SINGULARITY AND COMPARISON THEOREMS FOR METRICS WITH POSITIVE SCALAR CURVATURE

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Abstract

In this thesis, we discuss metric properties of positive scalar curvature. Metrics with positive scalar curvature naturally arise from various geometric and physical problems. However, some basic questions of positive scalar curvature were unknown. Specifically, can one conclude that the scalar curvature of a metric is positive, just based on measurement by the metric, without taking any derivative?

Such questions are usually answered via geometric comparison theorems. They are also built upon a good understanding of the singular set, along which a sequence of metrics with uniformly bounded curvature degenerate.

The primary contributions of this thesis are twofold: Firstly, we study the effect of uniform Euclidean singularities on the Yamabe type of a closed, boundary-less manifold. We show that, in all dimensions, edge singularities with cone angles $\leq 2\pi$ along codimension-2 submanifolds do not affect the Yamabe type. In three dimensions, we prove the same for more general singular sets, which are allowed to stratify along 1-skeletons, exhibiting edge singularities (angles $\leq 2\pi$) and arbitrary L^∞ isolated point singularities. Secondly, we establish a geometric comparison theorem for 3-manifolds with positive scalar curvature, answering affirmatively a dihedral rigidity conjecture by Gromov. For a large collections of polyhedra with interior non-negative scalar curvature and mean convex faces, we prove that the dihedral angles along its edges cannot be everywhere less or equal than those of the corresponding Euclidean model, unless it is isometric to a flat polyhedron.

From the viewpoint of metric geometry, our results show that $R \geq 0$ is faithfully captured by polyhedra. They suggest the study of "$R \geq 0$" with weak regularity assumptions, and the limit space of manifolds with scalar curvature lower bounds.

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Chapter 1

Introduction

1.1 Riemannian geometry and curvature

We first review some basic notions in Riemannian geometry. Let M^n be a smooth manifold, g a Riemannian metric on M. That is, at each point p, g_p is a positive definite symmetric $(0,2)$ tensor. In this thesis we will consider the various regularity assumptions of the metric g. We say a metric g is of C^k (or $C^{k,\alpha}$) regularity, if in local coordinates $\{x^1, \cdots, x^n\}$, the metric components $g_{ij} = g(\partial_{x^i}, \partial_{x^j})$ are C^k (or $C^{k,\alpha}$) functions. Given a metric g, there exists a unique covariant, torsion-free connection, the Levi-Civita connection ∇. We now define several notions of curvature.

Definition 1.1.1 (Intrinsic curvature)**.** *1. The Riemann curvature tensor is defined as the covariant* 4*-tensor*

$$\begin{aligned}&\mathrm{Rm}_M(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W})\\ &\qquad\triangleq \left\langle \nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{Z} - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{Z} - \nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z}, \mathbf{W}\right\rangle, \mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W} \in \Gamma(TM).\end{aligned}$$

The sectional curvature of a two-plane $L \subset T_pM$ is then defined as

$$\mathrm{Rm}_M(L) \triangleq \frac{\mathrm{Rm}_M(\mathbf{X},\mathbf{Y},\mathbf{Y},\mathbf{X})}{\|\mathbf{X}\|^2\|\mathbf{Y}\|^2 - \langle \mathbf{X},\mathbf{Y}\rangle^2}.$$

2. *The Ricci curvature tensor is defined as the covariant* 2*-tensor*

$$\mathrm{Ric}_M(\mathbf{X},\mathbf{Y}) \triangleq \mathrm{tr}_g \mathrm{Rm}_M(\mathbf{X},\cdot,\cdot,\mathbf{Y}), \mathbf{X},\mathbf{Y} \in \Gamma(TM);$$

   Call $\mathrm{Ric}_M(\mathbf{X},\mathbf{X})$ *the Ricci curvature in the direction* $\mathbf{X} \in T_pM$*.*

3. *The scalar curvature is defined as the function*

$$R_M \triangleq \mathrm{tr}_g \mathrm{Ric}_M .$$

4. *When* $\dim M = 2$*, the Gauss curvature is defined as the function*

$$K_M \triangleq \frac{1}{2} R_M.$$

Simply connected Riemannian manifolds with constant sectional curvature are called space forms. It is well known that an n-dimensional space form must be isometric to the flat Euclidean space $\mathbf{R}^n$, or a sphere S^n_k, or a hyperbolic plane $\mathbf{H}^n_k$, where k is the (constant) sectional curvature.

There are also extrinsic notions of curvature, that describe the extrinsic geometric properties of a submanifold of the ambient manifold M.

Definition 1.1.2 (Extrinsic curvature)**.** *Let* (M^n, g) *be a Riemannian manifold and* Σ^{n-1} *be a hypersurface with unit normal vector field* ν*. The second fundamental form of* Σ *with respect to* ν *is a symmetric covariant* 2*-tensor on* Σ*, defined as*

$$\mathrm{II}_\Sigma(\mathbf{X},\mathbf{Y}) \triangleq \langle \nabla_{\mathbf{X}} \nu, \mathbf{Y} \rangle , \mathbf{X},\mathbf{Y} \in \Gamma(T\Sigma).$$

The mean curvature of Σ *with respect to* ν *is the function on* Σ *defined by*

$$H_\Sigma \triangleq \mathrm{div}_\Sigma \nu = \mathrm{tr}_\Sigma \mathrm{II}_\Sigma,$$

and the mean curvature vector of Σ *is defined by*

$$\mathbf{H}_\Sigma \triangleq -H_\Sigma \nu.$$

Remark 1.1.3. *Under the convention that we take, the unit sphere in $\mathbf{R}^n$ has mean curvature $n-1$ with respect to outward unit normal. Generally, the mean curvature vector $\mathbf{H}_\Sigma$ is independent of the choice of ν.*

The significance of mean curvature is captured by the fact that the mean curvature measures the rate of change of volume. Precisely,

Proposition 1.1.4. *Let Σ^{n-1} be a submanifold of a Riemannian manifold (M^n, g). Let $\phi : \Sigma \times (-\varepsilon, \varepsilon) \to M$ be an one-parameter family of diffeomorphisms generated by the vector field $\mathbf{X}$. Then*

$$\left.\frac{d}{dt}\right|_{t=0} |\phi_t(\Sigma)| = -\int_\Sigma \langle \mathbf{H}_\Sigma, \mathbf{X} \rangle \, d\,\mathrm{Vol}_\Sigma,$$

where $|\phi_t(\Sigma)|$ is the $(n-1)$-dimensional volume of Σ.

As a straightforward consequence, a hypersurface $\Sigma \subset M$ is stationary for the area functional among all deformations, if and only if $\mathbf{H}_\Sigma \equiv 0$.

1.2 The triangle comparison theorem for sectional curvature

A classical question in Riemannian geometry is to characterize curvature conditions via metric properties. For instance, can one conclude that some curvature of a Riemannian manifold is positive, by just taking measurement by the metric itself?

As a first example, we observe that a hypersurface $\Sigma^{n-1} \subset M^n$ is mean convex- its mean curvature with respect to a unit normal vector field ν is nonnegative- may be concluded by only measuring the area. Precisely, we have

Proposition 1.2.1. *A hypersurface $\Sigma^{n-1} \subset M^n$ has nonnegative mean curvature with respect to a unit normal vector field ν, if for any point $p \in \Sigma$, there exists an open neighborhood U of p on Σ and some $\varepsilon > 0$, such that any normal deformation supported in an ε-neighborhood of U in the direction of ν increases the $(n-1)$-dimensional volume.*

Proof. Suppose the contrary. Assume there exists an open set U of Σ where its mean curvature on U is negative. Let X be a vector field supported in U, and pointing in the same direction as ν. Then by the first variational formula of volume,

$$\left.\frac{d}{dt}\right|_{t=0} |\phi_t(\Sigma)| = -\int_\Sigma H_\Sigma \langle X, \nu \rangle < 0,$$

where ϕ_t is the flow generated by X. Therefore the deformations $\phi_t(\Sigma)$ has smaller $(n-1)$-volume than Σ, contradiction. $\square$

Call the property defined in the above Proposition "one-sided area minimizing among small deformations". We conclude that

$$\Sigma \text{ is one-sided area minimizing among small deformations} \Rightarrow H_\Sigma \geq 0.$$

Moreover, the notion of "one-sided area minimizing among small deformations" makes sense without taking any derivative of the metric g, and therefore may be defined for C^0 metrics.

A natural question in the same spirit is to characterize intrinsic curvature being bounded via the metric g. Such questions are usually answered by comparison theorems. We start by reviewing the following well-known theorem, proved by Alexandrov [2] in 1951.

Theorem 1.2.2 (Triangle comparison for sectional curvature)**.** *Let (M^n, g) be a Riemannian manifold. Then for any $k \in \mathbf{R}$, the following two conditions are equivalent:*

- *The sectional curvature $\mathrm{Rm}_M \geq k$, for any two-plane in the tangent space of any point on M.*
- *The triangle comparison principle holds: take any geodesic triangle Δpqr. Let $\Delta p'q'r'$ be the triangle in the space form g_k with equal side lengths. Then for any points $x \in \overline{qr}$ and $x' \in \overline{q'r'}$ such that x, x' bisects the corresponding sides into the same ratio, $\mathrm{dist}_g(p, x) \geq \mathrm{dist}_{g_k}(p', x')$.*

Moreover, if $\mathrm{Rm}_M \geq k$, and the equality in triangle comparison holds for any choice *of Δpqr and $x \in \overline{qr}$, the universal cover of (M, g) is isometric to the space form of constant curvature k.*

Notice that, the triangle comparison principle makes sense for any **metric spaces** (M, d), where the there exists a notion of distance between each pair of points. Using this triangle comparison principle, Alexandrov introduced the definition of "sectional curvature bounded from below by k" on metric spaces (M, d)- the so-called Alexandrov spaces.

Similar questions for different curvature types have captured mathematicians' interests. The notion of "Ricci curvature" bounded from below has been successfully introduced by Lott-Villani [29] and independently by Sturm [55, 56, 57], via an entropy comparison theorem in optimal transport.

As one purpose of this thesis, we study a similar question for scalar curvature, and verify a comparison principle which was first proposed by Gromov [19].

1.3 Polyhedra and singular rigidity phenomena for positive scalar curvature

As triangle comparison theorems characterize sectional curvature lower bounds, Gromov proposed that polyhedra should be of great importance for the study of scalar curvature. We define:

Definition 1.3.1. *Let P be a flat polyhedron in $\mathbf{R}^n$. A closed Riemannian manifold M^n with non-empty boundary is called a P-type polyhedron, if it admits a Lipschitz diffeomorphism $\phi : M \to P$, such that ϕ^{-1} is smooth when restricted to the interior, the faces and the edges of P. We thus define the faces, edges and vertices of M as the image of ϕ^{-1} when restricted to the corresponding objects of P.*

The first case Gromov investigated in [19] was the following comparison principle for Riemannian **cubes**. We set it as our first goal:

Goal 1.3.2. *Let $(M, g)^n$ be a Riemannian cube with $R(g) \geq 0$ in the interior. Assume that each face of M is mean convex with respect to the outward unit normal vector. Then the dihedral angle of M cannot be everywhere less than $\frac{\pi}{2}$.*

As mentioned before, the property of a hypersurface being "mean convex" is implied by the property "one-sided volume minimizing", and hence makes sense for metrics that are only C^0. Also the dihedral angles is measured by just the metric itself. As a result, one may use this comparison principle as the definition of "scalar curvature is nonnegative" for C^0 metrics g.

Gromov has also proposed a strategy to approach Goal 1.3.2, which we will describe below. For simplicity, let us assume that $\dim M = 3$, as the same argument with obvious adjustment works for all dimensions. Assume that there does exist a cube M^3 with interior nonnegative scalar curvature, mean convex faces and everywhere acute dihedral angles. Take the doubling of M across its right face. Take the resulted cube, and make the doubling of it across its top face. Take the resulted cube, and make the doubling of it across its front face. After these three times of doubling, the resulted cube $\tilde{M}$ has isometric opposite faces. Identify the opposite faces of $\tilde{M}$ and obtain a torus T^3 with a singular metric $\tilde{g}$. Due to the geometric assumptions, the metric $\tilde{g}$ has positive scalar curvature away from a stratified singular set $S = F^2 \cup L^1 \cup V^0$, where:

1. $\tilde{g}$ is smooth on both sides from F^2. The mean curvatures of F^2 from two sides satisfy a positive jump;

2. $\tilde{g}$ is an edge metric along L^1 with angle less than 2π;

3. $\tilde{g}$ is bounded measurable across isolated vertices V^0.

Gromov then observed that the geometric conditions on F^2 and L^1 should imply that the scalar curvature of $\tilde{g}$ is nonnegative, and that the dimension of V^0 is too low to effect scalar curvature. Notice that the torus do not admit any smooth metric with nonnegative scalar curvature, unless the metric is flat [42][18]. Moreover, we have the following stronger result concerning the rigidity of scalar curvature:

Theorem 1.3.3 (See [22, 40]). *Let M^n, $n \geq 3$, be closed. Then*

$$\sigma(M) > 0 \iff M \text{ carries a smooth metric } g \text{ with } R(g) > 0. \tag{1.3.1}$$

Moreover,

$$R(g) \geq 0 \text{ and } \sigma(M) \leq 0 \implies \operatorname{Ric}(g) \equiv 0. \tag{1.3.2}$$

Gromov's strategy leads naturally to the following question:

Question 1.3.4 (Weakly nonnegative scalar curvature, globally)**.** *Suppose g is an L^∞ metric on M that is smooth away from a compact subset $\mathcal{S} \subset M$. What conditions on $\mathcal{S}$, g, ensure that*

$$\begin{aligned} &R(g) \geq 0 \text{ on } M \setminus \mathcal{S} \text{ and } \sigma(M) \leq 0 \\ &\qquad\qquad \implies g \text{ extends smoothly to } M \text{ and } \operatorname{Ric}(g) \equiv 0? \end{aligned} \tag{1.3.3}$$

Said otherwise, under what conditions on $\mathcal{S}$, g does M carry singular *metrics with nonnegative scalar curvature (on the regular part), but no such* smooth *metrics?*

We focus on the metric which is of class L^∞, as defined below:

Definition 1.3.5 (Uniformly Euclidean (L^∞) metrics)**.** *We define the class of L^∞ metrics on a closed manifold M to consist of all measurable sections of* $\operatorname{Sym}_2(T^*M)$ *such that*

$$\Lambda^{-1} g_0 \leq g \leq \Lambda g_0 \text{ a.e. on } M$$

for some smooth metric g_0 on M and some $\Lambda > 0$.

