# Representation Theory of Symmetric Groups

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These are lecture notes that I typed up for Dr. Susanne Danz's course on Representation Theory of Finite Groups. It was offered as an extra units Part C course at Oxford during Hilary Term of the 2010-2011 academic year. Some books that may be used as supplementary texts are the following:

- The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions (Sagan, B. E.)
- The Representation Theory of Symmetric Groups (James, G., Kerber, A.)
- Young Tableaux (Fulton, W.)
- Representation Theory: A First Course (Fulton, W., Harris, J.)
- Enumerative Combinatorics (Stanley, R.)

Here is an overview of the course (quoted from the course page):

> The representation theory of symmetric groups is a special case of the representation theory of finite groups. Whilst the theory over characteristic zero is well understood, this is not so over fields of prime characteristic. The course will be algebraic and combinatorial in flavour, and it will follow the approach taken by G. James. One main aim is to construct and parametrise the simple modules of the symmetric groups over an arbitrary field. Combinatorial highlights include combinatorial algorithms such as the Robinson-Schensted-Knuth correspondence. The final part of the course will discuss some finite-dimensional representations of the general linear group $\mathrm{GL}_n(\mathbb{C})$, and connections with representations of symmetric groups. In particular we introduce tensor products, and symmetric and exterior powers.

Here is a synopsis of the course (also quoted from the course page):

> Counting standard tableaux of fixed shape: Young diagrams and tableaux, standard-tableaux, Young-Frobenius formula, hook formula. Robinson-Schensted-Knuth algorithm and correspondence. Construction of fundamental modules for symmetric groups: Action of symmetric groups on tableaux, tabloids and polytabloids; permutation modules on cosets of Young subgroups. Specht modules, and their standard bases. Examples and applications. Simplicity of Specht modules in characteristic zero and classification of simple $\mathfrak{S}_n$-modules over characteristic zero. Characters of symmetric groups, Murnaghan-Nakayama rule. Submodule Theorem, construction of simple $\mathfrak{S}_n$-modules over a field of prime characteristic. Decomposition matrices. Examples and applications. Some finite-dimensional $\mathrm{GL}_n(\mathbb{C})$-modules, in particular the natural module, its tensor powers, and its symmetric and exterior powers. Connections with representations of $\mathfrak{S}_n$ over $\mathbb{C}$.

I should note that these notes are not polished and hence might be riddled with errors. If you notice any typos or errors, please do contact me at `charchan@stanford.edu`.

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# Chapter 0

# Background and Motivation

We begin with a revision of basic representation theory.

**Definition 0.1.** Let $G$ be a finite group and $F$ a field.

(a) $FG := \{\sum_{g\in G} \alpha_g g : \alpha_g \in F\}$ is called the *group algebra* of $G$ over $F$.

(b) An $FG$-module $M$ is a finite dimensional $F$-vector space endowed with a map $FG \times M \to M$ with the properties:

   (i) $(ab)m = a(bm)$ for $a, b \in FG, m \in M$.

   (ii) $(a+b)m = am + bm$ for $a, b \in FG, m \in M$.

   (iii) $a(m+n) = am + an$ for $a \in FG, m, n \in M$.

   (iv) $1m = m$ for $m \in M$.

(c) An $FG$-module $M$ is called simple if $M \neq 0$, and $M$ and $0$ are the only submodules of $M$.

**Remark 0.2.** By Jordan-Holder, every $FG$-module has a filtration with quotients being simple modules. Simple $FG$-modules are thus the "building blocks" of all $FG$-modules.

**Aim.** Our aim over the course of this term will be to construct explicitly a transversal for isomorphism classes of simple $F\mathfrak{S}_n$-modules where $\mathfrak{S}_n$ is the symmetric group of degree $n$ and $F$ is a characteristic 0 field. In the case that $\mathrm{char}(F) = p > 0$, we still get a parametrisation for the isomorphism classes of simple $F\mathfrak{S}_n$-modules. However, in spite of all this, there is still no known effective way of constructing these modules. (This is the fundamental open problem in representation theory.) For $\mathfrak{S}_n$, there is a tie to combinatorics, which is the reason we get such nice results. Hence our study here will have a bit of a combinatorial flavour.

**Remark 0.3.** We will use the language of modules, but recall that this is equivalent to matrix representations.

A matrix representation of $G$ over $F$ is a group homomorphism $\Delta : G \to \mathrm{GL}_n(F)$ for some $n \in \mathbb{N}$. Given $\Delta$, the $F$-vector space $F^n$ (with standard basis $e_1, \ldots, e_n$) becomes an $FG$-module via $g \cdot x := \Delta(g)x$ for $g \in G, x \in F^n$.

Conversely, if $M$ is an $FG$-module with basis $b_1, \ldots, b_n$, we get a matrix representation of $G$ over $F$ by defining $\Delta : G \to \mathrm{GL}_n(F), g \mapsto \Delta(g)$, where the matrix $\Delta(g)$ is defined as $\Delta(g)_{ij} = \alpha_{ij}$, $gb_j = \sum_{i=1}^{n} \alpha_{ij} b_i$.

A matrix representation of $G$ over $F$ is *irreducible* if the corresponding $FG$-module is simple.

**Remark 0.4.** We say that $g, h \in G$ are $G$-conjugate if there is some $x \in G$ such that $g = xhx^{-1}$. We write $g =_G h$. This defines an equivalence relation that splits $G$ into equivalence classes called conjugacy classes. The set of all conjugacy classes will be denoted $\mathrm{CL}(G)$.

Let $p$ be a prime. We say that $g \in G$ is $p$-regular if $p \nmid |\langle g \rangle|$. Otherwise, $g$ is called $p$-singular. The set of $p$-regular conjugacy classes of $G$ is denoted by $\mathrm{CL}_p(G)$.

**Theorem 0.5.** *Let $F$ be an algebraically closed field.*

*(a) If* $\mathrm{char}(F) \nmid |G|$ *then the number of isomorphism classes of simple $FG$-modules is* $|\,\mathrm{CL}(G)|$.

*(b) If* $\mathrm{char}(F) \mid |G|$ *then the number of isomorphism classes of simple $FG$-modules is* $|\,\mathrm{CL}_p(G)|$.

**Remark 0.6.** We will see that in the case of $\mathfrak{S}_n$, the assumption that $G$ is algebraically closed is unnecessary.

Now that we have revised some preliminary concepts, we can proceed to the content of this course. We start by discussing some group-theoretic properties of the symmetric group $\mathfrak{S}_n$.

# Chapter 1

# The Symmetric Group

**Remark 1.1.** Here are some basics of the symmetric group.

(a) The set of all bijections $\{1, \ldots, n\} \to \{1, \ldots, n\}$ with composition of maps forms a finite group. We call this group the symmetric group of degree $n$ and it is denoted by $\mathfrak{S}_n$.

(b) $\sigma \in \mathfrak{S}_n$ can be represented by $\begin{pmatrix} 1 & \cdots & n \\ \sigma(1) & \cdots & \sigma(n) \end{pmatrix}$. We will take the convention of composing permutations from right to left (this is most natural as we will think of functions as acting from the left) and so taking $\pi, \sigma \in \mathfrak{S}_n$, we have $\pi \cdot \sigma = \begin{pmatrix} 1 & \cdots & n \\ \pi(\sigma(1)) & \cdots & \pi(\sigma(n)) \end{pmatrix}$.

(c) Every $\sigma \in \mathfrak{S}_n$ can be written as the product of disjoint cycles. We call these $k$-cycles. A 2-cycle is also called a *transposition*.

**Definition 1.2.** Take $\sigma \in \mathfrak{S}_n$ with $\sigma = \sigma_1 \cdots \sigma_k$, where $\sigma_1, \ldots, \sigma_k$ are disjoint cycles of lengths $\lambda_1, \ldots, \lambda_k$. We may assume that $\lambda_1 \geq \cdots \geq \lambda_k$. Then $\lambda(\sigma) := (\lambda_1, \ldots, \lambda_k)$ is called the *cycle type* of $\sigma$.

**Remark 1.3.** Any permutation can be written as a product of transpositions. This can certainly be done in many, many ways, but no matter how it is done, the parity of the number of transpositions is invariant. In this way, we get a group homomorphism:

$$\begin{aligned} \mathrm{sgn} : \mathfrak{S}_n &\to \{1, -1\}, \\ \sigma &\mapsto \begin{cases} 1 & \text{if } \sigma \text{ is the product of an even number of transpositions} \\ -1 & \text{otherwise.} \end{cases} \end{aligned}$$

The kernel of this map is the alternating group $\mathfrak{A}_n$.

**Definition 1.4.** A *composition* on $n$ is a sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of non-negative integers such that $\lambda_1 + \cdots + \lambda_k = n$. We identify $\lambda$ with $(\lambda_1, \ldots, \lambda_k, 0)$. The integers $\lambda_1, \ldots, \lambda_k$ are called *parts* of $\lambda$. The set of all *compositions* of $n$ is $\mathcal{C}_n$ .

A *partition* of $n$ is a composition $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathcal{C}_n$ such that $\lambda_1 \geq \cdots \geq \lambda_k \geq 0$. The number of nonzero parts of $\lambda$ is called the length $l(\lambda)$ of $\lambda$; the set of partitions of $n$ is denoted by $\mathcal{P}_n$. Setting $a_i := |\{j = 1, \ldots, k : \lambda_j = i\}|$, we write $\lambda = (n^{a_n}, \ldots, 1^{a_1})$.

For $p \in \mathbb{N}, \lambda \in \mathcal{P}_n$, we say $\lambda$ is $p$-regular if $a_i < p$ for all $i = 1, \ldots, n$. Otherwise, $\lambda$ is $p$-singular. The set of $p$-regular partitions of $n$ is denoted $\mathcal{P}_{n,p}$ .

**Example 1.5.** Consider $\mathfrak{S}_4$, the symmetric group of degree 4. We have the following table:

| $\mathcal{P}_n$ | $\mathcal{P}_{n,3}$ | $\mathcal{P}_{n,2}$ |
|---|---|---|
| $(4)$ | $(4)$ | $(4)$ |
| $(3,1)$ | $(3,1)$ | $(3,1)$ |
| $(2^2)$ | $(2^2)$ | |
| $(2,1^2)$ | $(2,1^2)$ | |
| $(1^4)$ | | |

**Proposition 1.6.** *Permutations $\sigma, \pi \in \mathfrak{S}_n$ are conjugate if and only if $\lambda(\sigma) = \lambda(\pi)$.*

**Definition 1.7.** For $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathcal{P}_n$ with $\lambda_k \neq 0$, let $C_\lambda := \{\sigma \in \mathfrak{S}_n : \lambda(\sigma) = \lambda\}$. These are precisely the $\mathfrak{S}_n$-conjugacy classes. Note that $C_\lambda$ is $p$-regular if and only if $p \neq \lambda_i$ for all $i$.

**Proposition 1.8.** *Let $p$ be a prime. Then a) $|\operatorname{CL}(\mathfrak{S}_n)| = |\mathcal{P}_n|$, and b) $|\operatorname{CL}_p(\mathfrak{S}_n)| = |\mathcal{P}_{n,p}|$.*

*Proof.* a) follows from Proposition 1.8. For b), take $x \in \mathbb{R}$ with $|x| < 1$, and consider the formal power series

$$P(x) = \frac{\prod_{i\geq 1}(1 - x^{ip})}{\prod_{i\geq 1}(1 - x^i)}. \tag{1.1}$$

On one hand, if we cancel all the factors of $(1 - x^{jp})$ in (1.1), we get

$$P(x) = \prod_{p\nmid i} \frac{1}{(1 - x^i)} = \prod_{p\nmid i}(1 + x^i + x^{2i} + \cdots).$$

The coefficient of $x^n$ is the number of expansions $n = a_1 1 + a_2 2 + \cdots + a_n n$ where $a_i \in \mathbb{N}_0$ for each $i = 1, \dots, n$ and where $p \nmid a_i$ if $a_i \neq 0$. This is exactly the number of partitions of $p$ whose parts are coprime to $p$, i.e. $|\operatorname{CL}_p(\mathfrak{S}_n)|$.

On the other hand, if for every $j \geq 1$ we divide $1 - x^{jp}$ by $1 - x^j$, we are left with

$$P(x) = \prod_{j\geq 1}(1 + x^j + \cdots + x^{(p-1)j}).$$

Then the coefficient of $x^n$ is the number of expansions $n = a_1 1 + \cdots + a_n n$ where $a_i < p$. This equals $|\mathcal{P}_{n,p}|$.

We conclude from these two interpretations of the coefficient of the $n$th degree term of $P(x)$ that we indeed have $|\operatorname{CL}_p(\mathfrak{S}_n)| = |\mathcal{P}_{n,p}|$, as desired. □

We now state a corollary of theorem 1.1 and proposition 1.2.

**Corollary 1.9.** *Let $F$ be an algebraically closed field of characteristic $p \geq 0$.*

*(a) If $p \nmid n!$ then the number of isomorphism classes of simple $F\mathfrak{S}_n$-modules is $|\mathcal{P}_n|$.*

*(b) If $p \mid n!$ then the number of isomorphism classes of simple $F\mathfrak{S}_n$-modules is $|\mathcal{P}_{n,p}|$.*

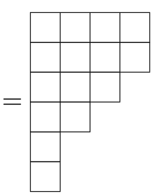
# Chapter 2

# Young Diagrams, Tableaux, and Tabloids

Now we will begin discussion of some combinatorial topics that will later aid in our study of representations of the symmetric group. We will discuss Young diagrams, Young tableaux, and Young tabloids, all of which are named after Alfred Young. Here, we take $n \in \mathbb{N}$.

**Definition 2.1.** Let $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathcal{P}_n$.

(a) The Young Diagram $[\lambda]$ is $[\lambda] := \{(i,j) \in \mathbb{N} \times \mathbb{N} : 1 \leq i \leq k, j = 1, \dots, \lambda_i\}$. If $(i,j) \in [\lambda]$ then $(i,j)$ is called a *node* (or *box*) of $[\lambda]$. For instance, if $\lambda = (4^2, 3, 2, 1^2)$, we would write

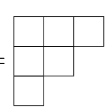
$[\lambda] =$ 

(b) A (Young) tableau of type $\lambda$ (a $\lambda$-*tableau*) is a bijection $t : [\lambda] \to \{1, \dots, n\}$. We write

$$t = \begin{array}{lllll} t(1,1) & \cdots & \cdots & \cdots & t(1,\lambda_1) \\ t(2,1) & \cdots & \cdots & t(1,\lambda_2) & \\ \vdots & \vdots & \iddots & & \\ t(k,1) & \cdots & t(k,\lambda_k) & & \end{array} .$$

(c) A Young tableau is called *standard* if the entries increase along rows, and down columns. The number of all standard $\lambda$-tableaux is denoted $f^\lambda$.

**Example 2.2.** Take $n = 6$ and $\lambda = (3,2,1)$. Then

$[\lambda] =$ 

And here are some examples of $(3,2,1)$-tableaux:

$$t_1 = \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & \\ 6 & & \end{array}, \quad t_2 = \begin{array}{ccc} 5 & 2 & 3 \\ 1 & 4 & \\ 6 & & \end{array}.$$

Here, $t_1$ is a standard tableau and $t_2$ is a non-standard tableau.

**Exercise.** Here is an exercise for if you are (extremely) bored. Write down all $(3,2,1)$-tableaux. How many are there? How many $\lambda$-tableaux are there for $\lambda = (n)$? $\lambda = (1^n)$?

