constructible and lower semi-continuous.



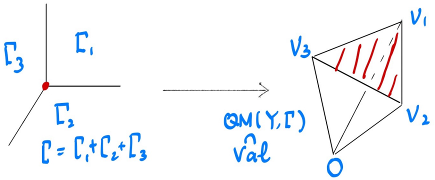
Proof.

1. (Two-divisor game) E is an lc place of the lc pair (X, D) for $D \sim_{\mathbb{Q}} -K_X$. Let E_1 be the divisor on $X_{\mathbb{A}^1} := X \times \mathbb{A}^1_t$ for $(\mathrm{ord}_E, \mathrm{ord}_t)$. E_1 is an lc place of $(X_{\mathbb{A}^1}, D_{\mathbb{A}^1} + X \times \{0\})$. By BCHM, we can extract E_1 and contract the divisor $X \times \{0\}$, and get a relative Fano $\mathcal{X} \to \mathbb{A}^1$, which satisfies $(\mathcal{X}, \mathcal{X}_0)$ is log canonical.
2. Let E_m be the divisor computes δ_m. It is the lc place of $(X, \delta_m D'_m)$ for any m-basis type divisor D'_m compatible with ord_{E_m}. D'_m can be chosen to be compatible with a general $\mathbb{Q}$-divisor $H \sim_{\mathbb{Q}} -K_X$. Then $D'_m = G_m + a_m H$ with $a_m \to \frac{1}{n+1}$. So E_m is an lc place of $(X, \delta_m G_m)$, as $G_m \sim_{\mathbb{Q}} -(1 - a_m)K_X$ with $\lim_m (1 - a_m)\delta_m < 1$.
3. Essentially after Birkar.

□

- (2) implies $\delta(X) = \lim_m \frac{A_X(E_m)}{S(E_m)}$ and (1) implies E_m arising from TC.

- If $(Y, \Gamma = \sum \Gamma_i)$ is simple normal crossing, for any component Z of $\bigcap_{i=1}^r \Gamma_i$, around the generic point $\eta(Z)$ of Z, we can define valuations v_α $(\alpha \in \mathbb{R}^r)$ such that $v_\alpha(f) = \min_{c_\beta \neq 0}\langle \alpha, \beta\rangle$, where the expansion of f at η is $\sum_{\beta \in \mathbb{N}^r} c_\beta z^\beta$ where $\Gamma_i = (z_i = 0)$.
- All such valuations are denoted as $\mathrm{QM}(Y, \Gamma)$, which is naturally identified as the cone over the dual complex $D(Y, \Gamma)$.
- The set of quasi-monomial valuations consists of $\mathrm{QM}(Y, \Gamma)$ for all (Y, Γ).



- $B = |-NK_X|$. After stratifying, we assume $(\mathcal{X} = X \times B, D_{\mathcal{X}}) \to B$ is a family of lc pairs with a simultaneous (fiberwise) resolution $(\mathcal{Y}, \Gamma_{\mathcal{Y}}) \to (\mathcal{X}, D_{\mathcal{X}})$. We work on a connected component of B.
- Over a fixed point $t \in B$, write $K_Y + \Gamma = \mu^*(K_X + D)$.
- For another $t_0 \in B$ with $K_{Y_0} + \Gamma_0 = \mu^*(K_X + D_0)$, we can canonically identify $D(Y, \Gamma^{=1}) = D(Y_0, \Gamma_0^{=1})$.
- (Invariance of log plurigenera, Hacon-McKernan-X.) For F, F_0 corresponding to the same point in $D(Y, \Gamma^{=1}) = D(Y_0, \Gamma_0^{=1})$, $S(F) = S(F_0)$. Thus we may assume all E_m to be over $t \in B$.
- Applying this argument to a family of Fano varieties $X \to B$, we get $(t \in B) \mapsto \min\{\frac{\dim X+1}{\dim X}, \delta(X_{\bar{t}})\}$ is constructible.
- For $\{v_m\}_m \subset D(Y, \Gamma^{=1})$ corresponding to $\{E_m\}_m$, a subsequence admits a limit v^*.
- A_X and S_X can be continuously extended to $\mathrm{QM}(Y, \Gamma)$. So v^* yields a quasi-monomial minimizer of $\frac{A_X(\cdot)}{S_X(\cdot)}$.

Stack of Fano varieties

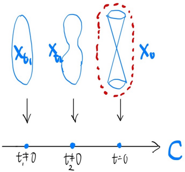
- (Boundedness, C. Jiang 17) Fix positive lower bounds v_0 and δ_0, all X with $v_0 \leq (-K_X)^n$ and $\delta_0 < \delta(X)$ are bounded.
- This boundedness can be reduced to Birkar's BAB type results. One can also show the Cartier index of X is bounded (X.-Zhuang 20), and apply Hacon-McKernan-X. Theorem.
- Together with the openness, we have the following: for a fixed volume V, there exist $1 = \delta_0 > \delta_1 > \cdots$ which converge to 0, and the stack $\mathfrak{X}^{\rm Fano}_{n,V}$ of all n-dimensional Fano varieties with $(-K_X)^n = V$ can be covered by increasing open substacks

$$\mathfrak{X}^{\rm K}_{n,V} = \mathfrak{X}^{\delta_0}_{n,V} \subset \mathfrak{X}^{\delta_1}_{n,V} \subset \cdots \subset \mathfrak{X}^{\rm Fano}_{n,V} ,$$

where $\mathfrak{X}^{\delta_i}_{n,V}$ parametrizes families of Fano varieties X with $\delta(X) \geq \delta_i$. Moreover, for $i > 0$, $\mathfrak{X}^{\delta_i}_{n,V} \setminus \mathfrak{X}^{\delta_{i-1}}_{n,V}$ parametrizes Fano varieties X with $\delta(X) = \delta_i$.

Tomorrow

- Started with a family of Fano varieties over a punctured disk $X^\circ \to C^\circ = C \setminus \{0\}$, what is optimal degeneration X_0?



- There exists a klt Fano degeneration X_0, but often non-unique.

Assume X_t are K-semistable,

1. Does there always exist a K-semistable degeneration?
2. Assuming the existence, is it unique?

We will continue.

Thank you very much!