We briefly discuss what is known on Question 1.3.4. Let us also survey the previous (singular) Positive Mass Theorem results because, in the smooth setting, [28, Section 6], [41, Proposition 5.4], and Theorem 1.3.3 *imply* the Positive Mass Theorem of Schoen-Yau [45] and Witten [64] for complete asymptotically flat (M^n, g) with $R(g) \geq 0$.

The case that is best understood is

$$\operatorname{codim}(\mathcal{S} \subset M) = 1,$$

where $\mathcal{S}$ is a closed embedded hypersurface with trivial normal bundle and where the ambient metric g induces the same smooth metric on $\mathcal{S}$ from both sides.

One *cannot* hope for Question 1.3.4 to be valid in such generality. To maintain any hope of validity, one must make an additional *geometric* assumption: the sum of mean curvatures of $\mathcal{S}$ computed with respect to the two unit normals as outward unit normals has to be nonnegative. We will discuss this condition in section 2.7.1.

There have been three approaches, all subject to the geometric assumption just described. The first, and closest in spirit to this thesis, is to combine the conformal method with arbitrarily fine desingularizations that are aware of the ambient geometry; this was first carried out in the positive mass setting by Miao [35]. The second is to use Ricci flow as a smoothing tool; see [46]. The third is to use spinors; see [47, 23] for positive mass theorems.

Codimension 2.

Much less is known when $\mathcal{S} \subset M$ is a closed embedded submanifold with

$$\operatorname{codim}(\mathcal{S} \subset M) \geq 2.$$

Nonetheless, one can *still* not expect Question 1.3.4 to be valid in such generality and needs to decide on additional geometric assumptions; see, e.g., section 2.7.2 for counterexamples.

One approach, which we won't pursue, is to strengthen the regularity assumptions on g; to that end, Shi-Tam [46] proved (using Ricci flow) that (1.3.3) is *true* if g is Lipschitz across $\mathcal{S}$.

In our work in codimension two, we opt to keep the low (L^∞) regularity assumption and instead study metrics whose singularities are of "edge" type (see Definition 2.1.2), which consolidate Gromov's tentative approach for the comparison theory together with the study of singularities in Einstein manifolds. Edge singularities have been studied intensively recently due to the Yau-Tian-Donaldson program in Kähler-Einstein geometry; see, e.g., [11, 12, 13, 59, 21], or [7] for non-complex-geometric results in (real) dimension four. See section 2.7.4 for examples of edge metrics.

Our first theorem deals with codimension two edge singularities in all dimensions, $n \geq 3$:

Theorem 1.3.6. *Let M^n be closed, with $\sigma(M) \leq 0$, and g a metric such that:*

1. $g \in L^\infty(M) \cap C^2_{\mathrm{loc}}(M \setminus \mathcal{S})$, *$\mathcal{S} \subset M$ is a codimension-2 closed submanifold, and g is an η-regular edge metric along $\mathcal{S}$ with $\eta > 2 - \frac{4}{n}$ and cone angles* $0 < 2\pi(\beta + 1) \leq 2\pi$,
2. $R(g) \geq 0$ *on* $M \setminus \mathcal{S}$.

Then g extends to a smooth Ricci-flat metric everywhere on M.

We note, in Section 2.7.2, that Theorem 1.3.6 would be *false* if one were to allow edge metrics with cone angles $> 2\pi$.

Despite recurring success in the study of Einstein metrics, the role of edge metrics in scalar curvature geometry has not been understood with depth. We expect general stratified singular sets with edge singularities along the codimension two strata to appear in the study of singular scalar curvature in a natural way. See, e.g., Akutagawa-Carron-Mazzeo [1] for the singular Yamabe problem in this setting.

Codimension 3.

Rick Schoen has conjectured that the situation is drastically different in codimension three than in codimensions one or two: one shouldn't need any additional regularity assumptions beyond L^∞ for (1.3.3) to hold true:

Conjecture 1.3.7. *Suppose g is an L^∞ metric on M that is smooth away from a closed, embedded submanifold $\mathcal{S} \subset M$ with* $\operatorname{codim}(\mathcal{S} \subset M) \geq 3$. *Then*

$$R(g) \geq 0 \text{ on } M \setminus \mathcal{S} \text{ and } \sigma(M) \leq 0 \implies g \text{ extends smoothly to } M \text{ and } \operatorname{Ric}(g) \equiv 0.$$

We confirm Conjecture 1.3.7, when $n \,(= \dim M) = 3$, as a corollary to our second theorem.

Corollary 1.3.8. *Let M^3 be closed, with $\sigma(M) \leq 0$. If $\mathcal{S} \subset M$ is a finite set, g is an $L^\infty(M) \cap C^{2,\alpha}_{\mathrm{loc}}(M \setminus \mathcal{S})$ metric, $\alpha \in (0, 1)$, and $R(g) \geq 0$ on $M \setminus \mathcal{S}$, then g is a smooth flat metric everywhere on M.*

Our second theorem is specific to the three-dimensional case, where we allow *stratified* singular sets of codimension two. We prove (see Definitions 2.1.2, 2.1.3):

Theorem 1.3.9. *Let M^3 be closed, with $\sigma(M) \leq 0$, and g a metric such that:*

1. $g \in L^\infty(M) \cap C^{2,\alpha}_{\text{loc}}(M \setminus \mathcal{S})$, $\alpha \in (0,1)$, *where $\mathcal{S} \subset M$ is a compact nondegenerate 1-skeleton, and is an η-regular edge metric along $\operatorname{reg} \mathcal{S}$ with $\eta > \frac{2}{3}$ and cone angles $0 < \delta \leq 2\pi(\beta+1) \leq 2\pi$,*

2. $R(g) \geq 0$ *on* $M \setminus \mathcal{S}$.

Then g extends to a smooth flat metric everywhere on M.

The techniques we use here also applies to asymptotically flat manifolds. As a result, we also obtain positive mass theorems, together with rigidity statement, in asymptotically flat manifolds with analogous singularities:

Theorem 1.3.10. *Let (M^n, g) be a complete asymptotically flat manifold, such that:*

1. $g \in L^\infty(M) \cap C^2_{\text{loc}}(M \setminus \mathcal{S})$, *$\mathcal{S} \subset M \setminus \partial M$ is a closed codimension two submanifold, and g is an η-regular edge metric along $\mathcal{S}$ with $\eta > 2 - \frac{4}{n}$ and cone angles $0 < 2\pi(\beta+1) \leq 2\pi$,*

2. $\partial M = \emptyset$, *or its mean curvature vectors vanish or point inside M,*

3. $R(g) \geq 0$ *on* $M \setminus \mathcal{S}$.

Then the ADM mass of each end of M is nonnegative. Moreover, if the mass of any end is zero, then $(M^n, g) \cong (\mathbf{R}^n, \delta)$.

Theorem 1.3.11. *Let (M^3, g) be a complete asymptotically flat three-manifold, such that:*

1. $g \in L^\infty(M) \cap C^{2,\alpha}_{\text{loc}}(M \setminus \mathcal{S})$, $\alpha \in (0,1)$, *with $\mathcal{S} \subset M \setminus \partial M$ a compact nondegenerate 1-skeleton, so that g is an η-regular edge metric along $\operatorname{reg} \mathcal{S}$ with $\eta > \frac{2}{3}$ and cone angles $0 < \delta \leq 2\pi(\beta+1) \leq 2\pi$,*

2. $\partial M = \emptyset$, *or its mean curvature vectors vanish or point inside M,*

3. $R(g) \geq 0$ *on* $M \setminus \mathcal{S}$*,*

Then the ADM mass each end of M *is nonnegative. Moreover, if the mass of any end is zero, then* $(M, g) \cong (\mathbf{R}^3, \delta)$*.*

This constructive approach has also been pursued in part for reasons of compatibility with the Sormani-Wenger [54] notion of "intrinsic flat" distance between Riemannian manifolds, which (see [53]) work of Gromov [19] suggests is the "correct" notion for taking limits of manifolds with lower scalar curvature bounds. See Section 2.7.5 for more discussion.

1.4 A variational approach to general polyhedra comparison theorems

Gromov's idea of the cube comparison theorem relies on the fact that cubes are the fundamental domains of the $\mathbf{Z}^n$ actions on $\mathbf{R}^n$, hence is not applicable to general polyhedra. An interesting question is then: which types of polyhedra share properties like those observed by Gromov for cube-type polyhedra in manifolds with nonnegative scalar curvature? In particular, it is conjectured by Gromov that an analogous comparison property should be satisfied by the regular tetrahedron in $\mathbf{R}^3$.

Another related question concerns the rigidity statement of Goal 1.3.2. Namely, if a cube M^n satisfies that $R(M) \geq 0$ in the interior, that the faces are all mean convex, and that the dihedral angles are everywhere less or equal than $\frac{\pi}{2}$, then is it necessary true that M is isometric to an Euclidean rectangular solid? More generally, one could ask what types of polyhedra are "mean convexly extremetal"? Surprisingly, this question is unsettled even in Euclidean spaces.

Conjecture 1.4.1 (Dihedral rigidity conjecture, section 2.2 of [19])**.** *Let* $P \in \mathbf{R}^n$ *be a convex polyhedron with faces* F_i*. Let* $P' \subset \mathbf{R}^n$ *be a* P*-type polyhedron with faces* F_i'*. If*

1. each F_i' *is mean convex, and*

2. *the dihedral angles satisfy* $\measuredangle'_{ij}(P') \leq \measuredangle_{ij}(P)$,

then P' *is flat.*

In the second part of this thesis we confirm this conjecture for a large collections of polyhedra in dimension 3. Let us define two general polyhedron types.

Definition 1.4.2. 1. *Let* $k \geq 3$ *be an integer. In* $\mathbf{R}^3$, *let* $B \subset \{x^3 = 0\}$ *be a convex* k*-polygon, and* $p \in \{x_3 = 1\}$ *be a point. Call the set*

$$\{tp + (1-t)x : t \in [0,1], x \in B\}$$

a (B, p)*-cone. Call* B *the base face and all the other faces side faces.*

2. *Let* $k \geq 3$ *be an integer. In* $\mathbf{R}^3$, *let* $B_1 \subset \{x^3 = 0\}, B_2 \subset \{x_3 = 1\}$ *be two* <u>*similar*</u> *convex* k*-polygons whose corresponding edges are parallel (i.e. the polygons are congruent up to scaling but not rotation). Call the set*

$$\{tp + (1-t)q : t \in [0,1], p \in B_1, q \in B_2\}$$

a (B_1, B_2)*-prism. Call* B_1, B_2 *the base faces and all the other faces side faces.*

If (M, g) *is a Riemannian polyhedron of* P*-type, where* P *is a* (B, p)*-cone (or a* (B_1, B_2)*-prism), we call* (M, g) *is of cone type (prism type, respectively).*

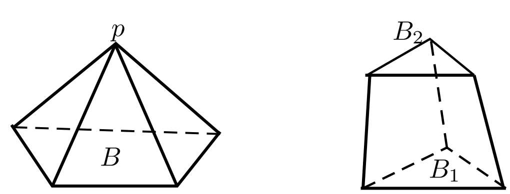

Figure 1.1: A (B, p)-cone and a (B_1, B_2)-prism.

The major objects we consider are Riemannian polyhedra (M^3, g) of cone type or prism type, as in Definition 1.4.2. Let us fix some notations that will be used throughout the thesis. We use $F_1, \cdots, F_k$ to denote the side faces of M; if M is of cone type, we use p to denote the cone vertex, and B to denote its base face; if M is of prism type, we use B_1, B_2 to denote its two bases. Let $F = \cup_{j=1}^k F_j$ be the union of all side faces. Our first theorem makes a comparison between Riemannian polyhedra with nonnegative scalar curvature and their Euclidean models:

Theorem 1.4.3. *Let (M^3, g) be a Riemannian polyhedron of P-type with side faces $F_1, \cdots, F_k$, where $P \subset \mathbf{R}^3$ is a cone or prism with side faces $F_1', \cdots, F_k'$. Denote γ_j the angle between F_j' and the base face of P (if P is a prism, fix one base face). Assume that everywhere along $F_j \cap F_{j+1}$,*

$$|\pi - (\gamma_j + \gamma_{j+1})| < \measuredangle(F_j, F_{j+1}) < \pi - |\gamma_j - \gamma_{j+1}|. \tag{1.4.1}$$

Then the strict comparison statement holds for (M, g). Namely, if $R(g) \geq 0$, and each F_j is mean convex, then the dihedral angles of M cannot be everywhere less than those of P.

In fact, it is not hard to argue as in [19] that the converse of Theorem 1.4.3 is also true: on a three-manifold with negative scalar curvature, one may construct a polyhedron which entirely invalidates the conclusions of Theorem 1.4.3. Thus the metric properties introduced by Theorem 1.4.3 faithfully characterize $R(g) \geq 0$.

A more refined analysis enables us to characterize the rigidity behavior for Theorem 1.4.3, thus answering Conjecture 1.4.1 for cone type and prism type polyhedra, with the very mild a priori angle assumptions (1.4.1). In fact, we obtain:

Theorem 1.4.4. *Under the same assumptions of Theorem 1.4.3 and the extra assumption that*

$$\gamma_j \leq \pi/2, j = 1, 2, \cdots, k, \quad \text{or} \quad \gamma_j \geq \pi/2, j = 1, 2, \cdots, k, \tag{1.4.2}$$

we have the rigidity statement. Namely, if $R(g) \geq 0$, each F_j is mean convex, and $\measuredangle_{ij}(M, g) \leq \measuredangle_{ij}(P, g_{Euclid})$, then (M, g) is isometric to a flat polyhedron in $\mathbf{R}^3$.