(d) The symmetric group $\mathfrak{S}_n$ acts on the set of all $\lambda$-tableaux in the most natural way possible—entry-wise. Take $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathcal{P}_n$, and take a $\lambda$-tableaux

$$t = \begin{matrix} x_{11} & \cdots & \cdots & x_{1,\lambda_1} \\ \vdots & \vdots & \iddots & \\ x_{k,1} & \cdots & x_{k,\lambda_k} & \end{matrix}.$$

Then for $\sigma \in \mathfrak{S}_n$, we define the action of $\sigma$ on $t$ (we act on the left) to be:

$$\sigma \cdot t = \begin{matrix} \sigma(x_{11}) & \cdots & \cdots & \sigma(x_{1,\lambda_1}) \\ \vdots & \vdots & \iddots & \\ \sigma(x_{k,1}) & \cdots & \sigma(x_{k,\lambda_k}) & \end{matrix}.$$

**Definition 2.3.** Let $\lambda \in \mathcal{P}_n$, and let $t$ be a $\lambda$-tableau. The *row stabilizer* of $t$ is

$$R_t := \{\sigma \in \mathfrak{S}_n : \sigma \text{ fixes every row of } t\}.$$

The *column stabilizer* of $t$ is

$$C_t := \{\sigma \in \mathfrak{S}_n : \sigma \text{ fixes every column of } t\}.$$

**Remark 2.4.** Both $R_t$ and $C_t$ are subgroups of $\mathfrak{S}_n$.

**Example 2.5.** Let $\lambda = (2,3) \in \mathcal{P}_5$ and let $t = \begin{matrix} 2 & 3 & 5 \\ 1 & 4 & \end{matrix}$. Then $R_t = \mathfrak{S}(\{2,3,5\}) \times \mathfrak{S}(\{1,4\})$ and $C_t = \mathfrak{S}(\{2,1\}) \times \mathfrak{S}(\{3,4\})$.

**Definition 2.6.** Let $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathcal{P}_n$ . The corresponding *standard Young subgroup* of $\mathfrak{S}_n$ is defined as

$$\mathfrak{S}_\lambda := \mathfrak{S}(\{1,\ldots,\lambda_n\}) \times \mathfrak{S}(\{\lambda_1+1,\ldots,\lambda_1+\lambda_2\}) \times \cdots \times \mathfrak{S}(\{\lambda_1+\cdots+\lambda_{k-1}+1,\ldots,\lambda_1+\cdots+\lambda_{k-1}+\lambda_k\})$$

A *Young subgroup* of type $\lambda$ is a subgroup of $\mathfrak{S}_n$ conjugate to $\mathfrak{S}_\lambda$.

**Remark 2.7.** Let $\lambda \in \mathcal{P}_n$ .

(a) A subgroup $Y \leq \mathfrak{S}_n$ is a Young subgroup if and only if there are, for some $k \in \mathbb{N}$, pairwise disjoint subsets $A_1, \ldots, A_k$ of $\{1, \ldots, n\}$ such that $A_i \neq \varnothing$, $\sqcup_{i=1}^k A_i = \{1, \ldots, n\}$ and $Y = \mathfrak{S}(A_n) \times \cdots \times \mathfrak{S}(A_k)$.

(b) Denote by $\lambda'$ the partition of $n$ whose Young diagram is obtained by transposing $[\lambda]$. We call $\lambda'$ the *conjugate partition of* $\lambda$. That is, for every $\lambda$-tableau $r$, $R_t$ is a Young subgroup of type $\lambda$, and $C_t$ is a Young subgroup of type $\lambda'$.

**Lemma 2.8.** *Let $\lambda \in \mathcal{P}_n$, $t$ a $\lambda$-tableau. Then $\pi R_t \pi^{-1} = R_{\pi t}$ and $\pi C_t \pi^{-1} = C_{\pi t}$ for all $\pi \in \mathfrak{S}_n$.*

*Proof.* This is just an easy verification. Denote the rows of $t$ by $A_1, \ldots, A_k$ and let $\sigma \in \mathfrak{S}_n$. Then

$$\begin{aligned} \sigma \in R_t &\iff \sigma(A_i) = A_i, i = 1, \ldots, k \\ &\iff [\pi\sigma\pi^{-1}](\pi(A_i)) = \pi(A_i), i = 1, \ldots, k \\ &\iff \pi\sigma\pi^{-1} \in R_{\pi t}. \end{aligned}$$

So $\pi\sigma\pi^{-1} = R_{\pi t}$. Similarly, $\pi C_t \pi^{-1} = C_{\pi t}$. $\square$

**Remark 2.9.** Let $\lambda \in \mathcal{P}_n$. Define a relation "$\sim$" on the set of tableaux:

$$t \sim \bar{t} \iff \text{there exists } \sigma \in R_t \text{ such that } \sigma \cdot t = \bar{t}.$$

This defines an equivalence relation: It is certainly reflexive. To check it is symmetric, see that $t \sim \bar{t}$ means $\sigma t = \bar{t}$ and so $t = \sigma^{-1}\bar{t}$, and hence $\sigma^{-1} = \sigma\sigma^{-1}\sigma^{-1} \in \sigma R_t \sigma^{-1} = R_{\sigma t} = R_{\bar{t}}$. The check for transitivity is left as an exercise.

**Definition 2.10.** Let $\lambda \in \mathcal{P}_n$ and let $t$ be a $\lambda$-tableau. We define $\{t\}$ as the equivalence class of $t$ under "$\sim$." We call $\{t\}$ a $\lambda$-*tabloid.* We write $\{t\}$ as $t$ with lines between rows. We call $\{t\}$ a *standard* $\lambda$-tabloid if it contains a standard $\lambda$-tableau.

**Example 2.11.** Consider $\lambda = (3,2) \in \mathcal{P}_5$ .

(a) If $t = \begin{matrix} 2 & 3 & 5 \\ 1 & 4 & \end{matrix}$ , then $\{t\} = \dfrac{\overline{\;2\;\;3\;\;5\;}}{\underline{\;1\;\;4\;\;\;\;\;}} = \dfrac{\overline{\;3\;\;2\;\;5\;}}{\underline{\;4\;\;1\;\;\;\;\;}}$. Here, $\{t\}$ is not a standard $\lambda$-tabloid.

(b) If $t = \begin{matrix} 3 & 1 & 5 \\ 4 & 2 & \end{matrix}$ , then $\{t\} = \dfrac{\overline{\;3\;\;1\;\;5\;}}{\underline{\;4\;\;2\;\;\;\;\;}} = \dfrac{\overline{\;1\;\;3\;\;5\;}}{\underline{\;2\;\;4\;\;\;\;\;}}$. Here, $\{t\}$ *is* a standard $\lambda$-tabloid.

**Definition 2.12.** Let $\lambda, \mu \in \mathcal{P}_n$ .

(a) We say $\lambda$ dominates $\mu$, and write $\mu \trianglelefteq \lambda$ if $\sum_{i=1}^{j} \mu_j \leq \sum_{i=1}^{j} \lambda_i$ for all $j = 1, 2, \ldots$. If $\mu \trianglelefteq \lambda$ and $\mu \neq \lambda$, then we write $\mu \triangleleft \lambda$.

(b) We write $\mu < \lambda$ if, for the smallest $i \in \mathbb{N}$ with $\lambda_i \neq \mu$, we have $\mu_i < \lambda$. If $\mu < \lambda$ or $\mu = \lambda$ then we write $\mu \leq \lambda$.

**Remark 2.13.** (a) The dominance order "$\trianglelefteq$" is a partial order on $\mathcal{P}_n$. The lexicographic order "$\leq$" is a total order on $\mathcal{P}_n$. Moreover "$\leq$" contains "$\trianglelefteq$" as relations. (That is to say $\trianglelefteq \Rightarrow \leq$.) The converse is not true. For example, $(3^2) < (4, 1^2)$, but $(3^2)$ and $(4, 1^2)$ are not comparable by "$\trianglelefteq$."

(b) For $\lambda, \mu \in \mathcal{P}$, we have $\mu \trianglelefteq \lambda \iff \lambda' \trianglelefteq \mu'$.

**Lemma 2.14** (Basic Combinatorial Lemma (BCL))**.** *Let $\lambda, \mu \in \mathcal{P}_n$, $t_1$ a $\lambda$-tableau, and $t_2$ a $\mu$-tableau. Suppose that for every $i \in \mathbb{N}$, the entries in the ith row of $t_2$ belong to mutually different columns of $t_1$. Then $\mu \trianglelefteq \lambda$. If $\lambda = \mu$ then there exists $\sigma \in R_{t_2}$, $\pi \in C_{t_1}$ such that $\sigma t_2 = \pi t_1$.*

We first give an example of this lemma.

**Example 2.15.** (a) Take $\lambda = (4,3,1), \mu = (3, 2^2, 1)$. Let $t_1$ be a $\lambda$-tableau and $t_2$ be a $\mu$-tableau as below:

$$t_1 = \begin{matrix} 1 & 5 & 3 & 6 \\ 4 & 2 & 7 & \\ 8 & & & \end{matrix} \quad , \quad t_2 = \begin{matrix} 1 & 2 & 3 \\ 4 & 5 & \\ 6 & 7 & \\ 8 & & \end{matrix} \quad .$$

Then by the BCL, $\mu \triangleleft \lambda$.

(b) Take $\lambda = \mu = (3, 2^2, 1)$. And consider the following $\lambda$-tableaux $t_1$ and $t_2$:

$$t_1 = \begin{matrix} 1 & 2 & 3 \\ 4 & 5 & \\ 6 & 7 & \\ 8 & & \end{matrix} \quad , \quad t_2 = \begin{matrix} 6 & 3 & 2 \\ 1 & 4 & \\ 8 & 7 & \\ 5 & & \end{matrix} \quad .$$

Then we can take $\sigma = (23)(45) \in R_{t_2}, \pi \in C_{t_1}$ such that $\sigma t_2 = \pi t_1$.

Now we prove the BCL.

*Proof of BCL.* Place the $\mu_1$ entries from the first row of $t_2$ into different columns of $[\lambda]$. So $[\lambda]$ must have at least $\mu_1$ columns, i.e. $\mu_1 \leq \lambda_1$. Next, we put the $\mu_2$ entries from the second row of $t_2$ into different columns of $[\lambda]$ with the first $\mu_1$ numbers already filled in. Every column of the resulting diagram has at most 2 entries. This forces $\mu_1 + \mu_2 \leq \lambda_1 + \lambda_2$. Inductively, we get $\mu \trianglelefteq \lambda$.

If $\lambda = \mu$, then we choose $\sigma \in R_{t_2}$ to rearrange the entries in $t_2$ such that $x$ is in column $i$ of $\sigma t_2$ if and only if $x$ is in column $i$ of $t_1$. Then we choose $\pi \in \sigma_{t,n}$ such that $y$ belongs to the $j$th row of $\pi t_1$ if and only if $y$ is in the $j$th row of $\sigma t_2$. □

This is actually a really obvious statement. There's really nothing to the proof at all. It basically says: Do what you can. QED.

# Chapter 3

# Permutation Modules and Specht Modules

We now discuss permutation modules and define Specht modules. Let $F$ be any field and let $G$ be a finite group.

**Remark 3.1.** (a) Let $\Omega$ be a finite $G$-set (so $G$ acts from the left). The corresponding $FG$-module $F\Omega$ is the $F$-vector space with basis $\Omega$. We call $\Omega$ the "natural" $F$-basis of $F\Omega$.

(b) If $\Omega$ and $\Omega'$ are $G$-sets, then $\Omega \times \Omega'$ is also, with diagonal action

(c) For any $FG$-modules $M$ and $N$, we have $\mathrm{Hom}_{FG}(M, N)$, the $F$-space of $F$-linear maps $M \to N$ commuting with the $FG$-action.

**Lemma 3.2.** *Assume $\Omega$ and $\Omega'$ are finite $G$-sets. Let $\varphi : F\Omega \to F\Omega'$ be an $F$-linear map with corresponding matrix $(a_{\omega',\omega})_{\omega'\in\Omega',\omega\in\Omega}$. Then*

*(a) $\varphi \in \mathrm{Hom}_{FG}(F\Omega, F\Omega')$ if and only if for all $g \in G, \omega \in \Omega, \omega' \in \Omega'$, we have $a_{\omega',\omega} = a_{g\omega',g\omega}$.*

*(b) $\mathrm{Hom}_{FG}(F\Omega, F\Omega')$ has an $F$-basis labelled by the $G$-orbits on $\Omega' \times \Omega$. That is, if $\theta$ is such a $G$-orbit, then $b_\theta = (a_{\omega',\omega})$ with*

$$a_{\omega',\omega} = \begin{cases} 1, & (\omega', \omega) \in \theta \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* Let $g \in G$, and let $\omega \in \Omega$. On one hand, we have

$$\varphi(g \cdot \omega) = \sum_{\omega'\in\Omega'} a_{\omega',g\omega}\omega',$$

and on the other hand, we have

$$g\varphi(\omega) = g \sum_{\omega'\in\Omega'} a_{\omega',\omega}\omega' = \sum_{\omega\in\Omega'} a_{\omega',\omega}g\omega' = \sum_{\omega'\in\Omega'} a_{g^{-1}\omega',\omega}\omega'.$$

Hence we have $\varphi(g\omega) = g\varphi(\omega)$ if and only if $a_{\omega',g\omega} = a_{g^{-1}\omega',\omega}$ for all $\omega' \in \Omega'$, $\omega \in \Omega$. □

**Example 3.3.** If $\Omega = \Omega'$ is the set of 2-element subsets of $\{1, \dots, n\}$, $G = \mathfrak{S}_n$, $n \geq 4$, then there are 3 orbits on $\Omega \times \Omega'$:

$$\theta_1 := \{(\{a, b\}, \{c, d\}) : \{a, b\} = \{c, d\}\},$$
$$\theta_2 := \{(\{a, b\}, \{c, d\}) : |\{a, b\} \cap \{c, d\}| = 1\},$$
$$\theta_3 := \{(\{a, b\}, \{c, d\}) : |\{a, b\} \cap \{c, d\}| = 0\}.$$

Now we will look at Young permutation modules.

Let $F$ be a field, $\lambda \in \mathcal{P}_n$, and let $\Omega^\lambda$ be the set of $\lambda$-tabloids. For $\pi \in \mathfrak{S}_n$ and $\{t\} \in \Omega^\lambda$, we define $\pi\{t\} := \{\pi t\}$. The next lemma shows that this is an $\mathfrak{S}_n$-action, and thus turns $\Omega^\lambda$ into an $\mathfrak{S}_n$-set.

**Lemma 3.4.** *Let $\lambda \in \mathcal{P}_n$, $t_1$ and $t_2$ $\lambda$-tableaux, $\pi \in \mathfrak{S}_n$. If $\{t_1\} = \{t_2\}$ then $\pi\{t_1\} = \pi\{t_2\}$.*

*Proof.* If $\{t_1\} = \{t_2\}$ then for some $\sigma \in R_t$ we have $t_2 = \sigma t_1$. Thus

$$\pi t_2 = \pi\sigma\pi^{-1}\pi t_1 \in \pi R_{t_1}\pi^{-1} = R_{\pi t_1}.$$

We conclude that $\{\pi t_2\} = \{\pi t_1\}$. □

**Definition 3.5.** Let $\lambda \in \mathcal{P}_n$. Then the permutation $F\mathfrak{S}_n$-module $F\Omega^\lambda$ is called a *Young permutation module* and it is denoted by $M^\lambda$.

> Just clarifying: After picking $\lambda \in \mathcal{P}_n$, we have an action of $\mathfrak{S}_n$ on $\Omega^\lambda$, the set of $\lambda$-tabloids. Pairing this action with the vector space attached to the permutation $F\mathfrak{S}_n$-module $F\Omega$ gives us the Young permutation module $M^\lambda$.

**Proposition 3.6.** *Let $\lambda \in \mathcal{P}_n$, $\lambda = (\lambda_1, \ldots, \lambda_k)$. Then $M^\lambda$ is a cyclic $F\mathfrak{S}_n$-module, generated by any tabloid $\{t\} \in \Omega^\lambda$. Moreover* $\dim_F(M^\lambda) = \frac{n!}{\lambda_1! \cdots \lambda_k!}$.