The angle assumption (1.4.1) may be regarded as a mild regularity assumption on (M, g). It is satisfied, for instance, by any small C^0 perturbation of the Euclidean polyhedron model P. Moreover, assumption (1.4.1) is vacuous, if all the angles γ_j

are $\pi/2$. In this case, the Euclidean model is a prism with orthogonal base and side faces, and we are able to obtain the prism inequality in section 5.4, [19] as a special case.

Motivated by the Schoen-Yau dimension reduction argument [42], we have also been able to generalize Theorem 1.4.3 and Theorem 1.4.4 in higher dimensions. We will study them in the future.

Now let us indicate the strategy of the proof for Theorem 1.4.3 and Theorem 1.4.4. Let us start with the cube comparison theorem in dimension $n = 2$. In this case, the cubes will be squares.

Proposition 1.4.5. *Let M^2 be a square such that the Gauss curvature of M is positive in its interior. Assume also that each edge of M is convex, namely, its geodesic curvature with respect to outward unit normal is nonnegative. Then the four inner angles of M cannot be all acute.*

Proof 1. The first proof we give is based on the Gauss-Bonnet theorem. By the Gauss-Bonnet theorem,

$$\int_M K_M dA + \int_{\partial M} k_g ds + \sum_{j=1}^{4} (\pi - \alpha_j) = 2\pi,$$

where K_M, k_g, α_j are the interior Gauss curvature, the boundary geodesic curvature, and the inner angles at the corners of M. We immediately see that

$$K_M \geq 0, k_g \geq 0, \alpha_j < \pi/2, j = 1, \cdots, 4,$$

will yield a contradiction. □

Proof 2. The disadvantage of Proof 1 is that it is hard to generalize to higher dimensions, due to the lack of a relevant Gauss-Bonnet theorem. The second proof we present here is variational in nature, and is also generalizable. Assume, for the sake of contradiction, that a square M exists with

$$K_M \geq 0, k_g \geq 0, \alpha_j < \pi/2, j = 1, \cdots, 4.$$

We may assume, without loss of generality, that the geodesic curvature k_g is strictly positive. This may be achieved by pushing out the boundary curves of M a bit, while keeping the inner angles at the corners acute.

Denote the four corners a, b, c, d, and we use $\overline{ab}$ to denote the edge connecting the vertex a and b. Consider the variational problem

$$\inf\{|\gamma| : \gamma : [0,1] \to M \text{ a smooth curve}, \gamma(0) \in \overline{ab}, \gamma(1) \in \overline{cd}, \gamma \cap \overline{bc}, \overline{ad} = \emptyset\}.$$

In other words, we consider the shortest distance between the edges $\overline{ab}$ and $\overline{cd}$, among all the curves that are disjoint from $\overline{bc}$ and $\overline{ad}$. By Arzelà-Ascoli theorem, the infimum is achieved by a curve γ. The fact that the inner angles at the four vertices are all less than $\pi/2$ implies that the length-minimizing curve cannot touch the four vertices, otherwise by moving away from the vertices, one may strictly decrease the length of the curve. Also, the fact that the curves $\overline{bc}, \overline{ad}$ are convex means that the interior of γ does not touch $\overline{ad}$ or $\overline{bc}$. As a result, γ is a length-minimizing curve. Denote p, q the two endpoints of γ.

Let N be the normal vector of γ, dl the arc-length parameter. Then the second variation of length in the direction of fN is given by

$$\delta^2|\gamma|(fN) = \int_\gamma [(f')^2 - (K + k_\gamma^2)f^2]dl - (k_g(p) + k_g(q))f^2,$$

where k_γ is the geodesic of γ, k_g is the geodesic curvature of the boundary $\overline{bc}, \overline{ad}$ with respect the outward unit normal vector field. Taking $f = 1$, we find that

$$\delta^2|\gamma|(N) = -\int_\gamma (K + k_\gamma^2) - (k_g(p) + k_g(q)) < 0,$$

therefore deforming in the direction of N will decrease the length of γ, a contradiction to the minimality of γ. $\square$

The proof for Theorem 1.4.3 and Theorem 1.4.4 in dimension 3 may be viewed as a generalization of the above argument. Precisely, consider the following energy functional:

$$\mathcal{F}(E) = \mathcal{H}^2(\partial E \cap \mathring{M}) - \sum_{j=1}^{k} (\cos \gamma_j) \mathcal{H}^2(\partial E \cap F_j), \tag{1.4.3}$$

and the variational problem

$$\mathcal{I} = \inf\{\mathcal{F}(E) : E \in \mathscr{E}\}, \tag{1.4.4}$$

here $\mathscr{E}$ is the collection of contractible open subset E' such that: if M is of cone type, then $p \in E'$ and $E' \cap B = \emptyset$; if M is of prism type, then $B_2 \subset E'$ and $E' \cap B_1 = \emptyset$. If the solution to (1.4.4) is regular, its boundary $\Sigma^2 = \partial E \cap \mathring{M}$ is called a capillary minimal surface. That is, Σ is a minimal surface that contacts each side face F_j at constant angle γ_j. The existence, regularity and geometric properties of capillary surfaces have attracted a wealth of research throughout the rich history of geometric variational problems. We refer the readers to the book of Finn [15] for a beautiful and thorough introduction.

Our first observation is that $\mathcal{I}$ is always finite: since M is compact, we deduce that

$$\mathcal{I} \geq -\sum_{j=1}^{k} (\cos \gamma_j) \mathcal{H}^2(F_j) > -\infty.$$

Thus a minimizing sequence exists. The existence and boundary regularity of the solution to (1.4.4) was treated by Taylor [58] (see page 328-(6); see also the discussion for more general anisotropic capillary problems by De Philippis-Maggi [14]). Using the language of integral currents, Taylor proved the existence of the minimizer Σ, and that Σ is C^∞ regular up to its boundary, where ∂M is smooth. However, the variational problem (1.4.4) has obstacles: the base face(s) of M. To overcome this difficulty, we apply the interior varifold maximum principle [52] and a new boundary maximum principle, and reduce (1.4.4) to a variational problem without obstacles. We then adapt ideas from Simon [49] and Lieberman [25], and obtain a $C^{1,\alpha}$ regularity property of Σ at its corners. This is the only place we need to use the angle assumption (1.4.1). Next, we unveil the connection between interior scalar curvature, the boundary mean curvature and the dihedral angle captured by the variational problem (1.4.4), and derive various geometric consequences with Σ. We prove Theorem 1.4.3 with the second variational inequality and the Gauss-Bonnet formula. We then proceed to the proof of Theorem 1.4.4, where an analysis for the "infinitesimally rigid" minimal capillary surface Σ is carried out, with the idea pioneered by Bray-Brendle-Neves [9]. The new challenge here is to deal with the case when $\mathcal{I} = 0$. We develop a new general existence result of constant mean curvature capillary foliations near the vertex p, and establish the dynamical behavior of such foliations in nonnegative scalar curvature.

Chapter 2

Positive scalar curvature rigidity with skeleton singularities

In this chapter we study rigidity phenomena for positive scalar curvature, and prove Theorem 1.3.9, Theorem 1.3.6, Theorem 1.3.11 and Theorem 1.3.10.

2.1 Edge singularities

The starting point of our discussion is the classical example of *isolated conical singularities* on two-dimensional Riemannian manifolds.

Assume M is a closed Riemann surface, $\{p_1, \ldots, p_k\} \subset M$, and g is an $L^\infty(M) \cap C^2_{\mathrm{loc}}(M \setminus \{p_1, \ldots, p_k\})$ metric. We call p_i, $i = 1, \ldots, k$, an isolated conical singularity with cone angle $2\pi(\beta_i + 1)$, $\beta_i \in (-1, \infty)$, if around p_i there exist coordinates so that

$$g = dr^2 + (\beta_i + 1)^2 r^2 d\theta^2. \tag{2.1.1}$$

See Figure 2.2 for a graphical illustration of a model isolated conical singularity.

Remark 2.1.1. *In complex geometry one often works with the complex variable*

$$z = [(\beta_i + 1)r]^{1/(\beta_i+1)} e^{\sqrt{-1}\theta} \in \mathbf{C} \setminus \{0\},$$

asserting that $g = |z|^{2\beta_i}|dz|^2$, $z \neq 0$. *We will not pursue this here.*

The Gauss-Bonnet formula in this setting of isolated conical singularities is

$$\int_{M\setminus\{p_1,\ldots,p_k\}} K_g \, d\,\mathrm{Area}_g - 2\pi \sum_{i=1}^{k} \beta_i = 2\pi\chi(M). \tag{2.1.2}$$

This can be seen, for instance, by excising arbitrarily small disks around the conical points and taking limits. (See also Lemma 2.2.1 below.)

As a straightforward corollary of (2.1.2), the presence of conical singularities all of whose cone angles are $\leq 2\pi$ does *not* affect the Yamabe type of M. On the other hand, conical singularities with cone angle bigger than 2π can affect the Yamabe type in the negative. We give an example in Section 2.7.2.

Let's proceed to the more interesting higher dimensional analog. A natural extension of the previous situation to higher dimensions leads to the definition of an edge singularity. Qualitatively, the singular metric g may be viewed as a family of two-dimensional conical metrics along a smooth $(n-2)$-dimensional submanifold.

Definition 2.1.2 (Edge singularities)**.** *Let* $N^{n-2} \subset M^n$ *be a codimension-2 submanifold (without boundary). We call* g *an* η*-regular edge metric along* N *with data* $(\eta, \beta, \sigma, \omega, \varrho, h)$, *where* $\eta \in (0,\infty)$, $\beta : N \to (-1,\infty)$ *is* C^2, σ *is a* C^2 *1-form on* N, ω *is a* C^2 *metric on* N, $\varrho : N \to (0,\infty)$ *is* C^2 *on* N, h *is a* C^2 *symmetric 2-tensor on* U, *if for some open set* $U \supseteq N$,

$$g = dr^2 + (\beta+1)^2 r^2 (d\theta + \sigma)^2 + \omega + r^{1+\eta} h \text{ on } U \setminus N, \tag{2.1.3}$$

$$\{(r,\theta,y) : r < \varrho(y), \theta \in \mathbf{S}^1, y \in N\} \subseteq U, \tag{2.1.4}$$

and

$$\begin{aligned} &\|\beta\|_{C^2(N)} + \|\sigma\|_{C^2(N)} + \|\omega\|_{C^2(N)} + \|(\det\omega)^{-1}\|_{C^0(N)} \\ &\qquad + \|\varrho\|_{C^1(N)} + \|\varrho^{-1-\eta}\partial^2\varrho\|_{C^0(N)} + \|h\|_{C^2(U)} < \infty. \end{aligned} \tag{2.1.5}$$

Specifically, we require that U *can be covered with Euclidean local coordinate charts*

$(x^1, x^2, y^1, \ldots, y^{n-2})$, where $re^{\sqrt{-1}\theta} = x^1 + \sqrt{-1}x^2$ and $(y^1, \ldots, y^n) \in N$, in which

$$\|\beta\|_{L^\infty(N)} + \|\partial_i \beta\|_{C^0(N)} + \|\partial_i \partial_j \beta\|_{L^\infty(N)} < \infty,$$

$$\|\sigma_i\|_{L^\infty(N)} + \|\partial_i \sigma_j\|_{L^\infty(N)} + \|\partial_i \partial_j \sigma_k\|_{L^\infty(N)} < \infty,$$

$$\|(\det \omega_{ij})^{-1}\|_{L^\infty(N)} + \|\omega_{ij}\|_{L^\infty(N)} + \|\partial_i \omega_{jk}\|_{L^\infty(N)} + \|\partial_i \partial_j \omega_{k\ell}\|_{L^\infty(N)} < \infty,$$

$$\|h_{\alpha\beta}\|_{L^\infty(U)} + \|\partial_\alpha \omega_{\beta\gamma}\|_{L^\infty(Y)} + \|\partial_\alpha \partial_\beta \omega_{\gamma\delta}\|_{L^\infty(U)} < \infty,$$

$$\|\varrho\|_{L^\infty} + \|\partial_i \varrho\|_{L^\infty(N)} + \|\varrho^{-\eta} \partial_i \partial_j \varrho\|_{L^\infty(N)} < \infty.$$

Latin indices only run through $(y^1, \ldots, y^{n-2})$ on N, while Greek indices run through all coordinates $(x^1, x^2, y^1, \ldots, y^{n-2})$ on U.

This definition is taken from [7, (1.1)-(1.2)], and has corresponding analogs in Kähler-Einstein geometry. The ϱ structural requirement did not appear in [7], which only considered compact manifolds, but it is needed here for our general smoothing procedure in case N is noncompact. (Notice that the ϱ-requirement is trivially true when N is compact.) It is a mild requirement that stipulates that our domain of validity of the cone expansion does not degenerate too wildly near the endpoints.

We conclude our collection of definitions with the notion of skeletons:

Definition 2.1.3 (Skeletons). *We say that a compact subset $\mathcal{S} \subset M$ is an $(n-2)$-skeleton if $\mathcal{S} = N_1 \cup \cdots \cup N_k$, where $N_1, \ldots, N_k \subset M$ are compact submanifolds-with-boundary (possibly empty), each with dimension $\leq n-2$, and which are such that $N_\ell \cap N_{\ell'} \subset \partial N_\ell \cup \partial N_{\ell'}$ for all ℓ, ℓ'. We denote*

$$\operatorname{reg} \mathcal{S} := \bigcup \Big\{ \mathcal{S} \cap W : W \subset U \text{ is open and } \mathcal{S} \cap W \text{ is a smooth}$$
$$(n-2) - \text{dimensional submanifold (without boundary)} \Big\},$$

and $\operatorname{sing} \mathcal{S} := \mathcal{S} \setminus \operatorname{reg} \mathcal{S}$. *A skeleton $\mathcal{S}$ is said to be* nondegenerate *if there are no two inner-pointing conormals of $\partial N_\ell \subset N_{\ell'}$ and $\partial N_\ell \subset N_{\ell'}$ $(\ell \neq \ell')$ that coincide.*

One could ostensibly also want to allow higher stratum singularities (i.e., codimension one) away from $\mathcal{S}$ (e.g., in the spirit of Miao [35], [46]). We do not pursue this direction in this thesis.

2.2 Smoothing edge singularities, I

We will prove the following smoothing lemma.

Lemma 2.2.1. *Let $W \subset M$ be a precompact open set containing a nondegenerate $(n-2)$-skeleton $\mathcal{S} \subset M$, and suppose that $g \in C^{2,\alpha}_{\mathrm{loc}}(W \setminus \mathcal{S})$, $\alpha \in [0,1]$, is an η-regular edge metric along $\operatorname{reg}\mathcal{S}$ with data $(\eta, \beta, \sigma, \omega, \varrho, h)$ satisfying*

$$0 < \Lambda^{-1} \leq \inf_{\operatorname{reg}\mathcal{S}} 2\pi(\beta+1) \leq \sup_{\operatorname{reg}\mathcal{S}} 2\pi(\beta+1) \leq 2\pi, \tag{2.2.1}$$

and

$$\begin{aligned}(\eta - 2 + \tfrac{4}{n})^{-1} + \|(\det\omega)^{-1}\|_{L^\infty} + \sum_{j=1}^{2} \|\partial^j \beta\|_{L^\infty} + \sum_{j=0}^{1} \|\partial^j \varrho\|_{L^\infty} \\ + \|\varrho^{-\eta}\partial^2\varrho\|_{L^\infty} + \sum_{j=0}^{2} + \|\partial^j\sigma\|_{L^\infty} + \|\partial^j\omega\|_{L^\infty} + \|\partial^j h\|_{L^\infty} \leq \Lambda. \end{aligned} \tag{2.2.2}$$

See Definition 2.1.2 for the notation. If $R(g) \geq 0$ on $W \setminus \mathcal{S}$, then for every $W' \subset\subset W$ containing the ϱ-normal tubular neighborhood of $\operatorname{reg}\mathcal{S}$ and every $\gamma > 0$, there exist

$$\varepsilon_1 = \varepsilon_1(n, \Lambda, \gamma, \operatorname{dist}_g(W', \partial W)),\ c_1 = c_1(n, \Lambda, \operatorname{dist}_g(W', \partial W)), \\ \delta = \delta(n, \Lambda, \operatorname{dist}_g(W', \partial W)) > 0,$$

such that for every $\varepsilon \in (0, \varepsilon_1]$, there is a metric $\widehat{g}_\varepsilon$ on W such that:

1. *$\widehat{g}_\varepsilon$ is $C^{2,\alpha}_{\mathrm{loc}}(W \setminus \operatorname{sing}\mathcal{S})$;*
2. *$\widehat{g}_\varepsilon = g$ on $W \setminus (W' \cap B^g_\varepsilon(\operatorname{reg}\mathcal{S}))$;*
3. *$\|R(\widehat{g}_\varepsilon)_-\|_{L^{\frac{n}{2}+\delta}(W,g)} \leq \gamma$;*
4. *$c_1^{-1} g \leq \widehat{g}_\varepsilon \leq c_1 g$ on W;*

5. *if $p \in \operatorname{reg}\mathcal{S}$ is such that $\beta(p) < 0$ and $\mu > 0$ is such that*

$$|\beta(p)|^{-1} + \varrho(p)^{-1} \leq \mu,$$

then

$$R(\widehat{g}_\varepsilon) \geq c_2\varepsilon^{-2}$$

on $B^g_{c_2}(p) \cap B^g_{c_3\varepsilon}(\operatorname{reg}\mathcal{S}) \setminus B^g_{c_3\varepsilon/2}(\operatorname{reg}\mathcal{S})$, with

$$c_2 = c_2(n, \Lambda, \operatorname{dist}_g(W', \partial W), \mu) > 0,$$
$$c_3 = c_3(n, \Lambda, \operatorname{dist}_g(W', \partial W), \mu) > 0,$$

and $B^g_{c_2}(p) \subset W'$.

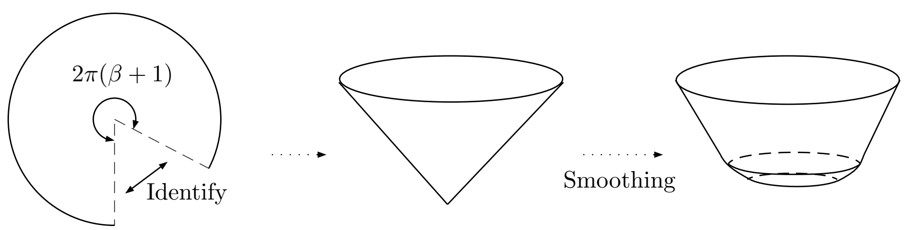

Figure 2.1: An illustration of a two-dimensional cone metric, and its smoothing procedure (Lemma 2.2.1). Roughly speaking, we glue a flat disk onto the conical singularity, such that the metric is L^∞, and the Gauss curvature is positive in the buffer region.

It is worth first looking at the special case in which $\mathcal{S} = \overline{N}$ for some embedded $(n-2)$-dimensional submanifold N (without boundary), and the edge singularity datum h is identically zero, i.e.,

$$g = dr^2 + (\beta+1)^2 r^2 (d\theta + \sigma)^2 + \omega \text{ in } U \setminus N. \tag{2.2.3}$$

As we'll see, all the interesting complications already arise in this situation.

Let's temporarily divert our attention to metrics $\widetilde{g}$ of the computationally simpler form

$$\widetilde{g} = f^2 dr^2 + r^2(d\theta + \sigma)^2 + \widetilde{\omega} \text{ on } U \setminus N. \tag{2.2.4}$$

Here, $f = f(r, y)$. We require the structural conditions

$$\|(\det \widetilde{\omega})^{-1}\|_{L^\infty} + \sum_{j=0}^{2} \|\partial^j \sigma\|_{L^\infty} + \|\partial^j \widetilde{\omega}\|_{L^\infty} \\ + \|f^{-1}\|_{L^\infty} + \|r\partial_r f\|_{L^\infty} + \|r\partial f\|_{L^\infty} + \|r^{2-\eta}\partial^2 f\|_{L^\infty} \leq \widetilde{\Lambda}. \quad (2.2.5)$$

All partial derivatives except the one explicitly denoted ∂_r are taken only with respect to $(y^1, \ldots, y^{n-2}) \in N$, but not in the two transversal polar directions.

Proposition 2.2.2. *The scalar curvature $R(\widetilde{g})$ of metrics $\widetilde{g}$ of the form* (2.2.4)*, which are subject to the structural assumptions* (2.2.5)*, satisfies*

$$r^{2-\eta}|R(\widetilde{g}) - 2r^{-1}f^{-3}\partial_r f| \leq c(n, \widetilde{\Lambda}), \quad (2.2.6)$$

at all points $(r, \theta, y^1, \ldots, y^{n-2})$ with $r \leq R_0 = R_0(n, \widetilde{\Lambda})$.

Proof. We circumvent a brute force computation by a slicing technique motivated by (2.7.1). The family of hypersurfaces

$$N_r := \{r = \text{const}\} \cap U,\ r > 0,$$

forms a codimension-1 foliation of $U \setminus N$, which is orthogonal to the ambient vector field

$$\nu_r := f^{-1}\partial_r$$

with respect to $\widetilde{g}$. In particular, the Gauss equation traced twice over N_r gives:

$$R(\widetilde{g}|_{N_r}) = R(\widetilde{g}) - 2\langle \mathrm{Ric}(\widetilde{g}), \nu_r \otimes \nu_r \rangle + H_{N_r}^2 - |A_{N_r}|^2, \quad (2.2.7)$$

where H_{N_r}, A_{N_r} denote the mean curvature and second fundamental form of $N_r \subset (U, \widetilde{g})$. On the other hand, the Jacobi equation implies

$$\partial_r(H_{N_r}) = -\Delta_{\widetilde{g}|N_r} f - (\langle \mathrm{Ric}(\widetilde{g}), \nu_r \otimes \nu_r \rangle, +|A_{N_r}|^2)f. \quad (2.2.8)$$

Together, (2.2.7)-(2.2.8) yield:

$$R(\widetilde{g}) = R(\widetilde{g}|_{N_r}) - 2f^{-1}\partial_r(H_{N_r}) - 2f^{-1}\Delta_{\widetilde{g}|N_r} f - H_{N_r}^2 - |A_{N_r}|^2. \tag{2.2.9}$$

This is the quantity we wish to estimate, written out in terms of the slicing technique. Let's fix $r > 0$ small and estimate the right hand side of (2.2.9).

Recall that

$$\widetilde{g}|_{N_r} = r^2(d\theta + \sigma)^2 + \omega, \tag{2.2.10}$$

and that, by the definition of second fundamental forms (here, $\mathcal{L}$ is the Lie derivative on 2-tensors),

$$A_{N_r} = \tfrac{1}{2}\mathcal{L}_{\nu_r}\widetilde{g} = \tfrac{1}{2}f^{-1}\mathcal{L}_{\partial_r}(\widetilde{g}|_{N_r}) = \tfrac{1}{rf}r^2(d\theta + \sigma)^2. \tag{2.2.11}$$

It will be convenient to pick out vector fields

$\mathbf{v}_1, \ldots, \mathbf{v}_{n-2} \in \Gamma(TN^{n-2})$ to be an

$(\omega - (r\sigma)^2)$-orthonormal frame on N^{n-2}.

We emphasize, that these are orthonormal on N^{n-2} with a metric *other* than the model metric $\omega \in \mathrm{Met}(N^{n-2})$. This modified metric was chosen specifically because, now,

$r^{-1}\partial_\theta, \mathbf{v}_1 - \sigma(\mathbf{v}_1)\partial_\theta, \ldots, \mathbf{v}_{n-2} - \sigma(\mathbf{v}_{n-2})\partial_\theta$

are a $\widetilde{g}_{N_r}$-orthonormal frame on N_r.

By repeated use of (2.2.11), we find:

$$A_{N_r}(r^{-1}\partial_\theta, r^{-1}\partial_\theta) = \tfrac{1}{rf}; \tag{2.2.12}$$

$$\begin{aligned}
&A_{N_r}(\mathbf{v}_\ell - \sigma(\mathbf{v}_\ell)\partial_\theta, \mathbf{v}_m - \sigma(\mathbf{v}_m)\partial_\theta)\\
&\quad = A_{N_r}(\mathbf{v}_\ell, \mathbf{v}_m) - \sigma(\mathbf{v}_\ell)A_{N_r}(\partial_\theta, \mathbf{v}_m)\\
&\qquad - \sigma(\mathbf{v}_m)A(\mathbf{v}_\ell, \partial_\theta) + \sigma(\mathbf{v}_\ell)\sigma(\mathbf{v}_m)A_{N_r}(\partial_\theta, \partial_\theta)\\
&\quad = \tfrac{r}{f}\sigma(\mathbf{v}_\ell)\sigma(\mathbf{v}_m) - 2\tfrac{r}{f}\sigma(\mathbf{v}_\ell)\sigma(\mathbf{v}_m) + \tfrac{r}{f}\sigma(\mathbf{v}_\ell)\sigma(\mathbf{v}_m)\\
&\quad = 0 \text{ for } \ell, m \in \{1, \ldots, n-2\};
\end{aligned} \tag{2.2.13}$$

$$\begin{aligned}
&A_{N_r}(r^{-1}\partial_\theta, \mathbf{v}_\ell - \sigma(\mathbf{v}_\ell)\partial_\theta)\\
&\quad = A_{N_r}(r^{-1}\partial_\theta, \mathbf{v}_\ell) - A_{N_r}(r^{-1}\partial_\theta, \sigma(\mathbf{v}_\ell)\partial_\theta)\\
&\quad = \tfrac{r}{f}\sigma(\mathbf{v}_\ell) - \tfrac{r}{f}\sigma(\mathbf{v}_\ell)\\
&\quad = 0 \text{ for } \ell \in \{1, \ldots, n-2\}.
\end{aligned} \tag{2.2.14}$$

Altogether, (2.2.12)-(2.2.14) imply

$$|A_{N_r}| = H_{N_r} = \tfrac{1}{rf},$$

and thus

$$2f^{-1}\partial_r(H_{N_r}) + H_{N_r}^2 + |A_{N_r}|^2 = -2r^{-1}f^{-3}\partial_r f. \tag{2.2.15}$$

In particular, three out of five terms in (2.2.9) cancel out.