*Proof.* Problem Sheet 1. □

**Example 3.7.** (a) If $\lambda = (n)$, then $M^\lambda \cong F$, the trivial $F\mathfrak{S}_n$-module.

(b) If $\lambda = (1^n)$, then $M^\lambda \cong F\mathfrak{S}_n$, the regular $F\mathfrak{S}_n$-module.

(c) $M^{(n-1,1)}$ is isomorphic to the natural permutation $F\mathfrak{S}_n$-module.

(d) For $0 < k < n$, $M^{(n-k,k)} \cong F\Omega^{\{k\}}$, where $\Omega^{\{k\}}$ is the set of all $k$-element subsets of $\{1, \ldots, n\}$.

**Definition 3.8.** Let $\lambda \in \mathcal{P}_n$, and let $t$ be a $\lambda$-tableau with column stabilizer $C_t$. Set

$$\kappa_t := \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\pi \in F\mathfrak{S}_n$$
$$e_t := \kappa_t \cdot \{t\} \in M^\lambda.$$

We call $e_t$ a *$\lambda$-polytabloid.* If $\lambda$ is a standard tableau, then we call $e_t$ a *standard $\lambda$-polytabloid.*

**Remark 3.9.** Note that the polytabloid $e_t$ depends on the choice of $\lambda$-tableau $t$ and not only on the tabloid $\{t\}$. To illustrate this, take, for instance, $\lambda = (2,1)$. Let $t = \begin{array}{cc} 1 & 2 \\ 3 & \end{array}$. Then we have $C_t = \mathfrak{S}(\{1,3\})$ and so

$$e_t = \begin{array}{cc} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} - \begin{array}{cc} \hline 3 & 2 \\ \hline 1 & \\ \hline \end{array}.$$

Now take $t' = \begin{array}{cc} 2 & 1 \\ 3 & \end{array}$. We have $C_t = \mathfrak{S}(\{2,3\})$ and hence

$$e_{t'} = \begin{array}{cc} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} - \begin{array}{cc} \hline 3 & 1 \\ \hline 2 & \\ \hline \end{array}.$$

In particular, we have $e_t \neq e_{t'}$ in spite of the fact that $\{t\} = \{t_1\}$.

**Lemma 3.10.** *Let $\lambda \in \mathcal{P}_n$ and let $t$ be a $\lambda$-tableau with columns $C_1, \dots, C_s$. Then*

$$\kappa_t = \kappa_{C_1} \cdots \kappa_{C_s} \text{ where } \kappa_{C_i} := \sum_{\pi \in \mathfrak{S}(C_i)} \operatorname{sgn}(\pi)\pi, i = 1, \dots, s.$$

*Proof.* We know that $C_1 = \mathfrak{S}(C_1) \times \cdots \times \mathfrak{S}(C_s)$. So arguing with induction on $s$, it is sufficient to show that for any two subgroups $H, K \leq \mathfrak{S}_n$ with disjoint support, we have

$$\sum_{\sigma \in HK} \operatorname{sgn}(\sigma)\sigma = \sum_{\pi \in H} \operatorname{sgn}(\pi)\pi \cdot \sum_{\rho \in K} \operatorname{sgn}(\rho)\rho.$$

This is true since every $\sigma \in HK$ has a unique factorization $\sigma = \pi\rho = \rho\pi$ for some $\pi \in H, \rho \in K$. Also, $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\pi)\operatorname{sgn}(\rho)$. The result follows. □

**Lemma 3.11.** *Let $\lambda \in \mathcal{P}_n$ and let $t$ be a $\lambda$-tableau. Then*

*(a)* $\kappa_{\pi t} = \pi \kappa_t \pi^{-1}$ *for* $\pi \in \mathfrak{S}_n$.

*(b)* $\kappa_t \cdot \pi = \pi \cdot \kappa_t = \operatorname{sgn}(\pi) \cdot \kappa_t$ *for* $\pi \in C_t$.

*(c)* $e_{\pi t} = \pi e_t$ *for* $\pi \in \mathfrak{S}_n$.

*Proof.* For a, we already know $C_{\pi t} = \pi C_t \pi^{-1}$ for all $\pi \in \mathfrak{S}_n$. So

$$\kappa_{\pi t} = \sum_{\sigma \in C_{\pi t}} \operatorname{sgn}(\sigma)\sigma = \sum_{\sigma \in C_t} \operatorname{sgn}(\pi\sigma\pi^{-1})\pi\sigma\pi^{-1} = \pi \left( \sum_{\sigma \in C_t} \operatorname{sgn}(\sigma)\sigma \right) \pi^{-1} = \pi\kappa_t\pi^{-1}.$$

For b, if $\pi \in C_t$ then

$$\pi\kappa_1 = \pi \sum_{\sigma \in C_t} \operatorname{sgn}(\sigma)\sigma = \sum_{\sigma \in C_t} \operatorname{sgn}(\pi)\operatorname{sgn}(\pi\sigma)\pi\sigma = \operatorname{sgn}(\pi)\kappa_t,$$

and similarly, we obtain that $\kappa_t\pi = \operatorname{sgn}(\pi)\kappa_1$, and this proves b.

For c, we use a, and so for all $\pi \in \mathfrak{S}_n$, we have

$$\pi e_t = \pi\kappa_t\{t\} = \kappa_{\pi t}\pi\{t\} = \kappa_{\pi t}\{\pi t\} = e_{\pi t}.$$

This completes the proof. □

**Definition 3.12.** Let $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathcal{P}_n$ and let $t$ be the following $\lambda$-tableau:

$$t = \begin{array}{llllll} 1 & 2 & \cdots & \cdots & \cdots & \lambda_1 \\ \lambda_1 + 1 & \lambda_1 + 2 & \cdots & \cdots & \lambda_1 + \lambda_2 & \\ \vdots & \vdots & & & \iddots & \\ \lambda_1 + \cdots + \lambda_{k-1} + 1 & \lambda_1 + \cdots + \lambda_{k-1} + 2 & \cdots & \lambda_1 + \cdots + \lambda_k & & \end{array}.$$

The *Specht* $F\mathfrak{S}_n$*-module* is the submodule of $M^\lambda$ generated by $e_t$. We denote it by $S^\lambda := S^\lambda_F$.

This is all fine and dandy, but I still have no idea what this means! We have a Young permutation module, and then we have $e_t = \kappa_t \cdot \{t\} = \left(\sum_{\pi \in C_t} \operatorname{sgn}(\pi)\pi\right) \cdot \{t\}$ generating a submodule of the Young permutation module. Oh, and $C_t$ is the column stabilizer.

**Remark 3.13.** If $s$ is any $\lambda$-tableau then $s = \pi t$ for some $\pi \in \mathfrak{S}_n$. Thus $\pi e_t = e_s$. So $S^\lambda = F\mathfrak{S}_n \cdot e_t = F\mathfrak{S}_n \cdot e_s$. Hence $S^\lambda$ is a cyclic $F\mathfrak{S}_n$-module generated by any $\lambda$-polytabloid.

The $\lambda$-polytabloids are a spanning set for $S^\lambda$. We will see later that the standard $\lambda$-polytabloids form an $F$-basis of $S^\lambda$.

Note that thus far, we have made no assumptions about the field $F$. That is, our discussion is invariant under the choice of $F$. However, the structure of the module $S^\lambda$ is extremely dependent on the choice of $F$.

We continue from the introduction of Specht modules from last lecture by giving some examples and then beginning our discussion of properties of Specht modules. We take $\lambda \in \mathcal{P}_n$, and $t$ a $\lambda$-tableau. Let $S^\lambda := F\mathfrak{S}_n \cdot e_t \subseteq M^\lambda$.

**Example 3.14.** (a) If $\lambda = (n)$, then $S^\lambda = M^\lambda \cong F$.

(b) If $\lambda = (1^n)$, then $t = \begin{smallmatrix} 1 \\ \vdots \\ n \end{smallmatrix}$, the unique standard $\lambda$-tableau, and $C_t = \mathfrak{S}_n, \kappa_t = \sum_{\pi \in \mathfrak{S}_n} \mathrm{sgn}(\pi)\pi$. If $\sigma \in \mathfrak{S}_n$, then $\sigma e_t = \sigma \kappa_t \{t\} = \mathrm{sgn}(\sigma) e_t$. So $S^\lambda$ is a one-dimensional $F\mathfrak{S}_n$-module, with $\mathfrak{S}_n$ acting by multiplication with sgn.

(c) The characterization for $S^\lambda$ where $\lambda = (n-1, 1)$ is on Sheet 2.

# Chapter 4

# Properties of Specht Modules

Now we proceed to talk about properties of Specht modules. We take $n \in \mathbb{N}$ and $F$ to be some field.

**Theorem 4.1.** *Let $\lambda, \mu \in \mathcal{P}_n$. Let $t$ be a $\lambda$-tableau.*

*(a) If $\kappa_t \cdot M^\mu \neq 0$ then $\mu \trianglelefteq \lambda$.*

*(b) As $F$-vector spaces, $\kappa_t \cdot M^\lambda = Fe_t$.*

*Proof.* If $\kappa_t \cdot M^\lambda \neq 0$ then for some $\mu$-tabloid $\{\bar{t}\}$, we have $\kappa_t \cdot \{\bar{t}\} \neq 0$. We proceed by proving the following claim that will allow us to apply the Basic Combinatorial Lemma (BCL).

*Claim.* For every $i \in \mathbb{N}$, the numbers in row $i$ of $\bar{t}$ belong to different columns of $t$.

It is clear that this holds when $\lambda = (n)$ or $\mu = (1^n)$. Hence we may assume that this is not the case. Let $a, b$ be numbers in the same row of $\{\bar{t}\}$. Then

$$(1 - (ab))\{\bar{t}\} = \{\bar{t}\} - (ab)\{\bar{t}\} = 0. \tag{4.1}$$

Suppose (for a contradiction) that $a, b$ are in the same column of $\{\bar{t}\}$ so that $(a, b) \in C_t$. Let $\{\pi_1, \ldots, \pi_n\}$ be a transversal for the left cosets $C_t/\langle(a, b)\rangle$. So $C_t = \sqcup_{i=1}^m \pi \cdot \langle(a, b)\rangle$ and

$$\begin{aligned}
\kappa_t &= \sum_{i=1}^m (\operatorname{sgn}(\pi_i)\pi_i + \operatorname{sgn}(\pi_i(ab))\pi_i(ab)) \\
&= \sum_{i=1}^m \operatorname{sgn}(\pi_i)\pi_i - \operatorname{sgn}(\pi_i)\pi_i(ab) \\
&= \sum_{i=1}^m \operatorname{sgn}(\pi_i)\pi_i(1 - (ab)) =: x \in F\mathfrak{S}_n.
\end{aligned}$$

By (4.1), this implies $\kappa_t\{\bar{t}\} = x(1 - (ab))\{\bar{t}\} = 0$, which is a contradiction. This proves the claim, and applying the BCL, we have $\mu \trianglelefteq \lambda$, proving a.

For b, assume $\lambda = \mu$. By the proof of a, we can apply the BCL. So there are some $\sigma \in R_t, \pi \in C_t$ such that $\sigma\bar{t} = \pi t$. Thus $\{\bar{t}\} = \pi\{t\}$, and $\kappa_t\{\bar{t}\} = \kappa_t\{\pi t\} = \operatorname{sgn}(\pi)\kappa_t \cdot \{t\} = \operatorname{sgn}(\pi)e_t$. Hence $\kappa_t \cdot M^\lambda \subseteq Fe_t$, and since $Fe_t$ is a one-dimensional $F$-vector space and $\kappa_t \cdot M^\lambda \neq 0$, then we must have equality. □

**Corollary 4.2.** *Let $\lambda, \mu \in \mathcal{P}_n$ . Let $\varphi \in \operatorname{Hom}_{F\mathfrak{S}_n}(M^\lambda, M^\mu)\backslash\{0\}$.*

*(a) If $S^\lambda \nsubseteq \ker(\varphi)$, then $\mu \trianglelefteq \lambda$.*

*(b) If $\lambda = \mu$, then $\varphi\big|_{S^\lambda}$ is multiplication by some $\alpha \in F$.*

*Proof.* For a, assume that $S^\lambda \not\subseteq \ker(\varphi)$ and let $u \in S^\lambda$ be such that $\varphi(u) \neq 0$. We can write $u = \sum_t \alpha_t e_t$ where $t$ varies over all $\lambda$-tableaux and $\alpha_t \in F$. Since

$$\varphi(e_t) = \varphi(\kappa_t \cdot \{t\}) = \kappa_t \varphi(\{t\}) \in \kappa_t \cdot M^\mu,$$

we get $0 \neq \varphi(u) = \sum_t \alpha_t \varphi(e_t) \in \sum_t \kappa_t M^\mu$. Hence there is some $\lambda$-tableau $t$ with $\kappa_t M^\mu \neq 0$. By the previous theorem, we have $\mu \trianglelefteq \lambda$.

For b, let $\lambda = \mu$. If $t$ is a $\lambda$-tableau, then by b of the previous theorem, we have $\kappa_t M^\lambda = Fe_1$. Since $\varphi(e_1) \in \kappa_t M^\lambda$, there is some $\alpha \in F_n$ with $\varphi(e_t) = \alpha e_t$. We know $S^\lambda = F\mathfrak{S}_n \cdot e_t$, hence we can conclude that $\varphi(u) = \alpha u$ for all $u \in S^\lambda$, as desired. □

**Theorem 4.3** (Maschke)**.** *Let $G$ be a finite group such that* $\mathrm{char}(F) \nmid |G|$. *Then every $FG$-module $M$ is semisimple. (i.e. If $M_1$ is a submodule of $M$ then there is a submodule $M_2$ of $M$ such that $M = M_1 \oplus M_2$.)*

**Theorem 4.4.** *Suppose* $\mathrm{char}(F) \nmid |\mathfrak{S}_n| = n!$ *and let $\lambda, \mu \in \mathcal{P}_n$.*

*(a)* $S^\lambda \cong S^\mu \iff \lambda = \mu$.

*(b)* $S^\lambda$ *is a simple module.*

*(c)* $\dim(\mathrm{Hom}_{F\mathfrak{S}_n}(S^\lambda, M^\lambda)) = 1 = \dim(\mathrm{End}_{F\mathfrak{S}_n}(S^\lambda))$.

*Proof.* For a, the ($\Leftarrow$) direction is clear. For ($\Rightarrow$), suppose $\varphi : S^\lambda \to S^\mu$ is an $F\mathfrak{S}_n$-isomorphism. By 4.3, there is a submodule $T^\lambda$ of $M^\lambda$ such that $M^\lambda = S^\lambda \oplus T^\lambda$. (Call this equation $(*)$.) Let $\pi_\lambda : M^\lambda \twoheadrightarrow S^\lambda$ be the canonical projection and let $\iota_\mu : S^\mu \hookrightarrow M^\mu$ be the inclusion map. Then $\psi := \iota)\mu \circ \varphi \circ \pi_\lambda : M^\lambda \to M^\mu$ is in $\mathrm{Hom}_{F\mathfrak{S}_n}(M^\lambda, M^\mu)$ with $S^\lambda \not\subseteq \ker(\psi)$. Thus $\mu \trianglelefteq \lambda$, by the corollary. Similarly, $\lambda \trianglelefteq \mu$, so indeed $\lambda = \mu$.

For b, assume that $S^\lambda$ is not simple. Then there are proper non-zero submodules $U_1, U_2$ of $S^\lambda$ such that $S^\lambda = U_1 \oplus U_2$. As above, we have $M^\lambda = U_1 \oplus U_2 \oplus T^\lambda$. We define an $F\mathfrak{S}_n$-homomorphism $\varphi : M^\lambda \to M^\lambda, u \mapsto \iota_1(\pi_1(u))$ where $\iota_1 : U_1 \hookrightarrow M^\lambda$ and $\pi_1 : M^\lambda \twoheadrightarrow U_1$ are the canonical injection and projection maps, respectively.

For c, let $\varphi \in \mathrm{Hom}_{F\mathfrak{S}_n}(S^\lambda, M^\lambda)\backslash\{0\}$. With the notation as in a, $\varphi$ extends to a homomorphism $\hat{\varphi} \in \mathrm{Hom}_{F\mathfrak{S}_n}(M^\lambda, M^\lambda)\backslash\{0\}$: $\hat{\varphi} = \varphi \circ \pi_\lambda$. Moreover $\hat{\varphi}\big|_{S^\lambda} = \varphi$. Thus by b of the corollary, there exists some $\alpha \in F$ such that $\varphi = \alpha - \iota_\lambda$, where $\iota_\lambda$ is the inclusion map $S^\lambda \hookrightarrow M^\lambda$. So

$$\dim(\mathrm{Hom}_{F\mathfrak{S}_n}(S^\lambda, M^\lambda)) = 1 = \dim(\mathrm{End}_{F\mathfrak{S}_n}(S^\lambda)).$$ □

**Corollary 4.5.** *Let $F$ be an (algebraically closed) field with* $\mathrm{char}(F) \nmid |\mathfrak{S}_n|$. *Then $\{S_F^\lambda : \lambda \in \mathcal{P}_n\}$ is a transversal for the isomorphism classes of simple $F\mathfrak{S}_n$-modules.*

*Proof.* This follows from the preceding theorem and the first corollary of Lecture 2. □

Note that the above statement is true without the hypothesis that $F$ is algebraically closed. However, the corollary used the fact that $F$ was algebraically closed, which is why the corollary is stated as it is.

**Remark 4.6.** The assumption on $\mathrm{char}(F)$ is essential in the last theorem here as well as in the corollary above. If $\mathrm{char}(F) \,\big|\, |\mathfrak{S}_n|$ then $S^\lambda$ is, in general, no longer simple. If $\mathrm{char}(F) = 2$, then $S^\lambda$ can even be decomposable. Moreover, in $\mathrm{char}(F) = 2$, then it does happen that $S^\mu \cong S^\lambda$ for $\lambda \neq \mu$. (This last phenomenon only happens when $\mathrm{char}(F) = 2$.)

**Theorem 4.7** (Wedderburn)**.** *Let $G$ be a finite group, and let $F$ be a group of* $\mathrm{char}(F) \nmid |G|$. *Let $\{s_1, \ldots, s_n\}$ be a transversal for the isomorphism classes of simple $FG$-modules. Then*

$$|G| = \sum_{i=1}^{n} (\dim(s_i))^2.$$

**Definition 4.8.** Let $\lambda \in \mathcal{P}_n$, and let $\{t_1\}$ and $\{t_2\}$ be $\lambda$-tabloids. We write $\{t_1\} < \{t_2\}$ if there is some $i \in \mathbb{N}$ such that

- for every $i < j$, $j$ belongs to the same row of $\{t_1\}$ and $\{t_2\}$
- $i$ occurs in $\{t_1\}$ in a higher row than it does in $\{t_2\}$.

**Example 4.9.** Let $\lambda = (3, 2)$. Then we have

$$\begin{array}{ccc} \hline 3 & 4 & 5 \\ \hline 1 & 2 & \\ \hline \end{array} < \begin{array}{ccc} \hline 2 & 4 & 5 \\ \hline 1 & 3 & \\ \hline \end{array} < \begin{array}{ccc} \hline 1 & 4 & 5 \\ \hline 2 & 3 & \\ \hline \end{array}$$

To see the first inequality, set $i = 3$. To see the second inequality, set $i = 2$.

**Remark 4.10.**
- This gives a total order on $\Omega^\lambda$, the set of $\lambda$-tabloids.
- Whenever $t$ is a $\lambda$-tableau such that the entries in $t$ increase down columns,

$$e_t = \{t\} \pm \{t_1\} \pm \cdots \pm \{t_m\},$$

for some $m \geq 1$ and $\{t_i\} < \{t\}$ for all $i = 1, \ldots, m$. Justification: $\{t_i\} = \pi_i\{t\}$ for some $\pi_i \in C_t$. Let $a \in \{1, \ldots, n\}$ be such that $\pi_i(b) = b$ for $b > a$ and $\pi(a) \neq a$. Then $a$ must in $\{t_i\}$ lie in a higher row than in $\{t\}$.

**Theorem 4.11.** *Let $\lambda \in \mathcal{P}_n$, then the set of all standard $\lambda$-polytabloids form an $F$-basis of $S^\lambda$. In particular,* $\dim(S^\lambda) = f^\lambda$.

**Remark 4.12.** We will only prove the assertion in the case where $F$ is algebraically closed and char $F \nmid |\mathfrak{S}_n|$, but this works in general.

*Proof.* We first prove linear independence. Suppose for a contradiction that $\sum \alpha_t e_t = 0$ (call this equation $(*)$), $\alpha_t \in F$, not all zero, where the sum runs over all standard $\lambda$-talbeaux $t$. Let $\{\bar{t}\}$ be maximal with respect to "<" such that $\bar{t}$ is standard and $\alpha_{\bar{t}} \neq 0$. By the above remark, the coefficient of $\{\bar{t}\}$ in $(*)$ is $\alpha_{\bar{t}}$. But the $\lambda$-tabloids are an $F$-basis of $M^\lambda$, and hence $\alpha_{\bar{t}} = 0$. This is a contradiction and hence we have linear independence.

Now we prove that this set is indeed a generating set. Since we assumed that $F$ is algebraically closed and $\mathrm{char}(F) \nmid |\mathfrak{S}_n|$, then Wedderburn's theorem together with the earlier statement (corollary) that the Specht modules form a transversal for the isomorphism classes of simple $F\mathfrak{S}_n$-modules, we have

$$n! = |\mathfrak{S}_n| = \sum_{\lambda \, \mathcal{P}_n} (\dim(S^\lambda))^2.$$

From the linear independence, we know that $\dim(S^\lambda) \geq f^\lambda$. What remains to be shown is that

$$n! = \sum_{\lambda \in \mathcal{P}_n} (f^\lambda)^2.$$

This will be done later as we now take a break from combinatorics to go into some more representation theory. □

**Remark 4.13.** One can show, using the "Garnis relations," that every $\lambda$-polytabloid is a $\mathbb{Z}$-linear combination of standard $\lambda$-polytabloids. With this we get directly that the latter are a basis of $S^\lambda$. (Reference: James' book, *The Representation Theory of Symmetric Groups.*)

**Theorem 4.14.** *Let $F$ be a field of* $\mathrm{char}(F) \nmid |\mathfrak{S}_n|$ *and let $\lambda \in \mathcal{P}_n$. Then*

$$M^\lambda \cong S^\lambda \oplus \bigoplus_{\lambda \lhd \mu} S^\mu m_{\lambda\mu}, \quad m_{\lambda\mu} \in \mathbb{N}_0$$

*Proof.* Since $\mathrm{char}(F) \nmid |\mathfrak{S}_n|$, $M^\lambda$ is semisimple by Maschke's theorem. So we can write

$$M^\lambda \cong \bigoplus_{\mu \in \mathcal{P}_n} m_{\lambda\mu} S^\mu \text{ for some } m_{\lambda\mu} \in \mathbb{N}_0.$$

If $m_{\lambda\mu} \neq 0$ then $\mathrm{Hom}_{F\mathfrak{S}_n}(S^\mu, M^\lambda) \neq 0$. Let $0 \neq \varphi \in \mathrm{Hom}_{F\mathfrak{S}_n}(S^\mu, M^\lambda)$. We write $M^\mu = S^\mu \oplus T^\mu$ for some $F\mathfrak{S}_n$-module $T^\mu$. Let $\phi_\mu : M^\mu \to S^\mu$ be the canonical projection. Then $\varphi \circ \pi_\mu : M^\mu \to M^\lambda$ is an $F\mathfrak{S}_n$-homomorphism by $S^\mu \not\subseteq \ker(\varphi \circ \pi_\mu)$. This forces $\lambda \trianglelefteq \mu$, by Corollary 4.2a. By the proelbm sheet, $m_{\lambda\mu} = \dim(\mathrm{Hom}(S^\mu, M^\lambda))$, so that $m_{\lambda\lambda} = 1$ by Theorem 4.4c. □

**Remark 4.15.**
- The multiplicities $m_{\lambda\mu}$ are called Kostka numbers (and they have an analogue in characteristic $p$).
- If $G \supseteq H$ are finite groups, then every $FG$-module $M$ can be viewed as an $FH$-module. We induce it by $\mathrm{Res}^G_H(M)$.

**Theorem 4.16** (Branching Rule)**.** *Let $F$ be any field, and let $\lambda \in \mathcal{P}_n$. Then $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(S^\lambda)$ has a filtration of $F\mathfrak{S}_{n-1}$-modules*

$$\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(S^\lambda) =: V_m \supset V_{m-1} \supset \cdots \supset V_1 \supset V_0 = \{0\},$$

*where $V_i/V_{i-1} \cong S^{\lambda(i)}$, and $\{\lambda(1), \dots, \lambda(m)\}$ is the set of partitions of $n-1$ whose Young diagrams are obtained by removing a node from $[\lambda]$, and $\lambda(j) \triangleleft \lambda(i)$ whenever $j < i$. In particular, if $\mathrm{char}(F) \nmid |\mathfrak{S}_n|$ then $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(S^\lambda) \cong \bigoplus_{i=1}^m S^{\lambda(i)}$.*

*Proof.* Let $r_1 < \cdots < r_m$ be natural numbers such that we can remove a node from the end of $r_i$ of $[\lambda]$. The resulting partitions of $n-1$ shall be $\lambda(1), \dots, \lambda(m)$. By construction, $\lambda(j) \triangleleft \lambda(i)$ for $j < i$. We define, for every $i = 1, \dots, m$, and $F\mathfrak{S}_{n-1}$-homomorphism $\theta_i \in \mathrm{Hom}_{F\mathfrak{S}_{n-1}}(M^\lambda, M^{\lambda(i)})$ via

$$\theta_i(\{t\}) := \begin{cases} 0, & n \text{ is not in the } r_i\text{th row of } t \\ \{\bar{t}\}, & \text{if } n \text{ is in the } r_i\text{th row of } t, \end{cases}$$

where $\{\bar{t}\}$ is $\{t\}$ with $n$ removed. If $t$ is a standard tableau then

$$\theta_i(e_t) = \begin{cases} e_{\bar{t}}, & n \text{ in the } r_i\text{th row of } t \\ 0, & n \text{ in the } r_j\text{th row of } t \text{ for some } j < i. \end{cases}$$

For $i = 1, \dots, m$, we define $V_i$ as the $F$-span of those $\lambda$-polytabloids $e_t$ where $n$ occurs in row $r_1, r_2, \dots,$ or $r_i$ of $t$. Then $V_{i-1} \subseteq \ker(\theta_i)$, and $(\theta_i(V_i) = S^{\lambda(i)}$, since $\theta_i(V_i)$ contains all standard $\lambda(i)$-polytabloids. We thus get a series of $F\mathfrak{S}_{n-1}$-modules

$$\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(S^\lambda) = V_m \supset V_m \cap \ker(\theta_m) \supseteq V_{m-1} \supset V_{m-1} \cap \ker(\theta_{m-1}) \supseteq \cdots \supseteq V_1 \supset V_1 \cap \ker(\theta_1) \supseteq 0.$$

(We need to check that the $V_i$ are actually $F\mathfrak{S}_{n-1}$-modules, which we will do at the beginning of next lecture.) We have $\dim(V_i/V_i \cap \ker(\theta_i)) = \dim(\theta_i(V_i)) = \dim(S^{\lambda(i)})$. On the other hand,

$$\dim(S^\lambda) = f^\lambda = \sum_{i=1}^m f^{\lambda(i)} = \sum_{i=1}^m \dim(S^{\lambda(i)}),$$

where the first equality holds from Theorem 4.11 and the second equality holds from Problem Sheet 1. So we must have $V_i = V_{i+1} \cap \ker(\theta_{i+1})$ for $i = 0, \dots, m-1$ (there is no room for more modules by the counting argument of what $\dim(S^\lambda)$ is), and we get the desired filtration. □

We begin by finishing the proof of the branching rule. We prove that if $V_j$ is spanned by standard $\lambda$-polytabloids $e_t$ with $n$ in one of the rows $r_1, \dots, r_j$ of $t$, then $V$ is an $F\mathfrak{S}_{n-1}$-module (where $j$ is some integer between 1 and $n$).

If $\sigma \in \mathfrak{S}_{n-1}$, then $\sigma e_t = e_{\sigma t} = \sum \alpha_s e_s$, where $\alpha_s \in \bar{t}$ and the sum runs over standard $\lambda$-tableaux $s$. Let $\{s\}$ be maximal with respect to the order "¡" such that $\alpha_s \neq 0$. Then $\{s\}$ occurs in $e_{\sigma t}$ with coefficient $\alpha_s$. Every tabloid in $e_s$ has $s$ in the same or higher row than $\{s\}$. Every tabloid in $e_t$, and thus in $e_{\sigma t}$, has $n$ in one of the rows $r_1, \ldots r_j$. So $e_s \in V_j$. Moreover, $e_s \in V_j$ whenever $\alpha_s \neq 0$. (In this case, $\{s'\} < \{s\}$ so $n$ occurs in $s'$ in the same or a higher row than in $s$.)

I don't understand the above proof at all.

# Chapter 5

# Character Tables

Now we will discuss character tables.

**Remark 5.1.** This is a recap from representation theory (previous course). Let $G$ be a finite group and let $F$ be any field.

(a) Let $\Delta : G \to \mathrm{GL}_n(F)$ be a matrix representation of $G$ over $F$. Then the character of $\Delta$ is

$$\chi_\Delta : G \to F, g \mapsto \mathrm{tr}(\Delta(g)).$$

This is well-defined; i.e. it does not depend on the choice of basis. Equivalent representations have the same character. Also, $\chi_\Delta$ is constant on $G$-conjugacy classes, and as such it is called a $F$-valued class function.

(b) If $F = \mathbb{C}$, then every representation of $G$ is uniquely determined by its character (up to equivalence).

(c) The character of the irreducible representations of $G$ over $\mathbb{C}$ are called (ordinary) irreducible characters of $G$; they form a $\mathbb{C}$-basis of the $\mathbb{C}$-span of $\mathbb{C}$-valued class functions. In particular, every ordinary character of $G$ can be written as a unique $\mathbb{C}$-linear combination of irreducible ones. (There is a correspondence between irreducible characters and the composition factors of the respective representation.) We denote the set of ordinary irreducible characters of $G$ by $\mathrm{Irr}(G)$.

(d) Recall that, for any character $\chi$ of $G$, we have $\chi(1) = \deg(\Delta)$, where $\Delta$ is any corresponding matrix representation.

(e) If $M$ is a $\mathbb{C}G$-module then we denote by $\chi_M$ the character of any matrix representation of $G$ afforded by $M$.

**Example 5.2.** (a) The trivial matrix representation $G \to \mathrm{GL}(1, \mathbb{C})$ corresponding to the trivial character $\mathbb{1}_G : G \to \mathbb{C}, g \mapsto 1$.

(b) Let $\Omega$ be a finite $G$-set, and let $\Delta_\Omega$ be the permutation representation with respect to the natural basis $\Omega$. The character of $\Delta_\Omega$ is the permutation character $\rho_\Omega$. Then, for $g \in G$, $\rho_\Omega(g) = |\text{fixed points}|$. With this, the orbit-counting formula from group theory can be rewritten.