Next, we seek to understand $R(\widetilde{g}|_{N_r})$, which denotes the scalar curvature of the $(n-1)$-dimensional manifold $(N_r, \widetilde{g}|_{N_r})$, with $\widetilde{g}|_{N_r}$ given explicitly in (2.2.10). We re-employ the slicing technique; this time we use the fact that

$$N_{r,\theta} := \{\theta = \text{const}\} \cap N_r$$

is a codimension-1 foliation of N_r, whose induced metrics are given by

$$\widetilde{g}|_{N_{r,\theta}} = \omega + (r\sigma)^2. \tag{2.2.16}$$

If $\nu_{r,\theta} \in \Gamma(TN_r)$ denotes the unit normal vector field to the foliation, then, arguing as before, we have

$$R(\widetilde{g}|N_r) = R(\widetilde{g}|N_{r,\theta}) - 2\langle \nu_{r,\theta}, \partial_\theta \rangle_{\widetilde{g}}^{-1} \mathcal{L}_{\nu_{r,\theta}}(H_{N_{r,\theta}}) \\ - 2\langle \nu_{r,\theta}, \partial_\theta \rangle_{\widetilde{g}}^{-1} \Delta_{\widetilde{g}|N_{r,\theta}} \langle \nu_{r,\theta}, \partial_\theta \rangle_{\widetilde{g}} - H^2_{N_{r,\theta}} - |A_{N_{r,\theta}}|^2. \quad (2.2.17)$$

Note that, unlike the previous slicing application, ∂_θ is no longer orthogonal to the foliation. Instead, the unit normal vector field $\nu_{r,\theta}$ is proportional to

$$\partial_\theta + \sum_{\ell=1}^{n-2} \alpha_\ell \mathbf{v}_\ell$$

for some coefficients $\alpha_1, \ldots, \alpha_{n-2} : N^{n-2} \to \mathbf{R}$; the vector fields $\mathbf{v}_\ell$ are the same as before. The coefficients $\alpha_1, \ldots, \alpha_{n-2}$ are such that

$$\langle \nu_{r,\theta}, \mathbf{v}_1 \rangle_{\widetilde{g}} = \ldots = \langle \nu_{r,\theta}, \mathbf{v}_{n-2} \rangle_{\widetilde{g}} = 0.$$

This is a uniformly invertible $(n-2) \times (n-2)$ linear system for small enough $r \leq r_0 = r_0(n, \widetilde{\Lambda})$. Recalling (2.2.5), the linear system readily implies:

$$\sum_{\ell=1}^{n-2} |\alpha_\ell| + r|\partial \alpha_\ell| + r^2 |\partial^2 \alpha_\ell| \leq c(n, \widetilde{\Lambda}) r^2. \quad (2.2.18)$$

In particular, the unit normal vector field is

$$\nu_{r,\theta} = (1 + \zeta) \left(\partial_\theta + \sum_{\ell=1}^{n-2} \alpha_\ell \mathbf{v}_\ell \right),$$

with

$$|\zeta| + r|\partial \zeta| + r^2 |\partial^2 \zeta| \leq c(n, \widetilde{\Lambda}) r^2. \quad (2.2.19)$$

Combined, (2.2.16), (2.2.18), and (2.2.19), imply a uniform bound on the right hand side of (2.2.17). Thus,

$$|R(\widetilde{g}|N_{r,\theta})| \leq c(n, \widetilde{\Lambda}). \quad (2.2.20)$$

Finally, the last remaining term of (2.2.9), $\Delta_{\widetilde{g}|N_r} f$, can be estimated directly by (2.2.5):

$$r^{2-\eta}|\Delta_{\widetilde{g}|N_r} f| \leq c(n, \widetilde{\Lambda}). \tag{2.2.21}$$

The proposition follows by plugging (2.2.15), (2.2.20), (2.2.21) into (2.2.9). □

Proof of Lemma 2.2.1. Let us first see how Proposition 2.2.2 fits into our simplified smoothing lemma situation, i.e., $\mathcal{S} = \overline{N}$ and $h \equiv 0$. Let's fix a smooth cutoff function $\zeta : [0, 1] \to [0, 1]$ such that

$$\zeta \equiv 0 \text{ on } [0, \tfrac{1}{3}],\ \zeta \equiv 1 \text{ on } [\tfrac{2}{3}, 1],\ 0 \leq \zeta' \leq 6,\ \zeta' = 1 \text{ on } [\tfrac{4}{9}, \tfrac{5}{9}].$$

Define $f_\varepsilon(r, y)$, $\varepsilon > 0$:

$$f_\varepsilon(r, y) := 1 + \zeta(\varepsilon^{-1}\varrho(y)^{-1}r)\Big[(1 + \beta(y))^{-1} - 1\Big]. \tag{2.2.22}$$

From (2.2.1), (2.2.2), and the defining properties of ζ:

$$f_\varepsilon \geq 1,\ 0 \leq r\partial_r f_\varepsilon \leq 6, \tag{2.2.23}$$

$$f_\varepsilon = 1 \text{ for } r \leq \tfrac{1}{3}\varepsilon\varrho,\ f_\varepsilon = (\beta + 1)^{-1} \text{ for } r \geq \tfrac{2}{3}\varepsilon\varrho, \tag{2.2.24}$$

$$r\partial_r f_\varepsilon(r, y) = (1 + \beta)^{-1} - 1 \text{ for } \varepsilon\tfrac{4}{9}\varrho(y) \leq r \leq \tfrac{5}{9}\varepsilon\varrho(y), \tag{2.2.25}$$

$$|r\partial f_\varepsilon| + |r^{2-\eta}\partial^2 f_\varepsilon| \leq c(\Lambda). \tag{2.2.26}$$

Setting

$$\widetilde{g}_\varepsilon := f_\varepsilon^2 dr^2 + r^2(d\theta + \sigma)^2 + (\beta + 1)^{-2}\omega, \tag{2.2.27}$$

it follows from (2.2.23)-(2.2.26) and (2.2.1)-(2.2.2) that $\widetilde{g}_\varepsilon$ is of the form (2.2.4) and satisfies the structural assumptions (2.2.5).

We'll verify that, for sufficiently small $\varepsilon > 0$, the conformal metric

$$\widehat{g}_\varepsilon := (\beta + 1)^2 \widetilde{g}_\varepsilon$$

is the metric postulated by Lemma 2.2.1. Without loss of generality,

$$\mathrm{dist}_g(W', \partial W) \geq 1, \ \varrho \leq 1 \text{ on } N.$$

Conclusions (1), (2), (4) of Lemma 2.2.1 is an immediate consequence of (2.2.2) and the definitions of $\widetilde{g}_\varepsilon$, $\widehat{g}_\varepsilon$. Now we prove conclusion (5). If p is as in the statement of the Lemma, then by Proposition 2.2.2 and (2.2.24),

$$r^{2-\eta}|R(\widetilde{g}_\varepsilon) - 2r^{-2}f_\varepsilon^{-3}((1+\beta(p))^{-1} - 1)| \leq c, \tag{2.2.28}$$

whenever $r \in [\frac{4}{9}\varepsilon\varrho(p), \frac{5}{9}\varepsilon\varrho(p)]$. This readily implies conclusion (5). Finally, we move on to conclusion (3). By Proposition 2.2.2, we have

$$R(\widetilde{g}_\varepsilon)_- \leq cr^{-2+\eta},$$

so the conformal metric $\widehat{g}_\varepsilon = (\beta+1)^2\widetilde{g}_\varepsilon$ satisfies

$$R(\widehat{g}_\varepsilon) = (\beta+1)^{\frac{n+2}{2}}\Big[\frac{4(1-n)}{n-2}\Delta_{\widetilde{g}_\varepsilon} + R(\widetilde{g}_\varepsilon)\Big](\beta+1)^{\frac{n-2}{2}}.$$

Since β has no dependence on r, θ, and is uniformly C^2 in $(y^1, \ldots, y^{n-2})$:

$$R(\widehat{g}_\varepsilon)_- \leq c(1 + R_-(\widetilde{g}_\varepsilon)) \leq c(1 + r^{-2+\eta}) \leq cr^{-2+\eta},$$

where the last inequality follows from our assumption that $\varrho \leq 1$. In particular, if we denote the $\varepsilon\varrho$-tubular neighborhood of N by U_ε, we have, from the coarea formula, that

$$\begin{aligned}
\|R(\widehat{g}_\varepsilon)_-\|^q_{L^q(W,g)} &= \|R(\widehat{g}_\varepsilon)_-\|^q_{L^q(U_\varepsilon,g)} \\
&\leq c\int_{U_\varepsilon}(r^{-2+\eta})^q d\operatorname{Vol}_g \\
&\leq c\int_N\int_0^{\varepsilon\varrho(y)} r^{q(-2+\eta)+1}\,dr\,d\mu_\omega(y) \\
&= c\int_N\left[\tfrac{r^{q(-2+\eta)+2}}{q(-2+\eta)+2}\right]_{r=0}^{\varepsilon\varrho(y)} d\mu_\omega(y) \\
&\leq c(\varepsilon\|\varrho\|_{C^0(N)})^{q(-2+\eta)+2},
\end{aligned}$$

provided

$$q(-2+\eta)+2>0 \iff q<\tfrac{2}{2-\eta}.$$

In the chain of inequalities above, c denotes a constant depending on n and Λ, which varies from line to line. Since $\eta \geq \Lambda^{-1}+2-\frac{4}{n}$, it follows that

$$q<\tfrac{2}{\frac{4}{n}-\Lambda^{-1}},$$

and conclusion (3) follows. This completes the proof of the lemma in the special case when $\mathcal{S}=\overline{N}$ and $h\equiv 0$.

Let's generalize to allow $h\not\equiv 0$ in

$$g = dr^2+(\beta+1)^2r^2(d\theta+\sigma)^2+\omega+r^{1+\eta}h.$$

We will regularize in two steps, leading up to

$$\widehat{g}_\varepsilon := (\beta+1)^2\widetilde{g}_\varepsilon+(\beta+1)^2 f_\varepsilon^2 r^{1+\eta}h,$$

where f_ε is as in (2.2.22) and $\widetilde{g}_\varepsilon$ as in (2.2.27). The first step, studying $(\beta+1)^2\widetilde{g}_\varepsilon$, is the step we carried out above. Now, a crude estimate that relies on (2.2.5) shows that when ξ is a $C^2_{\mathrm{loc}}(U\setminus N)$ 2-tensor, which in Euclidean coordinates (recall Definition

2.1.2) is controlled by

$$|\xi_{\alpha\beta}| + r|\partial_\alpha \xi_{\beta\gamma}| + r^2|\partial_\alpha\partial_\beta \xi_{\gamma\delta}| \leq \epsilon$$

and $\epsilon > 0$ sufficiently small, then

$$r^2 \big|R\big((\beta+1)^2 \widetilde{g}_\varepsilon + \xi\big) - R\big((\beta+1)^2\widetilde{g}_\varepsilon\big)\big| \leq c(n,\Lambda)\epsilon. \tag{2.2.29}$$

But note that

$$\xi := \widehat{g}_\varepsilon - (\beta+1)^2\widetilde{g}_\varepsilon = (\beta+1)^2 f_\varepsilon^2 r^{1+\eta} h$$

satisfies

$$|\xi_{\alpha\beta}| + r|\partial_\alpha \xi_{\beta\gamma}| + r^2|\partial_\alpha\partial_\beta \xi_{\gamma\delta}| \leq c(n,\Lambda) r^\eta,$$

and $\eta > 0$, which applied to (2.2.29) tells us that $R(\widehat{g}_\varepsilon)$ has precisely the same behavior now as in (2.2.28), so the result follows as before.

Finally, we deal with the most general case, where g can be of general edge type, and the skeleton $\mathcal{S}$ consists of more than just one piece; i.e., $\mathcal{S} = \overline{N}_1 \cup \ldots \cup \overline{N}_k$. Since we're assuming $\mathcal{S}$ is nondegenerate, it follows that the pieces $N_1, \ldots, N_k$ can be separated from each other with ϱ-tubular neighborhoods that decay with

$$\varrho \sim \operatorname{dist}_g(\cdot, \partial N_1 \cup \ldots \cup \partial N_k). \tag{2.2.30}$$

In particular, we may apply the lemma to each component $N_1, \ldots, N_k$ individually with a modified Λ that also accounts for the linear decay (2.2.30), and then glue all the metrics together since they agree away from their degenerating tubular neighborhoods by virtue of the rightmost equality in (2.2.24). $\square$

2.3 Almost positive scalar curvature

The following lemma will play a key and recurring role in this chapter, stating that $C^{2,\alpha}_{\text{loc}} \cap L^\infty$ metrics with little negative scalar curvature and sufficiently much positive scalar curvature are conformally equivalent to metrics with positive scalar curvature of the same regularity.

Lemma 2.3.1. *Suppose M^n is closed, g_0 is a smooth background metric on M, g is an $L^\infty(M) \cap C^{2,\alpha}_{\mathrm{loc}}(M \setminus \mathcal{S})$, $\alpha \in (0,1)$, $\mathcal{S} \subset M$ is compact, $\mathrm{Vol}_g(\mathcal{S}) = 0$, and $\Lambda^{-1} g_0 \leq g \leq \Lambda g_0$. If $\chi \in C^\alpha_{\mathrm{loc}}(M \setminus \mathcal{S}) \cap L^q(M, g)$ with $q > \frac{n}{2}$,*

$$\chi \leq R(g), \ \|\chi_-\|_{L^{n/2}(M,g)} \leq \delta_0,$$

then there exists $u \in C^{2,\alpha}_{\mathrm{loc}}(M \setminus \mathcal{S}) \cap C^0(M)$, $u > 0$, such that

$$\inf_{M\setminus\mathcal{S}} u^{\frac{4}{n-2}} R(u^{\frac{4}{n-2}} g) \geq \frac{1}{c_0^2 \mathrm{Vol}_g(M)} \left(\int_M \chi_+ \, d\mathrm{Vol}_g - c_0^4 \int_M \chi_- \, d\mathrm{Vol}_g \right)$$
$$\text{and } \sup_M u \leq c_0 \inf_M u, \quad (2.3.1)$$

where $\delta_0 = \delta_0(g_0, \Lambda) > 0$, $c_0 = c_0(g_0, \Lambda, q, \|\chi\|_{L^q(M,g,\Lambda)}) \geq 1$.