First, recall the character inner product. If $\chi_1, \chi_2$ are character of $G$, then

$$(\chi_1|\chi_2) = (\chi_1|\chi_2)_G = \frac{1}{|G|}\sum_{g\in G}\chi_1(g)\overline{\chi_2(g)} = \frac{1}{|G|}\sum_{g\in G}\chi_1(g)\chi_2^{-1}(g).$$

Then

$$(\rho_\Omega|\mathbb{1}_G)_G = \frac{1}{|G|}\sum_{g\in G}|\,\mathrm{Fix}_\Omega(g)| = |\{G\text{-orbits on }\Omega\}|.$$

**Definition 5.3.** Let $G$ be a finite group, and let $\{g_1, \dots, g_l\}$ be a transversal for the $G$-conjugacy classes. Let also $\operatorname{Irr}(G) = \{\chi_1, \dots, \chi_l\}$. The (ordinary) character table of $G$ is the $l \times l$ matrix with $(i, j)$-entry $\chi_i(g_j)$ for $i, j = 1, \dots, l$.

**Remark 5.4.** Here are some conventional notations that we too will use. Suppose $G = \mathfrak{S}_n$. Then, for $\lambda \in \mathcal{P}_n$, we denote by $\chi_\lambda$ the character of $\S^\lambda_\mathbb{C}$. For $\mu \in \mathcal{P}_n$, we denote by $\omega_\mu \in \mathfrak{S}_n$ an element of cycle type $\mu$. We label the rows of the character table of $\mathfrak{S}_n$ in lexicographic order, and the columns in reverse lexicographic order. For $\lambda \in \mathcal{P}_n$, let $\rho_\lambda$ be the permutation character corresponding to $M^\lambda_\mathbb{C}$.

**Proposition 5.5.** *Let $\lambda \in \mathcal{P}_n$. Then, for every $\sigma \in \mathfrak{S}_n$, we have $\chi_\lambda(\sigma) \in \mathbb{Z}$. Moreover, $\chi_\lambda$ can be written as a $\mathbb{Z}$-linear combination of permutation characters $\rho_\mu$ with $\lambda \trianglelefteq \mu$. (The coefficients that appear on the linear combination are exactly corresponds to the composition multiplicity.)*

*Proof.* We argue with reverse induction on the dominance order. For $\lambda = (n)$, we have $\chi_\lambda = \mathbb{1}$, which satisfies the assertions. Now let $\lambda \in \mathcal{P}_n$ be arbitrary. By Theorem 4.14, we can write

$$M^\lambda \cong S^\lambda \oplus \bigoplus_{\lambda \lhd \mu} m_{\lambda\mu} S^\mu, \quad m_{\lambda\mu} \in \mathbb{N}_0.$$

This implies $\rho_\lambda = \chi_\lambda + \sum_{\lambda \lhd \mu} m_{\lambda\mu}\chi_\mu$. As we have seen in Example 5.2, for $\sigma \in \mathfrak{S}_n$, $\rho_\lambda(\sigma) = |\operatorname{Fix}_{\Omega^\lambda}(\sigma)| \in \mathbb{Z}$, where $\Omega^\lambda$ is the set of $\lambda$-tabloids. By induction, also $\sum_{\lambda \lhd \mu} m_{\lambda\mu}\chi_\mu(\sigma) \in \mathbb{Z}$, thus $\chi_\lambda(\sigma) \in \mathbb{Z}$. By induction, we also know that $\sum m_{\lambda\mu}\chi_\mu \in {}_\mathbb{Z}\langle \rho_\mu : \lambda \lhd \mu \rangle$ (the $\mathbb{Z}$-span of $\langle \rho_\mu : \lambda \lhd \mu \rangle$). Hence $\chi_\lambda \in {}_\mathbb{Z}\langle \rho_\mu : \lambda \trianglelefteq \mu \rangle$. □

**Example 5.6.** (a) Character table of $\mathfrak{S}_3$. We have 3 irreducible characters:

$$\chi_{(3)} = \mathbb{1}_{\mathfrak{S}_3}, \quad \chi_{(2,1)}, \quad \chi_{(1^3)} = \operatorname{sgn}.$$

We know from Sheet 2 that $\rho_{(2,1)} = \chi_{(2,1)} + \chi_{(3)}$. A transversal for the $\mathfrak{S}_3$-conjugacy classes is $\{1, (12), (123)\}$. We get

$$\rho_{(2,1)}(1) = 3, \quad \rho_{(2,1)}((12)) = 1, \quad \rho_{(2,1)}((123)) = 0.$$

Hence the character table for $\mathfrak{S}_3$ is

| | 1 | (12) | (123) |
|---|---|---|---|
| (3) | 1 | 1 | 1 |
| (1, 2) | 2 | 0 | −1 |
| $(1^3)$ | 1 | −1 | 1 |

We continue with Example 5.6 from the previous lecture.

**Example 5.7.** (b) We construct the character table of $\mathfrak{S}_4$. Recall from Sheets 2 and 3:

$$M^{(3,1)} \cong S^{(3,1)} \oplus S^{(4)}$$
$$M^{(2^2)} \cong S^{(2^2)} \oplus S^{(3,1)} \oplus S^{(4)}$$
$$M^{(4)} \cong S^{(4)}$$

Hence we have

$$\rho_{(3,1)} = \chi_{(3,1)} + \chi_{(4)}, \quad \rho_{(2^2)} = \chi_{(2^2)} + \chi_{(3,1)} + \chi_{(4)}.$$

And we then have

| | 1 | (12) | (12)(34) | (123) | (1234) |
|---|---|---|---|---|---|
| $\rho_{(3,1)}$ | 4 | 2 | 0 | 1 | 0 |
| $\rho_{(2^2)}$ | 6 | 2 | 2 | 0 | 0 |

and hence

| | 1 | (12) | (12)(34) | (123) | (1234) |
|---|---|---|---|---|---|
| (4) | 1 | 1 | 1 | 1 | 1 |
| (3, 1) | 3 | 1 | −1 | 0 | −1 |
| $(2^2)$ | 2 | 0 | 2 | −1 | 0 |
| $(2, 1^2)$ | 3 | −1 | −1 | 0 | 1 |
| $(1^4)$ | 1 | −1 | 1 | 1 | −1 |

(We use the fact from Problem Sheet 4 that $\mathrm{sgn} \cdot \chi_{(3,1)}$ is again an irreducible character and so $(\mathrm{sgn} \cdot \chi_{(3,1)})(g) = \mathrm{sgn}(g) \cdot \chi_{(3,1)}(g)$.)

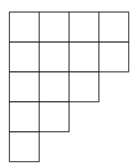
**Definition 5.8.** Let $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathcal{P}_n$, and let $\lambda'$ be the transposed (conjugate) partition.

(a) The $(i, j)$-hook of $[\lambda]$ consists of the $(i, j)$-node of $[\lambda]$, together with the $\lambda_i$-$j$ nodes to the right of it (arm of the hook) and the $(\lambda')_j$-$i$ nodes below it (leg of the hook).

(b) The length of the $(i, j)$-hook is $h_j(\lambda) := \lambda_i + \lambda'_j - i - j + 1$. Replacing every node $(i, j) \in [\lambda]$ by $h_{ij}(\lambda)$, we get the hook diagram of $\lambda$.

(c) A rim hook (skew hook) of $[\lambda]$ is a connected part of the rim of $[\lambda]$, which can be removed to leave the diagram of a partition.

**Example 5.9.** Set $\lambda = (4^2, 3, 2, 1) \in \mathcal{P}_{14}$ . Then

$[\lambda] =$  $\xrightarrow{\text{hook diagram}}$

| | | | |
|---|---|---|---|
| 8 | 6 | 4 | 2 |
| 7 | 5 | 3 | 1 |
| 5 | 3 | 1 | |
| 3 | 1 | | |
| 1 | | | |

There are also some examples of rim hooks that I don't know how to type.

**Lemma 5.10.** *Let $\lambda \in \mathcal{P}_n$. There is a bijection between the sets of hooks of $[\lambda]$ and rim hooks of $[\lambda]$.*

*Proof.* This is easy. Viewing the rim hook as starting from the bottom and going up, the rim hook that starts in the $j$th column and ends in the $i$th row (this uniquely defines a rim hook) corresponds to the $(i, j)$-hook. □

**Remark 5.11.** By the lemma, we can speak of the leg length of a rim hook $h$ as the leg length of the corresponding ordinary hook. Notation: $\mathrm{LL}(h)$.

**Definition 5.12.** Let $\lambda \in \mathcal{P}_n$. A removal sequence of $[\lambda]$ is a sequence

$$S = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(m+1)})$$

of partitions $\lambda^{(1)}, \ldots, \lambda^{(m+1)}$ such that

$$[\lambda^{(m+1)}] \subseteq [\lambda^{(m)}] \subseteq \cdots \subseteq [\lambda^{(1)}]$$

and $h_i := [\lambda^{(i)}] \setminus [\lambda^{(i+1)}]$ is a rim hook of $[\lambda^{(i)}]$ for $i = 1, \ldots, m$. For $i = 1, \ldots, m$, let $\mu_i$ be the length of $h_i$. The composition $\mu = (\mu_1, \ldots, \mu_m)$ is called the type of $S$. Let also $\mathrm{LL}(S) = \{0, 1\}$ such that $\prod_{i=1}^{M} (-1)^{\mathrm{LL}(h_i)} = (-1)^{\mathrm{LL}(S)}$.

**Theorem 5.13** (Munaghan-Nakayama Rule)**.** *Let $\lambda, \mu \in \mathcal{P}_n$ and let $\omega_\mu \in \mathfrak{S}_n$ be a permutation of cycle type $\mu$. Then*

$$\chi_\lambda(\omega_\mu) = \sum_S (-1)^{\mathrm{LL}(S)},$$

*where $S$ varies over all removal sequences of $[\lambda]$ of type $\mu$. The empty sum is interpreted as 0.*

**Example 5.14.** There were some examples worked out here. See written notes.

Here are some ideas about the proof of Munaghan-Nakayama. It is a very technical proof, so it will not be done rigorously.

We can break the proof down into two steps:

1. Reduction to the case where $\mu = (n)$, i.e. $\omega_\mu = (1, \dots, n)$.

2. One has to show

$$\chi_\lambda(\omega_\mu) = \begin{cases} (-1)^r, & \text{if } \lambda = (n-r, 1^r),\ 0 \le r \le n-1 \\ 0, & \text{otherwise.} \end{cases}$$

For the second task, we argue with reverse induction on the dominance order "$\trianglelefteq$." For $\lambda = (n)$ or $\lambda = (n-1, 1)$, the statement is true. Now let $\lambda$ be arbitrary and suppose the statement holds for all $\lambda \lhd \mu$. By Theorem 4.14, $\rho_\lambda = \chi_\lambda + \sum_{\lambda \lhd \nu} m_{\lambda\nu}\chi_\nu$. Since $\omega_\mu$ has no fixed points on $\{1, \dots, n\}$, $\omega_\lambda(\omega_\mu) = 0$. By induction, we have

$$\chi_\lambda(\omega_\mu) = -\sum_{\lambda \lhd \nu} m_{\lambda\nu}\chi_\nu(\omega_\mu) = -\sum_{\lambda \lhd (n-s, 1^s) = \nu} m_{\lambda\nu}\chi_\nu(\omega_\mu).$$

The $\omega_{\lambda\nu}$ can be computed via Young's rule. That is, $m_{\lambda,(n-s,1^s)} =$ number of ways to fill $[(n-s, 1^s)]$ with $\lambda_1 1's, \lambda_2 2's$, etc. in such a way that the entries in this "general tableau" strictly increase down columns and do not increase along rows $= \binom{r}{s}$. If $\lambda = (n-r, 1^r)$, $m_{(n-r,1^r),(n-s,1^s)} = \binom{r}{s}$. In this case, the statement reads

$$-\sum_{s=0}^{r-1} m_{\lambda,(n-s,1^s)}(-1)^s = -\sum_{s=1}^{r-1}\binom{r}{s}(-1)^s = \sum_{s=1}^{r-1}\binom{r}{s}(-1)^{s+1} = (-1)^r.$$

We need to do more work to finish this, but this gives the general feel of the proof.

# Chapter 6

# Induced Modules and the Branching Rule Revisited

Today we will discuss induced modules and then we will be able to formulate a branching rule for Specht modules, which relates the induction from a Specht $F\mathfrak{S}_{n-1}$-module up to $\mathfrak{S}_n$ to a direct sum of particular Specht $F\mathfrak{S}_{n-1}$-modules.

We begin in a very general setting. Let $F$ be any field, $H \leq G$ finite groups. If $M$ is an $FG$-module, then we can view it as and $FH$-module $\mathrm{Res}_H^G(M)$. If $N$ is an $FH$-module, we can construct an $FG$-module $\mathrm{Ind}_H^G(N)$ via induction.

**Remark 6.1.** Let $\Delta : H \to \mathrm{GL}(n, F)$ be a matrix representation. We construct $\mathrm{Ind}_H^G(\Delta) = \Gamma : G \to \mathrm{GL}(n|G:H|, F)$. Let $\mathscr{T} = \{t_1, \ldots, t_l\}$ be a transversal for the left cosets of $G/H$. For $g \in G$, we define

$$\Gamma(g) := \begin{pmatrix} \Delta(t_1^{-1}gt_1) & \Delta(t_1^{-1}gt_2) & \cdots & \Delta(t_1^{-1}gt_l) \\ \Delta(t_2^{-1}gt_1) & \Delta(t_2^{-1}gt_2) & \cdots & \Delta(t_2^{-1}gt_l) \\ \vdots & \vdots & \ddots & \vdots \\ \Delta(t_l^{-1}gt_1) & \Delta(t_l^{-1}gt_2) & \cdots & \Delta(t_l^{-1}gt_l) \end{pmatrix}$$

with $\Delta(t_i^{-1}gt_j) = 0$ if $t_i^{-1}gt_j \notin H$.

**Theorem 6.2.** *With the above:*

*(a)* $\mathrm{Ind}_H^G(\Delta)$ *is a matrix representation of* $G$*.*

*(b) It is independent of* $\mathscr{T}$*: if* $\{s_1, \ldots, s_l\}$ *is another transversal for* $G/H$*, the respective representation* $\Gamma'$ *is equivalent to* $\Gamma$*.*

*(c) If* $\Delta$ *and* $\Delta'$ *are equivalent representations of* $H$*, then* $\mathrm{Ind}_H^G(\Delta)$ *and* $\mathrm{Ind}_H^G(\Delta')$ *are equivalent.*

*Proof.* Let $g \in G$. Let $j \in \{1, \cdots, l\}$. Then there is a unique $t_i \in \mathscr{T}$ such that $gt_j \in t_iH$ and $t_k^{-1}gt_j \notin H$ for $k \neq i$ (i.e. each column has exactly one 1 and zeros elsewhere). If $t_i^{-1}gt_j \in H, t_i^{-1}gt_k \in H$, then $t_i = t_k$ (i.e. each row has exactly one 1 and zeros elsewhere). So in each row and column, we have exactly one nonzero block. Now we show multiplicativeness. That is, if $g, h \in G$, we have to check that $\Gamma(gh) = \Gamma(g)\Gamma(h)$; i.e.

$$\sum_{k=1}^{i} \Delta(t_i^{-1}gt_k)\Delta(t_k^{-1}ht_j) = \Delta(t_i^{-1}ght_j) \text{ for all } i, j.$$

Fixing $i, j$, and setting $a_k = t_i^{-1}gt_k$, $b_k = t_k^{-1}ht_j$, and $a_kb_k = c$, for $k = 1, \cdots, l$, we have two cases. If $\Delta(c) = 0$, then we have $c \in H$ and so for every $k = 1, \ldots, l$, either $a_k \notin H$ or $b_k \notin H$, and hence $\sum_{k=1}^{j} \Delta(a_k)\Delta(b_k) = 0$. If $\Delta(c) \neq 0$, then $c \in H$ and there is a unique $m \in \{1, \ldots, l\}$

such that $a_m \in H$. Thus $b_m \in H$. We get $\Delta(t_i^{-1} g t_m)\Delta(t_m^{-1} h t_j) = \Delta(t_i^{-1} g h t_j)$. It follow easily that $\Gamma(1)$ is the identity matrix. This proves a.

Let $\{s_1, \ldots, s_l\}$ be a transversal for $G/H$. For each $i = \{1, \ldots, l\}$, there is a unique $h_i \in H$ such that $t_i = s_i h_i$. Set

$$X := \begin{pmatrix} \Delta(h_1) & & 0 \\ & \ddots & \\ 0 & & \Delta(h_l) \end{pmatrix} \Longrightarrow X^{-1}\Gamma' X = \Gamma.$$

This proves b.

Finally, if $\Delta' = Y^{-1}\Delta Y$ for some $Y \in \mathrm{GL}(n, F)$, then

$$\mathrm{Ind}_H^G(\Delta') = \widetilde{Y}^{-1} \mathrm{Ind}_H^G(\Delta)\widetilde{Y} \text{ where } \widetilde{Y} = \begin{pmatrix} Y & & 0 \\ & \ddots & \\ 0 & & Y \end{pmatrix}. \qquad \square$$

**Example 6.3.** We give some examples on permutation representations.

(a) Let $G = \mathfrak{S}_3$ and let $H := \langle (23) \rangle$. Let $\Delta : H \to \mathrm{GL}(1, F)$ be the trivial representation and let $\{t_1, t_2, t_3\} = \{1, (12), (13)\}$ be a transversal for $G/H$. Then

$$\mathrm{Ind}_H^G(\Delta)((12)) = \begin{pmatrix} \Delta(12) & \Delta(1) & \Delta(132) \\ \Delta(1) & \Delta(12) & \Delta(13) \\ \Delta(123) & \Delta(13) & \Delta(23) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Computing similarly, we have

$$\mathrm{Ind}_H^G((123)) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus $\mathrm{Ind}_H^G(\Delta)$ is a permutation representation of $G$ afforded by the permutation module $F\{H, (12)H, (13)H\}$.

(b) For any finite groups $H \leq G$ and any transversal $\{t_1, \ldots, t_l\}$ for $G/H$, the induction $\mathrm{Ind}_H^G(\mathbb{1})$ of the trivial representation is the permutation representation of $G$ afforded by the permutation module $\{t_1 H, \ldots, t_l H\}$.

**Remark 6.4.** We discuss induced characters. With the notation of Remark 6.1, the character of the induced representation $\mathrm{Ind}_H^G(\Delta)$ is $\mathrm{Ind}_H^G(\chi_\Delta) =: \psi$ if $\chi_\Delta$ is the character of the representation $\Delta$. Then

$$\psi(g) = \sum_{i=1}^{l} \chi_\Delta(t_i^{-1} g t_i),$$

with $\chi_\Delta(t_i^{-1} g t_i) = 0$ whenever $t_i^{-1} g t_i \notin H$.

If $\mathrm{char}(F) \nmid |H|$ (for instance, if $F = \mathbb{C}$), we can rewrite this: for $h \in H$, $i = 1, \ldots, l$, $\chi_\Delta(t_i^{-1} g t_i) = \chi_\Delta(h^{-1} t_i^{-1} g t_i h)$. So

$$\psi(g) = \frac{1}{|H|} \sum_{i=1}^{l} \sum_{h \in H} \chi_\Delta(h^{-1} t_i^{-1} g t_i h) = \frac{1}{|H|} \sum_{x \in G} \chi_\Delta(x^{-1} g x).$$

**Theorem 6.5** (Frobenius Reciprocity)**.** *Let $H \leq G$ be finite groups. Let $\chi$ be a complex character of $H$. Let $\psi$ be a complex character of $G$. Then*

$$(\mathrm{Ind}_H^G(\chi) | \psi)_G = (\chi | \mathrm{Res}_H^G(\psi))_H.$$

*Proof.*

$$\begin{aligned}
(\operatorname{Ind}_H^G(\chi)|\psi)_G &= \frac{1}{|G|}\sum_{g\in G}\operatorname{Ind}_H^G(\chi)(g)\psi(g^{-1} = \frac{1}{|G|}\frac{1}{|H|}\sum_{g\in G}\sum_{x\in G}\chi(xgx^{-1}\psi(g^{-1}) \\
&= \frac{1}{|G|}\frac{1}{|H|}\sum_{x\in G}\sum_{y\in G}\chi(y)\psi(xy^{-1}x^{-1}) = \frac{1}{|G|}\frac{1}{|H|}\sum_{x\in G}\sum_{y\in G}\chi(y)\psi(y^{-1}) \\
&= \frac{1}{|H|}\sum_{y\in G}\chi(y)\psi(y^{-1}) = \frac{1}{|H|}\sum_{y\in H}\chi(y)\chi(y^{-1}) = (\chi|\operatorname{Res}_H^G(\psi))_H.
\end{aligned}$$

□

**Definition 6.6.** If $H \leq G$ are finite groups, and if $N$ is an $FH$-module with corresponding representation $\Delta$, then we define $\operatorname{Ind}_H^G(N)$ as the $FG$-module affording $\operatorname{Ind}_H^G(\Delta)$. Note: If $N_1 \cong N_2$ as $FH$-modules, then $\operatorname{Ind}_H^G(N_1) \cong \operatorname{Ind}_H^G(N_2)$.

**Corollary 6.7.** *Let $H \leq G$ be a finite group. Let $M$ be a simple $\mathbb{C}G$-module. Let $N$ be a simple $\mathbb{C}H$-module. Then the composition multiplicity of $N$ in $\operatorname{Res}_H^G(M)$ equals the composition multiplicity of $M$ in $\operatorname{Ind}_H^G(N)$.*

**Theorem 6.8** (Branching Rule for Specht modules)**.** *Let $\lambda \in \mathcal{P}_n$. Then*

$$\operatorname{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}}(S_{\mathbb{C}}^{\lambda}) \cong \bigoplus_{i=1}^{k} S_{\mathbb{C}}^{\nu(i)},$$

*where $\{\nu(1),\ldots,\nu(k)\}$ are the partitions of $n+1$ whose Young diagrams are obtained by some node to $[\lambda]$ such that $\nu(j) \lhd \nu(i)$ if $i < j$.*

*Proof.* This follows from Corollary 6.7, Theorem 4.16, and the fact that $x$ is an addable node of $[\lambda]$ if and only if $x$ is a removable node of $[\lambda]\cup\{x\}$. □

This lecture (as well as the next one) is just a lot of technicalities in a combinatorial sense. So get ready for an onslaught of products and sums and indexing notation.

# Chapter 7

# The Hook Formula

Take $n \in \mathbb{N}$. Recall that, for $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathcal{P}_n$, we have

$$f^\lambda = \sum_{i \geq 1} f^{\lambda \backslash i},$$

where $\lambda \backslash i = (\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_n)$. If $\lambda \backslash i$ is not a partition, then we set $f^{\lambda \backslash i} = 0$.

**Remark 7.1.** (a) Let $k \in \mathbb{N}$. Let $l_1, \dots, l_k \in \mathbb{Z}$. We set

$$\Delta_k(l_1, \dots, l_k) = \prod_{1 \leq i < j \leq n} (l_i - l_j).$$

This is the Vandermonde determinant; i.e.,

$$\Delta_k(l_1, \dots, l_k) = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ l_k & l_{k-1} & \cdots & l_1 \\ l_k^2 & l_{k-1}^2 & \cdots & l_1^2 \\ \vdots & \vdots & \ddots & \vdots \\ l_k^{k-1} & l_{k-1}^{k-1} & \cdots & l_1^{k-1} \end{pmatrix}$$

Given any composition $(\lambda_1, \dots, \lambda_k) = \lambda \in \mathcal{C}_n$, we set $l_j := l_j(\lambda) = \lambda_i + k - j$ and also

$$\overline{f}^\lambda := \frac{n! \Delta_k(l_1, \dots, l_k)}{l_1! \cdots l_k!}.$$

Note that we can put zeros at the end of $\lambda$ without changing $\overline{f}^\lambda$.

(b) Suppose $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathcal{P}_n$ . If, for $i \leq k$, $\lambda \backslash i$ is not a partition, then $\lambda_i = \lambda_{i+1}$. Hence

$$\lambda_i(\lambda \backslash i) = (\lambda_i - 1) + k - i = \lambda_{i+1} + k - (i+1) = l_{i+1}(\lambda \backslash i).$$

So $\Delta_k(l_1(\lambda \backslash i), \dots, l_k(\lambda \backslash i) = 0$ and thus $\overline{f}^{\lambda \backslash i} = 0$.

Recall the Young-Frobenius formula: For $\lambda \in \mathcal{P}_n$, $\lambda = (\lambda_1, \dots, \lambda_k)$, $l_j := \lambda_j + k - j$, we have

$$f^\lambda = \frac{n! \Delta_k(l_1, \dots, l_k)}{l_1! \cdots l_k!}.$$

Recall also the hook formula:

$$f^\lambda = \frac{n!}{\prod_{(i,j) \in [\lambda]} h_{ij}}.$$

We begin with a lemma.

**Lemma 7.2.** *Let $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathcal{P}_n$, and let $s \geq k$. For $j = 1, \ldots, s$, we set $l_j := \lambda_j + s - j$. Then*

$$\prod_{j=1}^{\lambda_i}(1 - t^{h_{ij}}) \prod_{j=i+1}^{s}(1 - t^{l_i - l_j}) = \prod_{j=1}^{\lambda_i + s - i}(1 - t^j).$$

Now we have a theorem.

**Theorem 7.3.** *Let $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathcal{P}_n$ and $s \geq k$. For $j = 1, \ldots, s$, let $l_j := \lambda_j + s - j$. Then*

*(a)* $\prod_{s \in [\lambda]}(1 - t^{h_x}) = \frac{\prod_{i=1}^{s} \prod_{j=1}^{l_i}(1-t^j)}{\prod_{i<j}(1-t^{l_i - l_j})}$, *and*

*(b)* $\prod_{x \in [\lambda]} h_x = \frac{\prod_{i=1}^{s} l_i!}{\prod_{i<j}(l_i - l_j)}$.

# Chapter 8

# The RSK Correspondence

Now we begin a new chapter. We discuss the Robinson-Schensted-Knuth correspondence, wherein we will construct a bijection between the symmetric group $\mathfrak{S}_n$ and the set of all pairs of standard $\lambda$-tableaux. As a consequence we will get

$$n! = \sum_{\lambda \in \mathcal{P}_n} (f^\lambda)^2,$$

and this will complete the proof that the Specht modules form a basis for $F\mathfrak{S}_n$-modules.

We continue discussing the Robinson-Schensted-Knuth (RSK) Algorithm. Last time, we described the insertion map

$$\text{ins} : \mathfrak{S}_n \to \sum_{\lambda \in \mathcal{P}_n} \{(P, Q) : P, Q \text{ standard } \lambda\text{-tableau}\}.$$

Now we prove a lemma that tells us that this insertion map does what we want it to do.

**Lemma 8.1.** *With the notation as in Algorithm 8.1, $T \leftarrow b$ is again a "generalised" standard tableau.*

**Theorem 8.2.** *Let $\sigma \in \mathfrak{S}_n$, and let $(P, Q) = \text{ins}(\sigma)$. Then there is some $\lambda \in \mathcal{P}_n$ such that $P$ and $Q$ are standard $\lambda$-tableau.*

**Remark 8.3.** One could show that, for $\sigma \in \mathfrak{S}_n$ and $\text{ins}(\sigma) = (P, Q)$, one has $\text{ins}(\sigma^{-1}) = (Q, P)$. A reference for this is Sagan, Theorem 3.6.6. In particular, this shows that $|\sigma| \le 2$ if and only if $P = Q$ and thus $|\{\sigma \in \mathfrak{S}_n : |\sigma \le 2\}| \le \sum_{\lambda \in \mathcal{P}_n} f^\lambda$.

Now we construct the inverse algorithm to the insertion algorithm.

**Algorithm 8.4** (The deletion algorithm)**.**

This is quite technical so we give an example.

**Example 8.5.**

We now verify that the deletion algorithm gives us what we want.

**Theorem 8.6.** *Let $\lambda \le n, \mu \in \mathcal{P}_k$, and let $T$ be a standard $\mu$-tableau (with entries in $\{1, \dots, n\}$). Let $x \in [\mu]$ be a removable node such that $T - x = (T^-, v)$. Then $T^-$ is again standard.*

I find that with these sorts of proofs, it's often much easier to explain it to someone in words (in person, with a blackboard) or just think about it independently rather than write out all the details. See, the thing is, I think that writing things down can get notationally cumbersome, which is something you don't have to deal with when you just think about it.

There is a correction that needs to be pointed out on Sheet 6: In question 2, the parts of the partition should be *consecutive* odd numbers and *consecutive* even numbers.

Now we continue discussing deletion in the RSK algorithm. (This will be the last bit of our work in combinatorics.)

**Theorem 8.7.** *The deletion and insertion algorithms are inverse to each other.*

*(a) Let $k \leq n$, $\mu \in \mathcal{P}_k$, let $x \in [\mu]$ be a removable note, and let $T$ be a standard $\mu$-tableau with entries in $\{1, \ldots, n\}$. Then if $T - x = (T^-, v)$, then $T^- \leftarrow v = T$.*

*(b) Let $k < n$, $\mu \in \mathcal{P}_k$, $a_1, \ldots, a_k, b \in \{1, \ldots, n\}$ pairwise distinct, and let $T = T(a_1, \ldots, a_k)$ be a standard $\mu$-tableau. Let $T^+ := T \leftarrow b$ be of shape $\mu^+ \in \mathcal{P}_{k+1}$, and let $x \in [\mu^+]$ be the removable node such that $[\mu^+] \backslash \{x\} = [\mu]$. Then $T^+ - x = (T, b)$.*

*Proof.* We again induct on the rows of the relevant tableau. □

**Algorithm 8.8.** The inverse algorithm.

**Theorem 8.9.** *Let $\sigma \in \mathfrak{S}_n, \lambda \in \mathcal{P}_n$, $P, Q$ standard $\lambda$-tableau. Then* $\mathrm{Del}(\mathrm{Ins}(\sigma)) = \sigma$ *and* $\mathrm{Ins}(\mathrm{Del}(\sigma)) = \sigma$. *Thus we have a bijection*

$$\mathfrak{S}_n \rightleftarrows \bigcup_{\lambda \in \mathcal{P}_n} \{(P, Q) : P, Q \text{ standard } \lambda\text{-tableaux}\}.$$

**Corollary 8.10.** *For $\lambda \in \mathcal{P}_n$, let $f^\lambda$ be the number of standard $\lambda$-tableaux. Then $n! = \sum_{\lambda \in \mathcal{P}_n} (f^\lambda)^2$.*

This completes also the proof that the Specht modules are a basis for $F\mathfrak{S}_n$-modules.

**Example 8.11.** Let $n = 7, \lambda = (3^2, 1)$. Let

$$P = \begin{array}{ccc} 1 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & & \end{array}, \quad Q = \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & & \end{array}, \quad \lambda = \begin{array}{|c|c|c|} \hline \phantom{0} & \phantom{0} & \phantom{0} \\ \hline \phantom{0} & \phantom{0} & \phantom{0} \\ \hline \phantom{0} & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

As an exercise, write out all the steps of the RSK-algorithm to get the permutation that corresponds to this pair of standard tableaux.

There are other uses of the RSK algorithm: Littlewood-Richardson, construction of Specht modules. There is a more direct way of seeing that the Specht modules are a basis of $F\mathfrak{S}_n$ modules. These are the Garnier relations.

# Chapter 9

# Modular Representation Theory

Now (finally!) we discuss modular representation theory. Our aim will be to get a parametrisation for the isomorphism classes of simple $F\mathfrak{S}_n$-modules, where $F$ is a field of characteristic $p > 0$. We first discuss some things for an arbitrary field.