Proof. We construct, using the direct method, the principal eigenvalue of the operator $-\frac{4(n-1)}{n-2}\Delta_g + \chi$ on S. Namely, we minimize

$$\|f\|^{-2}_{L^2(M,g)} \int_M \frac{4(n-1)}{n-2} \|\nabla^g f\|_g^2 + \chi |f|^2 \, d\mathrm{Vol}_g, \quad (2.3.2)$$

over $f \in L^2(M, g)$, $f \not\equiv 0$. From the Poincaré-Sobolev inequality,

$$\left(\int_M f^{\frac{2n}{n-2}} \, d\mathrm{Vol}_g \right)^{\frac{n-2}{n}} \leq C_1 \int_M \|\nabla^g f\|_g^2 + |f|^2 \, d\mathrm{Vol}_g$$
$$\implies \int_M \|\nabla^g f\|_g^2 \, d\mathrm{Vol}_g \geq C_1^{-1} \left(\int_M |f|^{\frac{2n}{n-2}} \, d\mathrm{Vol}_g \right)^{\frac{n-2}{n}} - \int_M |f|^2 \, d\mathrm{Vol}_g$$

for $C_1 = C_1(g_0, \Lambda) > 0$. From Hölder's inequality,

$$\int_M \chi |f|^2 \, d\mathrm{Vol}_g \geq -\delta_0 \left(\int_M |f|^{\frac{2n}{n-2}} \, d\mathrm{Vol}_g \right)^{\frac{n-2}{n}}$$

and the lower bound on (2.3.2) follows as long as we require δ_0 to be small enough depending on g_0, Λ. From functional analysis, minimizing (2.3.2) yields some $u \in$

$W^{1,2}(M, g)$, $u \geq 0$ g-a.e. on M, that satisfies, for some $\lambda \in \mathbf{R}$,

$$-\frac{4(n-1)}{n-2}\Delta_g u + \chi u = \lambda u \text{ on } M, \tag{2.3.3}$$

in the weak sense. From elementary elliptic PDE theory, $u \in C^{2,\alpha}_{\mathrm{loc}}(M \setminus \mathcal{S})$. From De Giorgi-Nash-Moser theory,

$$u \in C^{0,\theta}(M), \text{ and } \lambda \geq -\Lambda_1,\ \Lambda_1 = \Lambda(g_0, \Lambda, q, \|\chi\|_{L^q(M,g)}) > 0.$$

(The precise $\theta \in (0, 1)$ isn't relevant.) The inequality

$$\sup_M u \leq c_0 \inf_M u \tag{2.3.4}$$

with $c_0 = c_0(g_0, \Lambda, q, \|\chi\|_{L^q(M,g)})$ follows from Moser's Harnack inequality. From the variational characterization of (2.3.3) as a minimizer of (2.3.2), and from (2.3.4), we see that

$$\begin{aligned}
\lambda &= \|u\|^{-2}_{L^2(M,g)} \int_M \frac{4(n-1)}{n-2}\|\nabla^g u\|^2_g + \chi|u|^2 \, d\operatorname{Vol}_g \\
&\geq \|u\|^{-2}_{L^2(M,g)} \int_M \chi_+|u|^2 \, d\operatorname{Vol}_g - \|u\|^{-2}_{L^2(M,g)} \int_M \chi_-|u|^2 \, d\operatorname{Vol}_g \\
&\geq \inf_M u^2 \cdot \|u\|^{-2}_{L^2(M,g)} \int_M \chi_+ \, d\operatorname{Vol}_g - \sup_M u^2 \cdot \|u\|^{-2}_{L^2(M,g)} \int_M \chi_- \, d\operatorname{Vol}_g \\
&\geq c_0^{-2} \operatorname{Vol}_g(M, g)^{-1} \left(\int_M \chi_+ \, d\operatorname{Vol}_g - c_0^4 \int_M \chi_- \, d\operatorname{Vol}_g \right).
\end{aligned} \tag{2.3.5}$$

Thus, from the scalar curvature conformal transformation formula and (2.3.3),

$$\begin{aligned}
R(u^{\frac{4}{n-2}}g) &= u^{-\frac{n+2}{n-2}} \left(-\frac{4(n-1)}{n-2}\Delta_g u + R(g)u \right) \\
&= u^{-\frac{4}{n-2}}(R(g) - \chi + \lambda) \geq \lambda u^{-\frac{4}{n-2}} \text{ on } M \setminus \mathcal{S},
\end{aligned}$$

and the result follows from (2.3.5). $\square$

We obtain, as a direct corollary, the following rigidity result that extends a well-known (to the experts of the field) result from the smooth case to a general singular setting: nonnegative scalar curvature can be conformally transformed into positive scalar curvature, as long as the original metric isn't scalar-flat.

Corollary 2.3.2. *Suppose M^n is closed, g is an $L^\infty(M) \cap C^{2,\alpha}_{\mathrm{loc}}(M \setminus \mathcal{S})$ metric, $\alpha \in (0,1)$, $\mathcal{S} \subset M$ is compact, and $\mathrm{Vol}_g(\mathcal{S}) = 0$. If $R(g) \geq 0$ on $M \setminus \mathcal{S}$, and $R(g) \not\equiv 0$, then*

$$R(u^{\frac{4}{n-2}} g) > 0 \text{ on } M \setminus \mathcal{S}$$

for some $u \in C^{2,\alpha}_{\mathrm{loc}}(M \setminus \mathcal{S}) \cap C^0(M)$, $u > 0$.

Remark 2.3.3. *We will later show that for particular kinds of singular behavior, we can construct* everywhere *smooth metrics with positive scalar curvature, at the expense of leaving the conformal class of g. This is essentially the content of Theorems 1.3.9, 1.3.6, and Corollary 1.3.8.*

2.4 Smoothing edge singularities, II

Proposition 2.4.1. *Suppose M^n is closed, $\sigma(M) \leq 0$, $\mathcal{S} \subset M$ is a nondegenerate $(n-2)$-skeleton, and $g \in L^\infty(M) \cap C^{2,\alpha}_{\mathrm{loc}}(M \setminus \mathcal{S})$, $\alpha \in [0,1]$. Assume g is an η-regular edge metric along $\mathrm{reg}\,\mathcal{S}$ with $\eta > 2 - \frac{4}{n}$ and cone angles*

$$0 < \inf_{\mathrm{reg}\,\mathcal{S}} 2\pi(\beta+1) \leq \sup_{\mathrm{reg}\,\mathcal{S}} 2\pi(\beta+1) \leq 2\pi.$$

If $R(g) \geq 0$ on $M \setminus \mathcal{S}$ and either

1. *$R(g) \not\equiv 0$ on $M \setminus \mathcal{S}$, or*

2. *$2\pi(\beta+1) \not\equiv 2\pi$ on $\mathrm{reg}\,\mathcal{S}$,*

then there exists an $L^\infty(M) \cap C^{2,\alpha}_{\mathrm{loc}}(M \setminus \mathrm{sing}\,\mathcal{S})$ metric $\widetilde{g}$ with

$$R(\widetilde{g}) > 0 \text{ on } M \setminus \mathrm{sing}\,\mathcal{S}.$$

We need to introduce some more notation. For $s > 0$, define

$$\phi(\cdot; s) : \mathbf{R} \to \mathbf{R} \text{ with } \phi(x; s) = \begin{cases} x \text{ for } x \in (-\infty, s], \\ 2s \text{ for } x \in [3s, \infty), \end{cases} \tag{2.4.1}$$

with

$$\frac{\partial}{\partial x}\phi(x; s) \geq 0 \text{ and } \phi(x; s) \leq x \text{ for all } x \in \mathbf{R}, s > 0, \tag{2.4.2}$$

and, for $q \in (\frac{n}{2}, n)$ fixed for the remainder of the chapter, and $\varepsilon > 0$,

$$\zeta(\cdot; \varepsilon) : M \to \mathbf{R} \text{ with } \zeta(x; \varepsilon) = \begin{cases} 1 \text{ for } x \notin B^g_{2\varepsilon}(\mathcal{S}), \\ \varepsilon^{-2/q} \text{ for } x \in B^g_{\varepsilon}(\mathcal{S}), \end{cases} \tag{2.4.3}$$

such that

$$|\zeta(x; \varepsilon)| \leq \varepsilon^{-2/q} \text{ for all } x \in M, \varepsilon > 0. \tag{2.4.4}$$

Proof of Proposition 2.4.1. Let $\widehat{g}_\varepsilon$ be as in Lemma 2.2.1 above, for a choice of $\gamma > 0$ that is yet to be determined.

claim 2.4.2. *We have*

$$\limsup_{\varepsilon \to 0} \|\phi(R(\widehat{g}_\epsilon); \zeta(\cdot; \varepsilon))\|_{L^q(M,g)} < \infty.$$

Remark 2.4.3. *The L^q norm can be taken with respect to the measure induced by any one of g, $\widehat{g}_\varepsilon$, since they are uniformly equivalent by Lemma 2.2.1 (4).*

Proof of Claim. We estimate the integral by splitting up M into the region of negative scalar curvature (which is controlled by Lemma 2.2.1), the tubular neighborhood

$B^g_{2\varepsilon}(\mathcal{S})$ (which is controlled by the codimension of $\mathcal{S}$), and the remainder:

$$\begin{aligned}
&\|\phi(R(\widehat{g}_\varepsilon);\zeta(\cdot;\varepsilon))\|^q_{L^q(M,g)} \\
&\qquad \le \|R(\widehat{g}_\varepsilon)_-\|^q_{L^q(M,g)} + \int_{B^g_{2\varepsilon}(\mathcal{S})} |2\varepsilon^{-2/q}|^q \, d\operatorname{Vol}_g \\
&\qquad\qquad + \int_{M\setminus B^g_{2\varepsilon}(\mathcal{S})} |\min\{R(\widehat{g}_\varepsilon)_+, 2\}|^q \, d\operatorname{Vol}_g \\
&\qquad \le \gamma^q + 2^q\varepsilon^{-2}\operatorname{Vol}_g(B^g_{2\varepsilon}(\mathcal{S})) + 2^q \operatorname{Vol}_g(M),
\end{aligned}$$

which is uniformly bounded as $\varepsilon \to 0$ when $\mathcal{S}$ is an $(n-2)$-skeleton, i.e., one with codimension ≥ 2. □

We now apply Lemma 2.3.1 with $\widehat{g}_\varepsilon$ in place of g, $\operatorname{sing}\mathcal{S}$ in place of $\mathcal{S}$, and $\chi = \phi(R(\widehat{g}_\varepsilon);\zeta(\cdot;\varepsilon))$. Note that the constant δ_0 in Lemma 2.3.1 is independent of $\varepsilon \to 0$, since the metrics $\widehat{g}_\varepsilon$, g are all uniformly equivalent. It remains to show

$$\int_M \phi(R(\widehat{g}_\varepsilon);\zeta(\cdot;\varepsilon))_+ \, d\operatorname{Vol}_{\widehat{g}_\varepsilon} - c_0^4 \int_M \phi(R(\widehat{g}_\varepsilon);\zeta(\cdot;\varepsilon))_- \, d\operatorname{Vol}_{\widehat{g}_\varepsilon} > 0 \tag{2.4.5}$$

for sufficiently small $\varepsilon > 0$.

Separating the positive scalar curvature regions from the negative ones, recalling $R(\widehat{g}_\varepsilon) \ge 0$ on $M \setminus B^g_\varepsilon(\operatorname{reg}\mathcal{S})$ by Lemma 2.2.1 (2), and $\|R_-(\widehat{g}_\varepsilon)\|_{L^q} \le \gamma$ by Lemma 2.2.1 (3),

$$\begin{aligned}
&\int_M \phi(R(\widehat{g}_\varepsilon);\zeta(\cdot;\varepsilon))_+ \, d\operatorname{Vol}_{\widehat{g}_\varepsilon} - c_0^4 \int_M \phi(R(\widehat{g}_\varepsilon);\zeta(\cdot;\varepsilon))_- \, d\operatorname{Vol}_{\widehat{g}_\varepsilon} \\
&\qquad \ge \int_M \phi(R_+(\widehat{g}_\varepsilon);\zeta(\cdot;\varepsilon^{-1})) \, d\operatorname{Vol}_{\widehat{g}_\varepsilon} - c_0^4 \|R_-(\widehat{g}_\varepsilon)\|_{L^1(M,\widehat{g}_\varepsilon)} \\
&\qquad \ge \int_M \phi(R_+(\widehat{g}_\varepsilon);\zeta(\cdot;\varepsilon^{-1})) \, d\operatorname{Vol}_{\widehat{g}_\varepsilon} - c_0^4 \gamma \operatorname{Vol}_{\widehat{g}_\varepsilon}(B^g_\varepsilon(\mathcal{S}))^{(q-1)/q} \\
&\qquad \ge \int_M \phi(R_+(\widehat{g}_\varepsilon);\zeta(\cdot;\varepsilon^{-1})) \, d\operatorname{Vol}_{\widehat{g}_\varepsilon} - \gamma C_1 \varepsilon^{2(q-1)/q},
\end{aligned} \tag{2.4.6}$$

where $C_1 = C_1(\mathcal{S}, g_0, \Lambda) > 0$, and g_0 is some fixed background smooth metric on M.

Note that

$$\liminf_{\varepsilon\to 0}\int_M \phi(R_+(\widehat{g}_\varepsilon);\zeta(\cdot;\varepsilon))\,d\operatorname{Vol}_{\widehat{g}_\varepsilon}$$
$$\geq \liminf_{\varepsilon\to 0}\int_{M\setminus B^g_{2\varepsilon}(\operatorname{reg}\mathcal{S})} \phi(R_+(g);1)\,d\operatorname{Vol}_{\widehat{g}_\varepsilon} = \int_M \phi(R(g);1)\,d\operatorname{Vol}_g .$$

In particular, if $R(g)\not\equiv 0$ on $M\setminus\mathcal{S}$, then (2.4.6) implies (2.4.5), and we're done.

Alternatively, when $R(g)\equiv 0$ on $M\setminus\mathcal{S}$, suppose that $\beta\not\equiv 0$ on $\operatorname{reg}\mathcal{S}$. By Lemma 2.2.1 (5), there exists p with

$$\phi(R_+(\widehat{g}_\varepsilon);\zeta(\cdot;\varepsilon)) = 2\varepsilon^{-2/q} \text{ on } B^g_{c_3}(p)\cap B^g_{c_4\varepsilon}(\operatorname{reg}\mathcal{S})\setminus B^g_{c_4\varepsilon/2}(\operatorname{reg}\mathcal{S}),$$

and $B^g_{c_3}(p)\subset U$. Note that

$$\operatorname{Vol}_{\widehat{g}_\varepsilon}(B^g_{c_3}(p)\cap B^g_{c_4\varepsilon}(\operatorname{reg}\mathcal{S})\setminus B^g_{c_4\varepsilon/2}(\operatorname{reg}\mathcal{S}))\geq C_2\varepsilon^2,$$

with $C_2=C_2(\mathcal{S},g_0,\Lambda)>0$ since $\mathcal{S}$ is an $(n-2)$-skeleton, i.e., it has codimension ≥ 2. In sight of this, (2.4.6) implies

$$\int_M \phi(R(\widehat{g}_\varepsilon);\zeta(\cdot;\varepsilon))_+\,d\operatorname{Vol}_{\widehat{g}_\varepsilon} - c_0^4\int_M \phi(R(\widehat{g}_\varepsilon);\zeta(\cdot;\varepsilon))_-\,d\operatorname{Vol}_{\widehat{g}_\varepsilon}$$
$$\geq 2\cdot\varepsilon^{-2/q}\cdot C_2\varepsilon^2 - \gamma C_1\varepsilon^{2(q-1)/q},$$

which can be made to be positive provided γ is sufficiently small depending on $\mathcal{S}$, g_0, Λ. □

2.5 Smoothing point singularities in dimension 3

Our method for smoothing point singularities consists of:

1. "blowing up" the singularity;
2. excising the asymptotic end produced in the previous step by cutting along a particular minimal surface;

3. "filling in" the holes created in the two previous steps with regions of positive scalar curvature.