**Definition 9.1.** Let $G$ be a finite group, and let $\Omega = \{\omega_1, \ldots, \omega_d\}$ be a finite $G$-set. We define an $F$-bilinear form on the permutation module $F\Omega$:

$$\langle \cdot | \cdot \rangle : F\Omega \times F\Omega \to F, (\omega_i, \omega_j) \mapsto \delta_{ij}, \quad i, j = 1, \ldots, d.$$

For every $F$-subspace $U$ of $F\Omega$, we have the orthogonal space:

$$U^\perp = \{x \in F\Omega : \langle \chi | u \rangle = 0 \text{ for all } u \in U\}.$$

**Remark 9.2.** The $F$-bilinear form $\langle \cdot | \cdot \rangle$ is

- symmetric: $\langle x|y \rangle = \langle y|x \rangle$ for all $x, y \in F\Omega$.
- $G$-invariant: $\langle gx|gy \rangle = \langle x|y \rangle$ for all $x, y \in F\Omega$.
- non-degenerate: If $\langle x|y \rangle = 0$ for all $y \in F\Omega$, then $x = 0$.

If $U$ is an $FG$-submodule of $F\Omega$, then $U^\perp$ is an $FG$-module, since $\langle \cdot | \cdot \rangle$ is $G$-invariant.

**Lemma 9.3.** *Let $\lambda \in \mathcal{P}_n$, $x, y \in M^\lambda$, $t$ a $\lambda$-tableau. Then*

$$\langle \kappa_t x | y \rangle = \langle x | \kappa_t y \rangle,$$

*for $x, y \in M^\lambda$, where $\kappa_t = \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\pi$.*

*Proof.* We compute directly:

$$\langle \kappa_t x | y \rangle = \langle \sum_{\pi \in C_t} \operatorname{sgn}(\pi) \pi x | y \rangle = \sum_{\pi \in C_t} \operatorname{sgn}(\pi) \langle x | \pi^{-1} y \rangle = \langle x | \sum_{\pi \in C_t} \operatorname{sgn}(\pi) \pi y \rangle = \langle x | \kappa_t y \rangle. \qquad \square$$

**Theorem 9.4.** *Let $\lambda \in \mathcal{P}$ and let $U$ be an $F\mathfrak{S}_n$-submodule of $M^\lambda$. Then $S^\lambda \subseteq U$ or $U \subseteq (S^\lambda)^\perp$.*

Note that this statement is trivial when $p > n$ or when $\operatorname{char} F = 0$ since in these situations, $S^\lambda$ is simple and then the result is obvious. Hence this is only meaningful in the case when $p < n$, i.e. when $p \,\big|\, |\mathfrak{S}_n| = n!$.

*Proof.* Let $u \in U$ and let $t$ be a $\lambda$-tableau. By Theorem 4.1b (check the numbering here), there exists an $\alpha \in F$ such that $\kappa_t u = \alpha e_t$. If we can choose $u$ and $t$ in such a way that $\alpha \neq 0$, then

$$e_t = \alpha^{-1}\kappa_t u \in U,$$

and thus $S^\lambda \subseteq U$. If $\kappa u = 0$ for all $u \in U$ and all $\lambda$-tableaux $t$, then for every such $t$,

$$0 = \langle \kappa_t u | \{t\}\rangle = \langle u | \kappa_t\{t\}\rangle = \langle u | e_t\rangle.$$

Hence $U \subseteq (S^\lambda)^\perp$. This completes the proof. □

**Definition 9.5.** Let $\lambda \in \mathcal{P}_n$. We define an $F\mathfrak{S}_n$-module:

$$D^\lambda := S^\lambda/(S^\lambda \cap (S^\lambda)^\perp).$$

**Theorem 9.6.** *Let $\lambda \in \mathcal{P}_n$. If $S^\lambda \neq S^\lambda \cap (S^\lambda)^\perp$, then $S^\lambda \cap (S^\lambda)^\perp$ is the unique maximal $F\mathfrak{S}_n$-submodule of $S^\lambda$, and $D^\lambda$ is a simple $F\mathfrak{S}_n$-module.*

*Proof.* Let $U$ be some maximal submodule of $S^\lambda$. By Theorem 9.4, $U \subseteq S^\lambda \cap (S^\lambda)^\perp$. So if $S^\lambda \cap (S^\lambda)^\perp \neq S^\lambda$, then $U = S^\lambda \cap (S^\lambda)^\perp$, and $U$ is the unique maximal submodule of $S^\lambda$. □

Hence we have shown that for any $\lambda \in \mathcal{P}_n$, either $D^\lambda = 0$ or $D^\lambda$ is simple.

**Lemma 9.7.** *Let $\lambda = (n^{m_n}, \ldots, 1^{m_1}) \in \mathcal{P}_n$. Then*

*(a) If $t$ and $\tilde{t}$ are $\lambda$-tableaux, then*

$$\prod_{j=1}^{n}(m_j)! \,\Big|\, \langle e_t \mid e_{\tilde{t}}\rangle.$$

*(b) If $\tilde{t}$ is the tableau obtained by reversing the entries in the entire row of $t$, then*

$$\langle e_t \mid e_{\tilde{t}}\rangle = \prod_{j=1}^{n}(m_j!)^j.$$

*Proof.* We say that $\lambda$-tabloids $\{t_1\}$ and $\{t_2\}$ are equivalent it $\{t_2\}$ is obtained from $\{t_1\}$ by permuting rows.

(a) Since, for $j = 1, \ldots, n$, the partition $\lambda$ has $m_j$ rows of length $j$, each equivalent class of $\lambda$-tabloids has cardinality $\prod_{j=1}^{n} m_j!$. If $\{t_1\}$ occurs in $e_t$, then so does every equivalent tabloid $\{t_2\}$; furthermore, either $\{t_1\}$ and $\{t_2\}$ have the same or opposite coefficient, and whichever case occurs is only dependent on $\{t_1\}$ and $\{t_2\}$ (not on $t$).

Now let $t$ and $\tilde{t}$be two such $\lambda$-tableaux, and suppose that $\{t_1\}$ occurs in $e_t$ and $e_{\tilde{t}}$ with the same coefficient. Call this coefficient $\alpha \in \{1, -1\}$. Then all $\prod_{j=1}^{n}(m_j)!$ elements of the equivalence class of $\{t_1\}$ must occur with the same coefficient $\alpha$ in $e_t$ and $e_{\tilde{t}}$. Since $\langle\{t_2\}, \{t_2\}\rangle = 1 = \langle -\{t_2\}, -\{t_2\}\rangle$, then using the linearity of the inner product, we have that the contribution of the equivalence class of $\{t_1\}$ to $\langle e_t, e_{\tilde{t}}\rangle$ is $\prod_{j=1}^{n}(m_j)!$. Similarly, if $\{t_1\}$ occurs in $e_t$ and $e_{\tilde{t}}$ with opposite sign, then the contribution is $-\prod_{j=1}^{n}(m_j)!$. This proves (a).

(b) Let $C \leq C_t$ be the subgroup of $C_t$ consisting of all permutations $\pi \in C_t$ such that for all $i \in \{1, \ldots, n\}$, $i$ and $\pi(i)$ are in rows of equal lengths in $t$. Then

$$C \cong \prod_{j=1}^{n}(\mathfrak{S}_{m_j})^j,$$

so that $|C| = \prod_{j=1}^{n}(m_j!)^j$. If $\{t_1\}$ occurs in both $e_t$ and $e_{\tilde{t}}$, then $\{t_1\} = \{\pi t\}$ for some $\pi \in C$ and it occurs with the same coefficient. Thus $\langle e_t, e_{\tilde{t}}\rangle = \prod_{j=1}^{n}(m_j!)^j$, as desired. □

**Corollary 9.8.** *Let $\lambda \in \mathcal{P}_n$. Then $D^\lambda \neq 0$ if and only if $\lambda \in \mathcal{P}_{n,p}$.*

*Proof.* Assume $\lambda \notin \mathcal{P}_{n,p}$. Then letting $\lambda = (n^{m_n}, \dots, 1^{m_1})$, we have $m_j \geq p$ for some $j$, so $p \,\big|\, \prod_{j=1}^n m_j!$. Then by Lemma 9.7(a), $\langle e_t, e_{\tilde{t}} \rangle = 0$ for every $\lambda$-tableau $\tilde{t}$. But then this means $S^\lambda \subseteq (S^\lambda)^\perp$ and so $S^\lambda \cap (S^\lambda)^\perp = S^\lambda$. By definition, this means $D^\lambda = 0$.

Conversely, assume that $\lambda \in \mathcal{P}_{n,p}$, let $t$ be a $\lambda$-tableau, and let $\tilde{t}$ be the tableau obtained by reversing the entires in every row of $t$. If $\lambda = (n^{m_n}, \dots, 1^{m_1})$ then $m_j < p$ for every $j$ (by definition). So by Lemma 9.7(b), we have

$$\langle e_t, e_{\tilde{t}} \rangle = \prod_{j=1}^n (m_j!)^j \neq 0.$$

Therefore $e_t \notin (S^\lambda)^\perp$ and therefore by Theorem 9.6, $D^\lambda \neq 0$ and is simple. □

**Proposition 9.9.** *Consider $\lambda \in \mathcal{P}_{n,p}$, $\mu \in \mathcal{P}_n$. Let $U$ be an $F\mathfrak{S}_n$-submodule of $M^\mu$ and let $\varphi \in \mathrm{Hom}_{F\mathfrak{S}_n}(S^\lambda, M^\mu/U)$. Then*

*(a) $\varphi \neq 0 \Longrightarrow \mu \trianglelefteq \lambda$.*

*(b) If $\mu = \lambda$, then there is some $\alpha \in F$ such that $\varphi(s) = \alpha s + U$ for all $s \in S^\lambda$. In particular, if $\varphi \neq 0$, then $S^\lambda \not\subseteq U$.*

*Proof.* Write $\lambda = (n^{m_n}, \dots, 1^{m_1})$ with $0 \leq m_j \leq p-1$ for all $j$. Let $t$ and $\tilde{t}$ as in Lemma 9.7(b). By Theorem 4.1(b), $\kappa_t e_{\tilde{t}} \in \kappa_t M^\lambda = F e_t$, so $\kappa_t e_{\tilde{t}} = \beta e_t$ for some $\beta \in F$. Moreover,

$$\beta = \langle \beta e_t, \{t\} \rangle = \langle \kappa_t e_{\tilde{t}}, \{t\} \rangle = \langle e_{\tilde{t}}, \kappa_t \{t\} \rangle = \langle e_{\tilde{t}}, e_t \rangle = \prod_{j=1}^n (m_j!)^j \neq 0.$$

To prove (a), let $0 \neq \varphi \in \mathrm{Hom}_{F\mathfrak{S}_n}(S^\lambda, M^\mu/U)$. Then form some $x \in M^\mu$, we have $0 \neq \beta\varphi(e_t) = \kappa_t \varphi(e_{\tilde{t}}) = \kappa_t(x + U)$. Thus $\kappa_t M^\mu \not\subseteq U$. In particular, $\kappa_t M^\mu \neq 0$. By Theorem 4.1(a), this implies $\mu \trianglelefteq \lambda$.

For (b), let $\mu = \lambda$. By Theorem 4.1 and the above considerations, for some $\alpha' \in F$, we have $\beta\varphi(e_t) = \kappa_t x + U = \alpha' e_t + U$, and so $\varphi(s) = \alpha s + U$ for $s \in S^\lambda$ for $\alpha = \alpha'/\beta$. □

**Theorem 9.10.** *Let $\lambda, \mu \in \mathcal{P}_{n,p}$. Then $D^\lambda \cong D^\mu$ if and only if $\lambda = \mu$.*

*Proof.* Suppose $\psi : D^\lambda \to D^\mu$ is an $F\mathfrak{S}_n$-isomorphism. Then we get a homomorphism

$$\varphi : S^\lambda \twoheadrightarrow D^\lambda \xrightarrow{\psi} D^\mu = S^\mu/(S^\mu \cap (S^\mu)^\perp) \hookrightarrow M^\mu/(S^\mu \cap (S^\mu)^\perp)$$

where $\varphi$ is nonzero and $\varphi \in \mathrm{Hom}_{F\mathfrak{S}_n}(S^\lambda, M^\mu/(S^\mu \cap (S^\mu)^\perp)$. Thus $\mu \trianglelefteq \lambda$ by Proposition 9.9. Applying the same argument to $\psi^{-1}$ in place of $\psi$, we get $\lambda \trianglelefteq \mu$. Therefore we must have $\lambda = \mu$. □

We have finally proved all the ingredients needed for the following theorem.

**Theorem 9.11.** *Let $F$ be a(n algebraically closed) field of characteristic $p > 0$. Then $\{D^\lambda : \lambda \in \mathcal{P}_{n,p}\}$ is a transversal for the isomorphism classes of simple $F\mathfrak{S}_n$-modules.*

*Proof.* By Theorem 9.6, Corollary 9.8, Theorem 9.10, Corollary 1.9. □

# Chapter 10

# Properties of the Modules $D^\lambda$

## 10.1 Changing the Field

**Remark 10.1.** Let $G$ be a finite group and $F$ a field, and let $E \supseteq F$ be an extension field of $F$. Given a matrix representation $\Delta : G \to \mathrm{GL}(n, F)$, we can get a matrix representation of $G$ over $E$ via

$$\Delta^E : G \xrightarrow{\Delta} \mathrm{GL}(n, F) \hookrightarrow \mathrm{GL}(n, E).$$

If $M$ is an $FG$-module affording $\Delta$, we denote by $M^E$ the $EG$-module affording $\Delta^E$. By construction, $\dim_F(M) = \dim_E(M^E)$.

**Definition 10.2.** Let $D$ be a simple $FG$-module. If $D^E$ is simple for every extension field $E$ of $F$, then $D$ is called *absolutely simple*. If every simple $FG$-module is absolutely simple, then $F$ is called a *splitting field* for $G$.

**Example 10.3.** (a) $F = \overline{F}$ is a splitting field for every finite field.

(b) If $\mathrm{char}(F) \nmid n!$ then every Specht $F\mathfrak{S}_n$-module is absolutely simple.

(c) $D^\lambda_F$ is absolutely simple if $\mathrm{char}(F) = p \,\big|\, n!$ and $\lambda \in \mathcal{P}_{n,p}$. (This needs proof)

(d) In Question 3 on Sheet 3, we saw a simple $\mathbb{F}_2\mathfrak{A}_4$-module that is not absolutely simple.

**Proposition 10.4.** *Let $F$ be a field, $V$ a finite-dimensional $F$-vector space equipped with a non-degenerate bilinear form $\langle \cdot, \cdot \rangle : V \times V \to F$. Let $W$ be a subspace of $V$ with $F$-basis $\{e_1, \ldots, e_m\}$. Then*

$$\dim_F(W/W \cap W^\perp) = \mathrm{rk}((\langle e_i, e_j \rangle)_{i,j=1,\ldots,m}),$$

*where $(\langle e_i, e_j \rangle)_{i,j}$ is the Gram matrix of $W$ with respect to the basis $\{e_1, \ldots, e_m\}$.*

*Proof.* Let $W^* := \mathrm{Hom}_F(W, F)$ and let $\{e_1^*, \ldots, e_m^*\}$ be the basis of $W^*$ dual to $\{e_1, \ldots, e_m\}$. We define an $F$-linear map $\varphi : W \to W^*, w \mapsto \varphi_w$, where $\varphi_w(u) = \langle w, u \rangle$. Then

$$\varphi e_i = \sum_{j=1}^{m} \varphi e_i(e_j) e_j^* = \sum_{j=1}^{m} \langle e_i, e_j \rangle e_j^*.$$

So the matrix corresponding to $\varphi$ with respect to $\{e_1, \ldots, e_m\}$ and $\{e_1^*, \ldots, e_m^*\}$ is exactly the Gram matrix $(\langle e_i, e_j \rangle)_{i,j}$. Since $\ker(\varphi) = W \cap W^\perp$, then

$$\dim(W/W \cap W^\perp) = \dim(\mathrm{Im}(\varphi)) = \mathrm{rk}((\langle e_i, e_j \rangle)_{i,j}).$$ □

**Corollary 10.5.** *Let $\lambda \in \mathcal{P}_n$ and $F$ be any field. Assume that $D^\lambda_F \neq 0$. Then*

$$\dim_F(D^\lambda_F) = \mathrm{rk}(\textit{Gram matrix of } S^\lambda_F \textit{ with respect to the standard basis}).$$

*Moreover,*

$$\dim_F(D^\lambda_F) = \dim_E(D^\lambda_E),$$

*where $E$ is any field with* $\mathrm{char}\, E = \mathrm{char}\, F$.

*Proof.* The first assertion follows from Proposition 10.4. Recall that the standard $\lambda$-polytabloids form an $F$-basis of $S^\lambda_F$. Also, if $t$ is a standard $\lambda$-tableau, then every tabloid occuring in $e_t$ has coefficient 1 or $-1$. Therefore

$$\langle e_t, e_s \rangle \in \begin{cases} \mathbb{F}_p, & \mathrm{char}(F) = p \\ \mathbb{Q}, & \mathrm{char}(F) = 0, \end{cases}$$

where $t$ and $s$ are standard $\lambda$-tableaux. In particular, the rank of the Gram matrix depends only on the characteristic of $F$. This completes the proof. □

**Theorem 10.6.** *Let $F$ be a field of positive characteristic $p$ and let $\lambda \in \mathcal{P}_{n,p}$. Then $(D^\lambda_F)^E \cong D^\lambda_E$ for every extension field $E$ of $F$. In particular, every simple $F\mathfrak{S}_n$-module is absolutely simple.*

*Proof Outline.* We have $S^\lambda_E \cong (S^\lambda_F)^E$ since every $\lambda$-polytabloid is a $\mathbb{Z}$-linear combination of standard $\lambda$-polytabloids. So $S^\lambda_F$ affords a matrix representation $\Delta : \mathfrak{S}_n \to \mathrm{GL}(F)$. Also, $D^\lambda_F$ and $D^\lambda_E$ are the unique simple quotient modules of $S^\lambda_F$ and $S^\lambda_E$, respectively. Note that any quotient of $(D^\lambda_F)^E$ is a quotient module of $(S^\lambda_F)^E \cong S^\lambda_E$. If $(D^\lambda_F)^E$ were not simple, then $D^\lambda_E$ would be a (proper, nonzero) quotient module of $(D_F)^E$. But comparing dimensions using Corollary 10.5, we have

$$\dim_E(D^\lambda_E) = \dim_F(D^\lambda_F) = \dim_E((D^\lambda_F)^E).$$

So $(D^\lambda_F)^E$ must be simple and by uniqueness, we have $(D^\lambda_F)^E \cong D^\lambda_E$. □

**Example 10.7.** We give a small example to illustrate that the rank of the Gram matrix is highly dependent on the characteristic of the field. Consider $\lambda = (3, 2)$. We identify $\lambda$-tabloids with the sets containing their 2nd-row entries. We have $\lambda$-polytabloids

$$\begin{aligned} e_1 &= \{45\} + \{12\} - \{15\} - \{24\}, \\ e_2 &= \{35\} + \{12\} - \{15\} - \{23\}, \\ e_3 &= \{34\} + \{12\} - \{14\} - \{23\}, \\ e_4 &= \{25\} + \{13\} - \{15\} - \{23\}, \\ e_5 &= \{24\} + \{13\} - \{14\} - \{23\}, \end{aligned}$$

and so the Gram matrix of the Specht module $S^{(32)}$ with respect to the above basis of $\lambda$-polytabloids is

$$\begin{pmatrix} 4 & 2 & 1 & 1 & -1 \\ 2 & 4 & 2 & 2 & 1 \\ 1 & 2 & 4 & 1 & 2 \\ 1 & 2 & 1 & 4 & 2 \\ -1 & 1 & 2 & 2 & 4 \end{pmatrix}.$$

In characteristic 2, this matrix reduces to the rank 4 matrix

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

In characteristic 3, it reduces to the rank 1 matrix

$$\begin{pmatrix} 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & -1 & 1 \end{pmatrix}.$$

One can check easily that in characteristic 5, we get a rank 5 matrix, and similarly compute the rank of the Gram matrix in larger characteristics.

## 10.2 Self-duality

**Remark 10.8.** Let $F$ be a field an $V$ a finite dimensional $F$-vector space. Let $V^* = \mathrm{Hom}_F(V, F)$ denote the dual vector space. We set

$$W^\circ := \{f \in V^* : f(w) = 0, \text{ for all } w \in W\}.$$

**Proposition 10.9.** *Let $V$ be an $n$-dimensional $F$-vector space with a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle : V \times V \to F$. Let $U$ and $W$ be $F$-subspaces of $V$. Then we have the following facts.*

*(a) The $F$-lienar map $\varphi : V \to V^*, v \mapsto (u \mapsto \langle u, w \rangle)$ is an $F$-vector space homomorphism with $\varphi(W^\perp) = W^\circ$.*

*(b) $\dim(W) + \dim(W^\perp) = \dim(V)$ and $(W^\perp)^\perp = W$.*

*(c) If $U \subseteq W$ then $W^\perp \subseteq U^\perp$ and $\dim(W/U) = \dim(U^\perp/W^\perp)$.*

*(d) $U^\perp + W^\perp = (U \cap W)^\perp$ and $(U + W)^\perp = U^\perp \cap W^\perp$.*

**Lemma 10.10.** *Let $G$ be a finite group and $M$ and $FG$-module. Then $M^* = \mathrm{Hom}_F(M, F)$ becomes an $FG$-module via*

$$(g \cdot \varphi)(x) = \varphi(g^{-1}x),$$

*for all $\varphi \in M^*, g \in G, x \in M$.*

**Definition 10.11.** In the above situation, we say that $M$ is *self-dual* if $M \cong M^*$ as $FG$-modules.

**Proposition 10.12.** *Let $G$ be a finite group and let $M$ be an $FG$-module with a non-degenerate, symmetric, $G$-invariant $F$-bilinear form $\langle \cdot, \cdot \rangle : M \times M \to F$. Let $U \subseteq V \subseteq M$ be $FG$-modules. Then as $FG$-modules,*

$$V/U \cong (U^\perp/V^\perp)^*$$

*Proof.* We define

$$\varphi : V \to (U^\perp/V^\perp)^*, v \mapsto (\varphi_v : x + V^\perp \mapsto \langle v, x \rangle).$$

It is easy to check that $\varphi$ is well-defined: if $x, x' \in U^\perp$ are such that $x + V^\perp = x' + V^\perp$, then $\langle v, x \rangle = \langle v, x' \rangle$ for all $v \in V$. Since $\langle \cdot, \cdot \rangle$ is $F$-bilinear, then $\varphi$ and $\varphi_v$ (for any $v \in V$) are $F$-linear. We have

$$\ker \varphi = \{v \in V : \langle v, x \rangle = 0 \text{ for all } x \in U^\perp\} = V \cap (U^\perp)^\perp = V \cap U = U,$$

and thus

$$\dim(\mathrm{Im}(\varphi)) = \dim(V/U) = \dim(U^\perp/V^\perp) = \dim((U^\perp/V^\perp)^*).$$

From the above, we can conclude that $\varphi$ is an $F$-vector space isomorphism. But we claim that it is in fact an $FG$-isomorphism, and so the finish the proof we need to show that $\varphi$ is a $FG$-homomorphism.

Let $g \in G$. For $v \in V$ and $x \in U^\perp$, we have

$$\varphi_{gv}(x + V^\perp) = \langle gv, x \rangle = \langle v, g^{-1}x \rangle = \varphi_v(g^{-1}x + V^\perp) = (g \cdot \varphi_v)(x + V^\perp).$$

So $\varphi_{gv} = g \cdot \varphi_v$ and $\varphi$ is indeed an $FG$-homomorphism. ∎

**Theorem 10.13.** *In the notation of Proposition 10.12, $V/(V \cap V^{\perp})$ is a self-dual $FG$-module.*

*Proof.* Using Propositions 10.12 and 10.9, we have the following chain of isomorphisms:

$$V/(V \cap V^{\perp}) \cong (V + V^{\perp})/V^{\perp} \cong ((V^{\perp})^{\perp}/(V + V^{\perp})^{\perp})^{*} \cong (V/V^{\perp} \cap V)^{*}.$$ □

**Corollary 10.14.** *Let $F$ be a field. Then every simple $F\mathfrak{S}_n$-module is self-dual.*

*Proof.* If $\mathrm{char}(F) \nmid n!$, then every simple $F\mathfrak{S}_n$-module $D$ is isomorphic to $S^{\lambda}$ for some $\lambda \in \mathcal{P}_n$; moreover, $S^{\lambda} \cap (S^{\lambda})^{\perp} = 0$. Thus

$$D \cong S^{\lambda} \cong S^{\lambda}/(S^{\lambda} \cap (S^{\lambda})^{\perp})$$

is self-dual by Theorem 10.13. If $\mathrm{char}(F) \,\big|\, n!$, then $D \cong S^{\lambda}/S^{\lambda} \cap (S^{\lambda})^{\perp}$ for some $\lambda \in \mathcal{P}_{n,p}$. Theorem 10.13 gives the desired result. □

# Chapter 11

# Specht Modules in Positive Characteristic

**Lemma 11.1.** *Let $\lambda, \mu \in \mathcal{P}_n$. Let $\varphi \in \mathrm{Hom}_{\mathbb{Q}\mathfrak{S}_n}(M^\lambda_\mathbb{Q}, M^\mu_\mathbb{Q})$ be such that $\varphi(\{t\})$ is a $\mathbb{Z}$-linear combination of $\mu$-tabloids for every $\lambda$-tabloid $\{t\}$. Let $p$ be a prime. Identifying $\varphi$ with its matrix and reducing entries modulo $p$, we get $\overline{\varphi} \in \mathrm{Hom}_{\mathbb{F}_p\mathfrak{S}_n}(M^\lambda_{\mathbb{F}_p}, M^\mu_{\mathbb{F}_p})$. Then*

$$\ker(\varphi) = (S^\lambda_\mathbb{Q})^\perp \implies (S^\lambda_{\mathbb{F}_p})^\perp \subseteq \ker(\overline{\varphi}).$$

*Proof.* The idea of the proof is to construct a $\mathbb{Q}$-basis of $(S^\lambda_\mathbb{Q})^\perp$ such that every basis element is a $\mathbb{Z}$-linear combination of tabloids, and after reduction modulo $p$, we get an $\mathbb{F}_p$-basis of $(S^\perp_{\mathbb{F}_p})^\perp$.

Let $\{f_1, \ldots, f_k\}$ be any $\mathbb{Q}$-basis of $(S^\lambda_\mathbb{Q})^\perp$, and extend it to a $\mathbb{Q}$-basis of $M^\lambda_\mathbb{Q}$ $\{f_1, \ldots, f_m\}$, where $\{f_{k+1}, \ldots, f_m\}$ is the standard basis of $S^\lambda_\mathbb{Q}$. We set

$$N := (n_{ij}) \text{ with } n_{ij} := \langle f_i, \{t_j\}\rangle \text{ for } i, j = 1, \ldots, m,$$

where $\{t_1\}, \ldots, \{t_m\}$ are the $\lambda$-tabloids. We may suppose that $N$ has entries in $\mathbb{Z}$ and that the first $k$ rows are linearly independent modulo $p$. (We do this by Gaussian elimination to get the first $k$ rows to be of the form $(0, \ldots, 0, *, \ldots)$, where $*$ is a $p'$ element of $\mathbb{Z}$ and the number of 0s corresponds to the row.) Reducing all entries modulo $p$, we get $m$ vectors (not necessarily linearly independent) in $M^\lambda_{\mathbb{F}_p}$ whose last $m-k$ vectors form the standard basis of $M^\lambda_{\mathbb{F}_p}$ and whose first $k$ are linearly independent and contained in $(S^\lambda_{\mathbb{F}_p})^\perp$. Comparing dimensions, we have $\dim((S^\lambda_{\mathbb{F}_p})^\perp) = \dim(M^\lambda_{\mathbb{F}_p}) - \dim(S^\lambda_{\mathbb{F}_p})$, and this forces the reduction of $\{f_1, \ldots, f_k\}$ to indeed be a basis $\mathscr{B}$ of $(S^\lambda_{\mathbb{F}_p})^\perp$. We have $\overline{\varphi}(b) = 0$ for $b \in \mathscr{B}$, and hence $\overline{\varphi}((S^\lambda_{\mathbb{F}_p})^\perp) = 0$. This proves the lemma. □

**Theorem 11.2.** *Let $\lambda \in \mathcal{P}_n$ . For any field $F$, we have*

$$(S^{\lambda'}_F)^* \cong S^\lambda_F \otimes \mathrm{sgn}$$

*as $F\mathfrak{S}_n$-modules where* sgn *is the signature representation.*

*Proof.* Let $t$ be a $\lambda$-tableau, and let $t'$ be the $\lambda'$-tableau obtained by transposing $t$. We first consider the case when $F = \mathbb{Q}$. We set

$$\kappa_{t'} := \sum_{\pi \in C_{t'}} \mathrm{sgn}(\pi)\pi, \quad \rho_{t'} := \sum_{\sigma R_{t'}} \sigma$$

and define a $\mathbb{Q}$-linear map

$$\varphi : M^{\lambda'}_\mathbb{Q} \twoheadrightarrow S^\lambda_\mathbb{Q} \otimes \mathrm{sgn}, \{\pi t'\} \mapsto \pi_{\rho_{t'}} \otimes \{t\}, \text{ where } \pi \in \mathfrak{S}_n.$$

Note that

$$\varphi(\{\pi t'\}) = \sum_{\sigma \in R_{t'}} \pi\sigma \cdot \{t\} = \sum_{\sigma \in R_{t'} = C_t} \operatorname{sgn}(\pi)\operatorname{sgn}(\sigma)\pi\sigma\{t\} = \operatorname{sgn}(\pi)\pi\kappa_t\{t\} = \operatorname{sgn}(\pi)\kappa_{\pi t}\{\pi t\}.$$

Now, $\varphi$ is injective and also surjective. Moreover,

$$\varphi(\kappa_{t'}\{t'\}) = \sum_{\pi \in C_{t'} = R_t} \operatorname{sgn}(\pi)\varphi(\{\pi t'\}) = \rho_t\kappa_t\{t\}.$$

Since $\langle \cdot, \cdot \rangle$ is $\mathfrak{S}_n$-invariant,

$$\langle \rho_t\kappa_t\{t\}, \{t\}\rangle = \langle \kappa_t\{t\}, \rho_t\{t\}\rangle = \langle \kappa_t\{t\}, |R_t| \cdot \{t\}\rangle = |R_t| \in \mathbb{Q}\backslash\{0\}.$$

Thus $S_{\mathbb{Q}}^{\lambda'} \not\subseteq \ker(\varphi)$. By the submodule theorem, $\ker(\varphi) \subseteq (S_{\mathbb{Q}}^{\lambda'})^{\perp}$. We can conclude after comparing dimensions that $\ker(\varphi) = (S_{\mathbb{Q}}^{\lambda'})^*$. So we get

$$S_{\mathbb{Q}}^{\lambda} \otimes \operatorname{sgn} \cong M_{\mathbb{Q}}^{\lambda'}/(S_{\mathbb{Q}}^{\lambda'})^{\perp} \cong (S_{\mathbb{Q}}^{\lambda'})^*.$$

Therefore Lemma 11.1 is applicable to this $\varphi$ and the assertion holds for $F = \mathbb{F}_p$. For arbitrary $F$, we can argue by extension of scalars. □