Step (1) is inspired from works of Schoen-Yau (e.g. [45, 43]). Steps (2) and (3) are inspired by constructions that feature in the recent work of C. Mantoulidis and P. Miao [32].

A new key necessary ingredient in this work is the following new excision lemma for asymptotic ends with weak regularity at infinity:

Lemma 2.5.1. *Suppose g is a $C^{2,\alpha}_{\mathrm{loc}}$ metric on $\mathbf{R}^n \setminus B_1(\mathbf{0})$ with*

$$\Lambda^{-1}\delta \leq g \leq \Lambda\delta, \tag{2.5.1}$$

where δ is the standard flat metric on $\mathbf{R}^n$ and $\alpha \in [0,1]$. If

$$\mathscr{C} := \{\Omega \subset \mathbf{R}^3 \text{ bounded, open neighborhood of } B_1(\mathbf{0})\},$$

then $\inf\{\mathcal{H}^{n-1}_g(\partial\Omega) : \Omega \in \mathscr{C}\}$ *is attained by some $\Omega \subset B_R(\mathbf{0})$, $R = R(n,\Lambda)$.*

Proof. First, by a direct comparison argument, we have

$$\inf\{\mathcal{H}^{n-1}_g(\partial\Omega) : \Omega \in \mathscr{C}\} \leq \mathcal{H}^{n-1}_g(\partial B_1(\mathbf{0})) \leq c_1(n,\Lambda). \tag{2.5.2}$$

Let $\{\Omega_i\}_{i=1,2,\ldots} \subset \mathscr{C}$ be a minimizing sequence of domains for the left hand side of (2.5.2), for each of which we denote

$$r_i := \inf\{|\mathbf{x}| : \mathbf{x} \in \partial\Omega_i\}.$$

Here, $|\mathbf{x}|$ denotes the Euclidean length of a position vector $\mathbf{x} \in \mathbf{R}^n$. By another direct comparison and the area formula on $(\mathbf{R}^n, \delta)$,

$$\mathcal{H}^{n-1}_g(\partial\Omega_i) \geq c_2'(n,\Lambda)\mathcal{H}^{n-1}_\delta(\partial\Omega_i) \geq c_2'(n,\Lambda)\mathcal{H}^{n-1}_\delta(\partial B_{r_i}(\mathbf{0})) = c_2''(n,\Lambda)r_i^{n-1},$$

which together with (2.5.2) implies

$$r_i \leq c_2(n, \Lambda) \text{ for all } i = 1, 2, \ldots \tag{2.5.3}$$

For convenience, denote

$$r := \liminf_{i \to \infty} r_i \in [1, \infty),$$

where the finiteness is a byproduct of (2.5.3). Pass to a subsequence that attains the lim inf. For that subsequence, let

$$R_i := \sup\{|\mathbf{x}| : \mathbf{x} \in \partial\Omega_i\}$$

and

$$R := \limsup_{i \to \infty} R_i \in [1, \infty].$$

Without loss of generality, $R > r$. We seek to estimate R from above. Pass to yet another subsequence that attains the lim sup.

By a compactness argument, there will exist a closed $\Omega \subset \mathbf{R}^n \setminus B_1(\mathbf{0})$ containing $\partial B_1(\mathbf{0})$ such that, by definition of r, R,

$$\Sigma_t := \partial\Omega \cap \partial B_t(\mathbf{0}) \neq \emptyset \text{ for al } t \in (r, R), \tag{2.5.4}$$

and, by (2.5.2),

$$\mathcal{H}^{n-1}_g(\partial\Omega) \leq c_1(n, \Lambda). \tag{2.5.5}$$

For each $t \in (r, R)$, let $h(t)$ denote the $\mathcal{H}^{n-1}_g$-measure of the solution of the Plateau problem with prescribed boundary Σ_t; this is guaranteed to be nonzero by (2.5.4). We do not concern ourselves with the technicalities behind the existence of a feasible minimizer in the Plateau problem— we are content with the existence of a competitor with $\mathcal{H}^{n-1}_g(\cdot) \leq 2h(t)$, which is guaranteed, for instance, by the deformation theorem.

By (2.5.1) and the isoperimetric inequality on $(\mathbf{R}^n, \delta)$,

$$h(t)^{\frac{n-2}{n-1}} \leq c_3(n, \Lambda)\mathcal{H}^{n-2}_g(\Sigma_t) \text{ for all } t \in (r, R). \tag{2.5.6}$$

Moreover, we claim that

$$2h(t) \geq \mathcal{H}_g^{n-1}(\partial\Omega \setminus B_t(\mathbf{0})) \text{ for all } t \in (r, R); \tag{2.5.7}$$

indeed, if this were false for some t, then a direct replacement could produce $\Omega' \in \mathscr{C}$ with $\mathcal{H}_g^{n-1}(\partial\Omega') < \mathcal{H}_g^{n-1}(\partial\Omega) = c_1$, violating (2.5.2).

The coarea formula, (2.5.6), and (2.5.7), give:

$$\begin{aligned} 2h(t) &\geq \mathcal{H}_g^{n-1}(\partial\Omega \setminus B_t(\mathbf{0})) \\ &\geq \int_t^R \int_{\Sigma_s} |\nabla^T \operatorname{dist}_g(\mathbf{0}; \cdot)|^{-1} \, d\mathcal{H}_g^{n-2} \, ds \\ &\geq \int_t^R \mathcal{H}_g^{n-2}(\Sigma_s) \, ds \\ &\geq c_3^{-1} \int_t^R h(s)^{\frac{n-2}{n-1}} \, ds; \end{aligned}$$

the second equality follows from $|\nabla^T \operatorname{dist}(\mathbf{0}; \cdot)| \leq 1$. In other words, if $H(t)$ denotes the ultimate integral that appears above, we've shown that

$$|H'(t)|^{\frac{n-1}{n-2}} \geq (2c_3)^{-1} H(t) \text{ for all } t \in (r, R).$$

In fact, since $H' \leq 0$, we get

$$-H'(t) \geq c_4(n, \Lambda) H(t)^{\frac{n-2}{n-1}} \text{ for all } t \in (r, R).$$

Integrating, we find that there exists $R^\star = R^\star(n, \Lambda, r) \leq R^\star(n, \Lambda)$ such that $H(t) = 0$ for all $t \in [R^\star, R)$. This violates (2.5.4) unless $R \leq R^\star$, giving us an a priori bound on R.

Finally, the finiteness of R shows that the minimizing sequence Ω_i is trapped inside a fixed annulus, and the desired conclusion follows from standard compactness theorems in geometric measure theory. □

Proposition 2.5.2. *Suppose $n = 3$, $\mathcal{S} \subset M$ is finite, $\widetilde{g}$ is an $L^\infty(M) \cap C^{2,\alpha}_{\text{loc}}(M \setminus \mathcal{S})$ metric, $\alpha \in (0, 1)$, and $R(\widetilde{g}) > 0$ on $M \setminus \mathcal{S}$. Then, there exists a $C^{2,\alpha}(M)$ metric $\overline{g}$*

with $R(\overline{g}) > 0$ everywhere; i.e., $\sigma(M) > 0$.

Proof. We may assume, without loss of generality, that $\mathcal{S} \neq \emptyset$, for else there is nothing to do. For notational simplicity we relabel $\widetilde{g}$ as h. Let G denote the distributional solution of elliptic PDE

$$-8\Delta_h G + \phi(R(h); 1)G = \delta_{\mathcal{S}} \text{ on } M,$$

where $\delta_{\mathcal{S}}$ denotes the Dirac delta measure on $\mathcal{S}$, and ϕ is as in Section 2.4. Since h is uniformly Euclidean and $\phi(R(h); 1)$ is bounded, we know that

$$c_{0,G}^{-1} \operatorname{dist}_h(\cdot, \mathcal{S})^{-1} \leq G \leq c_{0,G} \operatorname{dist}_h(\cdot, \mathcal{S})^{-1} \text{ on } M \setminus \mathcal{S}, \tag{2.5.8}$$

for $c_{0,G} = c_{0,G}(M, h) > 0$, and, therefore,

$$c_G^{-1} \operatorname{dist}_g(\cdot, \mathcal{S})^{-1} \leq G \leq c_G \operatorname{dist}_g(\cdot, \mathcal{S})^{-1} \text{ on } M \setminus \mathcal{S}. \tag{2.5.9}$$

We refer the reader to [27] for the existence and the aforementioned blow up rate of Greens functions in this setting.

Consider, for small $\sigma > 0$, the conformal metric $h_\sigma = (1 + \sigma G)^4 h$ on $M \setminus \mathcal{S}$, which is $C^{2,\alpha}_{\mathrm{loc}}$, complete, noncompact, and whose scalar curvature satisfies

$$\begin{aligned} R(h_\sigma) &= (1 + \sigma G)^{-5}(-8\Delta_h(1 + \sigma G) + R(h)(1 + \sigma G)) \\ &= (1 + \sigma G)^{-5} R(h) + (1 + \sigma G)^{-5} \sigma (R(h) - \phi(R(h); 1))G > 0. \end{aligned}$$

Fix a family of disjoint open neighborhoods of the points in $\mathcal{S}$ (one for each point) labeled $\{U_p\}_{p \in \mathcal{S}}$, so that each $U_p \subset M$ is diffeomorphic to a 3-ball.

claim 2.5.3. *For every $p \in \mathcal{S}$, there exists a diffeomorphism*

$$\Phi_p : \mathbf{R}^3 \setminus B_1(0) \xrightarrow{\approx} U_p \setminus \{p\}$$

and a constant $c_{p,\sigma} > 0$ such that

$$c_{p,\sigma}^{-1}\delta \le {\Phi_p}^* h_\sigma \le c_{p,\sigma}\delta,$$

where δ denotes a flat metric on $\mathbf{R}^3 \setminus B_1(0)$.

Proof. From the manifold's smooth structure, there exists a diffeomorphism

$$\Psi_p : (B_1(0) \subset \mathbf{R}^3) \xrightarrow{\approx} (U_p \subset T),$$

such that $\Psi_p(0) = p$. We can then define

$$\Phi_p := \Psi_p \circ \iota,$$

where $\iota(x) = |x|^{-2}x$ is the inversion map on $\mathbf{R}^3 \setminus \{0\}$. With this definition for Φ_p, we see that

$$\begin{aligned} {\Phi_p}^* h_\sigma &= \iota^* \Psi_p^* (1 + \sigma G)^4 h \\ &= (1 + \sigma(G \circ \Psi_p \circ \iota))^4 (\iota^* {\Psi_p}^* h). \end{aligned}$$

Next, note that ${\Psi_p}^* h$ is uniformly Euclidean on $B_1(0)$, and thus certainly on $B_1(0) \setminus \{0\}$. By the scaling nature of ι, $\rho^4({\Psi_p}^* h)$ is uniformly Euclidean on $\mathbf{R}^3 \setminus B_1(0)$, with ρ denoting the standard radial polar coordinate on $\mathbf{R}^3$. The result then follows from the asymptotics in (2.5.9). □

claim 2.5.4. *For every $p \in \mathcal{S}$, and for every sufficiently small $\sigma > 0$, there exists a compact set $D_p \subset U_p$ whose boundary ∂D_p consists of a stable minimal 2-spheres in $(M \setminus \mathcal{S}, h_\sigma)$.*

Proof. Lemma 2.5.1 guarantee that for each $p \in \operatorname{sing} \mathcal{S}$ there exists a compact surface $\Sigma = \Sigma_\sigma$ in $(U_p \setminus \{p\}, h_\sigma)$, with least $\mathcal{H}^2_{h_\sigma}$-area among all surfaces in the same region that are homologous to ∂U_p. From standard regularity theory in geometric measure theory [48], Σ is regular and embedded away from ∂U_p. By a straightforward comparison argument and the smooth convergence $h_\sigma \to h$ away from $\mathcal{S}$, we know that

$$\lim_{\sigma\to 0} \mathcal{H}^2_{h_\sigma}(\Sigma_\sigma) = 0. \tag{2.5.10}$$

Next, denote

$$W_{p,\tau} := \{x \in U_p : \mathrm{dist}_g(x; \partial U_p) < \tau\},$$

where $\tau > 0$ is small. We will show that, for $\tau > 0$,

$$\Sigma \cap (W_{p,2\tau} \setminus W_{p,\tau}) = \emptyset,$$

as long as $\sigma > 0$ is sufficiently small (depending on τ).

Assume, by way of contradiction, that there exists a sequence $\sigma_j \downarrow 0$ such that the corresponding area-minimizing surfaces $\Sigma_j = \Sigma_{\sigma_j}$ are such that $\Sigma_j \cap (W_{p,2\tau} \setminus W_{p,\tau}) \neq \emptyset$ for all $j = 1, 2, \ldots$ For each j, pick $p_j \in \Sigma_j \cap (W_{p,2\tau} \setminus W_{p,\tau})$, and denote T_j the connected component of Σ_j in $W_{p,2\tau}$ containing the point p_j. By the local monotonicity formula in small regions of Riemannian manifolds, and the fact that $h_{\sigma_j} \to h$ smoothly away from $\operatorname{sing} \mathcal{S}$, we know that

$$\liminf_j \mathcal{H}^2_{h_{\sigma_j}}(T_j) > 0. \tag{2.5.11}$$

However, (2.5.11) contradicts (2.5.10).

Thus, $\Sigma \cap (W_{p,2\tau} \setminus W_{p,\tau}) = \emptyset$ as long as σ is sufficiently close to zero. This implies that

1. $\Sigma' \subset \overline{W}_{p,\tau}$, or
2. $\Sigma' \cap W_{p,2\tau} = \emptyset$,

for every connected component $\Sigma' \subset \Sigma$. Case (1) cannot occur for arbitrarily small $\sigma > 0$: there is a positive lower $\mathcal{H}^2_h$-area bound for all non-null-homologous surfaces in $\overline{W}_{p,\tau}$, in violation of (2.5.10). Thus, (2) holds for all connected components $\Sigma' \subset \Sigma$. Therefore,

$$\Sigma \cap W_{p,2\tau} = \emptyset \implies \Sigma \cap \partial U_p = \emptyset,$$

so Σ is a regular embedded minimal surface in $(U_p \setminus \{p\}, h_\sigma)$. Using $R(h_\sigma) > 0$ and the main theorem of [16], we conclude that Σ consists of stable minimal 2-spheres. The fact that Σ bounds a compact region follows from topological considerations. $\square$

Fix $\sigma > 0$ small enough so that the previous claim applies for all $p \in \mathcal{S}$, where the corresponding $\mathcal{H}^2_{h_\sigma}$-area minimizing surfaces are Σ_p, $p \in \mathcal{S}$. Denote $\Sigma := \cup_{p\in\mathcal{S}}\Sigma_p$.

Combining [31, Lemma 2.2.1] and [31, Corollary 2.2.13], we deduce that there exists a smooth manifold N^3, diffeomorphic to M^3, and a metric $\overline{h}$ on N which is uniformly C^2 on the complement of the image $\Sigma' \subset N$ of $\Sigma \subset M$, Lipschitz across Σ', and such that Σ' is minimal from both sides. Since the mean curvatures of Σ' from both sides agree, we may use the mollification procedure in [35, Proposition 4.1] to smooth out $\overline{h}$ to $\widetilde{h}$; the result follows by applying the conformal transformation in Lemma 2.3.1 to $\widetilde{h}$, with $\chi = R(\widetilde{h})$. $\square$

2.6 Proof of main theorems

Proof of Theorem 1.3.6. Notice that $\operatorname{sing}\mathcal{S} = \emptyset$, so, by Proposition 2.4.1, either

1. g is C^2 and $R(g) \equiv 0$ on M, or

2. there exists a C^2 metric $\widetilde{g}$ on M with $R(\widetilde{g}) > 0$ everywhere.

The latter contradicts the assumption $\sigma(M) \leq 0$, so the prior must be true. In that case, the result follows from Theorem 1.3.3. $\square$

Proof of Theorem 1.3.9. By Proposition 2.4.1, either

1. g is $L^\infty(M) \cap C^{2,\alpha}_{\mathrm{loc}}(M \setminus \operatorname{sing}\mathcal{S})$ with $R(g) \equiv 0$, or

2. there exists an $L^\infty(M) \cap C^{2,\alpha}_{\mathrm{loc}}(M \setminus \operatorname{sing}\mathcal{S})$ metric $\widetilde{g}$ with

$$R(\widetilde{g}) > 0 \text{ on } M \setminus \operatorname{sing}\mathcal{S}.$$

Proposition 2.5.2 rules out the second case when $\sigma(M) \leq 0$, so the first case holds.

claim 2.6.1. $\operatorname{Ric}(g) \equiv 0$ *on* $M \setminus \operatorname{sing}\mathcal{S}$.

Proof of claim. We argue by contradiction. Suppose that $\mathrm{Ric}(g) \neq 0$ at some $p \in M \setminus \operatorname{sing} \mathcal{S}$. Let U be some small smooth open neighborhood of p such that $U \subset\subset M \setminus \operatorname{sing} \mathcal{S}$; note that $g|_U \in C^{2,\alpha}(U)$.

Consider the Banach manifold

$$\mathcal{M}_g^{2,\alpha}(U) := \{\text{metrics } g' \in C^{2,\alpha}(U) : g - g' \equiv 0 \text{ on } \partial U\},$$

where $g - g' \equiv 0$ on ∂U is to be interpreted as the equality of tensors on $\overline{U}$ pointwise on the subset $\partial U \subset \overline{U}$; i.e., we aren't pulling back to ∂U. The scalar curvature functional

$$R : \mathcal{M}_g^{2,\alpha}(U) \to C^{0,\alpha}(U)$$

is a C^1 Banach map, with Fréchet derivative $\delta R(g') : T_{g'}\mathcal{M}_g^{2,\alpha}(U) \to C^{0,\alpha}(U)$ known to be given by

$$\delta R(g')\{h\} = -\Delta_{g'} \operatorname{Tr}_{g'} h + \operatorname{div}_{g'} \operatorname{div}_{g'} h - \langle h, \mathrm{Ric}(g')\rangle_{g'},$$

for all $g' \in \mathcal{M}_g^{2,\alpha}(U)$, $h \in T_{g'}\mathcal{M}_g^{2,\alpha}(U) \cong (T_0^{2,\alpha} \otimes T_0^{2,\alpha})^*(U)$. Here, $T_{g'}\mathcal{M}_g^{2,\alpha}(U)$ denotes the tangent space at g' to the Banach manifold $\mathcal{M}_g^{2,\alpha}(U)$, and $T_0^{2,\alpha}(U)$ denotes the space of contravariant $C^{2,\alpha}$ tensors that vanish on ∂U.

Fix some $h \in T_g\mathcal{M}_g^{2,\alpha}(U)$. Define $\gamma : [0,\delta) \subset \mathcal{M}_g^{2,\alpha}(U)$ to be a C^1 curve with $\gamma(0) = g$, $\gamma'(0) = -h$. By definition of Fréchet derivatives, the fact $R(g) \equiv 0$, and the trivial continuous embedding $C^{0,\alpha}(U) \hookrightarrow L^\infty(U)$, we have

$$\lim_{t \downarrow 0} \frac{\|R(\gamma(t)) - \Delta_g \operatorname{Tr}_g th + \operatorname{div}_g \operatorname{div}_g th - \langle th, \mathrm{Ric}(g)\rangle_g\|_{L^\infty(U)}}{t} = 0. \tag{2.6.1}$$

In particular, by observing that the Fréchet derivative contains two divergence terms that integrate to zero with respect to $d\,\mathrm{Vol}_g$, we have

$$\begin{aligned}
&\lim_{t\downarrow 0}\frac{1}{t}\int_U (R(\gamma(t)) - \langle th, \mathrm{Ric}(g)\rangle_g)\, d\,\mathrm{Vol}_g \\
&\quad = \lim_{t\downarrow 0}\frac{1}{t}\int_U (R(\gamma(t)) - \Delta_g \mathrm{Tr}_g\, th + \mathrm{div}_g \mathrm{div}_g\, th - \langle th, \mathrm{Ric}(g)\rangle_g)\, d\,\mathrm{Vol}_g \\
&\quad = 0, \qquad (2.6.2)
\end{aligned}$$

where the last equality follows from (2.6.1).

Consider the map $t \in (-\varepsilon, \varepsilon) \mapsto \lambda_1(t) = \lambda_1(-\frac{4(n-1)}{n-2}\Delta_{g_t} + R(g_t))$. Since g_t is an smooth (in t) family of $C^{2,\alpha}$ metrics, we know that $t \mapsto \lambda_1(t)$ is C^1; the corresponding first eigenfunctions, u_t, normalized to have $\|u_t\|_{L^2(M,g_t)} = 1$, form a C^1 path in $W^{1,2}(M)$. (See, e.g., [33, Lemma A.1]). Notice that u_0 is a constant, $R_0 \equiv 0$ on $M \setminus \mathrm{sing}\,\mathcal{S}$, and $\lambda_1(0) = 0$. Observe that

$$\begin{aligned}
\lambda'(0) &= \frac{d}{dt}\bigg|_{t=0}\int_M \left[\frac{4(n-1)}{n-2}|\nabla_{g_t} u_t|^2 + R(g_t)u_t^2\right] d\,\mathrm{Vol}_{g_t} \\
&= \int_M \frac{d}{dt}\bigg|_{t=0} R(g_t) d\,\mathrm{Vol}_g \\
&= \int_M \langle h, \mathrm{Ric}(g)\rangle\, d\,\mathrm{Vol}_g\,.
\end{aligned}$$

where we have used the fact that the only nonzero contribution of the derivative is from $\frac{d}{dt}R(g_t)$ and (2.6.2).

Suppose, now, that $(\mathrm{Ric}(g))_\sigma$ denotes a (tensorial) mollification of $\mathrm{Ric}(g)$ away from p, such that

$$\lim_{\sigma\to 0}\|(\mathrm{Ric}(g))_\sigma - \mathrm{Ric}(g)\|_{L^\infty(U)} = 0. \qquad (2.6.3)$$

If $\xi : M \to [0,1]$ is a smooth cutoff function such that $\xi(p) = 1$ and $\mathrm{spt}\,\xi \subset U$, then (2.6.3) implies

$$\lim_{\sigma\to 0}\langle \xi(\mathrm{Ric}(g))_\sigma, \mathrm{Ric}(g)\rangle_{L^2(U,g)} > 0. \qquad (2.6.4)$$

Together, (2.6.2), (2.6.4) imply that for all sufficiently small $\sigma > 0$,

$$\lambda(t) > 0,$$

for all $t \in (0, t_0(\sigma))$, when $h = \xi(\mathrm{Ric}(g))_\sigma \in T_g\mathcal{M}_g^{2,\alpha}(U)$; in fact, since $\gamma(t)$, g are all uniformly equivalent for small σ, t, we have

$$\int_U R(\gamma(t))\, d\,\mathrm{Vol}_{\gamma(t)} > 0 \text{ for } t \in (0, t_0(\sigma)).$$

Now fix a small σ. This implies that for any $t \in (0, t_0(\sigma))$, $\widetilde{g}_t = u_t^{\frac{4}{n-2}} g_t$ is a $L^\infty(M) \cap C^{2,\alpha}_{\mathrm{loc}}(M \setminus \operatorname{sing}\mathcal{S})$ metric with *positive* scalar curvature on its regular part; this contradicts Proposition 2.5.2 when $\sigma(M) \le 0$. □

Given the above, all that remains to be checked is that g is smooth across $\operatorname{sing}\mathcal{S}$. This follows (when $n = 3$) from the main theorem of Smith-Yang [51] on the removability of isolated singularities of Einstein metrics. □

We now turn our attention onto asymptotically flat manifolds and prove Theorem 1.3.10, 1.3.11. The idea is to take the smoothed metric g_ε in Lemma 2.2.1 and apply a conformal deformation to g_ε with small change of the ADM mass. Assume (M^n, g) is an asymptotically flat manifold, $\mathcal{S} \subset M$ is a compact nondegenerate $(n-2)$-skeleton which is η-regular along $\operatorname{reg}\mathcal{S}$.

Proof of Theorem 1.3.10. Notice that $\operatorname{sing}\mathcal{S} = \emptyset$. By Lemma 2.2.1, for every $\gamma > 0$, there exists constant ε_1 such that for every $\varepsilon \in (0, \varepsilon_1]$, there is a metric $\hat{g}_\varepsilon$ on M such that:

1. $\widehat{g}_\varepsilon$ is $C^2(M)$;

2. $\widehat{g}_\varepsilon = g$ on $M \setminus B^g_\varepsilon(\operatorname{reg}\mathcal{S})$;

3. $\|R(\widehat{g}_\varepsilon)_-\|_{L^{\frac{n}{2}+\delta}(M,g)} \le \gamma$;

By the maximum principle and the Poincaré-Sobolev inequality, we conclude that the elliptic boundary value system

$$\left\{\begin{matrix} \Delta_{\widehat{g}_\varepsilon} u_\varepsilon + c_n R(\widehat{g}_\varepsilon)_- u_\varepsilon = 0 \\ \lim_{x\to\infty} u_\varepsilon = 1 \\ u_\varepsilon = 0 \text{ on } \partial M \end{matrix}\right\}$$

has a unique solution u_ε, and $0 < u_\varepsilon < 1$. This follows as in [45]. The same argument as in [35, Proposition 4.1], moreover, shows that

$$\lim_{\varepsilon\to 0} \|u_\varepsilon - 1\|_{L^\infty(M)} = 0, \quad \|u_\varepsilon\|_{C^{2,\alpha}(K)} \leq C_K,$$

for each compact set $K \subset M \setminus \mathcal{S}$, where $C_K = C_K(g, \mathcal{S}, \Lambda, K)$.

Now define $\widetilde{g}_\varepsilon = u_\varepsilon^{\frac{4}{n-2}} \widehat{g}_\varepsilon$. By the choice of u_ε, $R(\widetilde{g}_\varepsilon) \geq 0$ everywhere. We then apply the argument of [35, Lemma 4.2] and conclude that

$$m_{ADM}(g) = \lim_{\varepsilon\to 0} m_{ADM}(\widetilde{g}_\varepsilon),$$

which is ≥ 0 by the smooth positive mass theorem [45, 43, 41].

If the cone angle along $\mathcal{S}$ is not identically 2π, then Lemma 2.2.1 additionally gives the following concentration behavior of scalar curvature:

$$R(\widehat{g}_\varepsilon) \geq C_1 \varepsilon^{-2} \text{ on } B^g_{c_2\varepsilon}(\mathcal{S}) \setminus B^g_{c_3\varepsilon}(\mathcal{S}),$$

where $C_1 = C_1(\mathcal{S}, g, \Lambda)$, $c_j = c_j(\mathcal{S}, g, \Lambda)$, $j = 2, 3$. Then [35, Proposition 4.2] implies that

$$\liminf_{\varepsilon\to 0} m_{ADM}(\widetilde{g}_\varepsilon) > 0,$$

and hence $m_{ADM}(g) > 0$. Hence if $m_{ADM}(g) = 0$, then g is smooth across $\mathcal{S}$, and therefore the rigidity conclusion of the smooth positive mass theorem in [41] implies that g is flat everywhere. $\square$

Proof of Theorem 1.3.11. Take $\widehat{g}_\varepsilon$ as in Lemma 2.2.1. By the maximum principle and the Poincaré-Sobolev inequality, the weak $W^{1,2}_{\text{loc}}(M)$ solution of

$$\begin{Bmatrix} \Delta_{\widehat{g}_\varepsilon} u_\varepsilon + c_n R(\widehat{g}_\varepsilon)_- u_\varepsilon = 0 \\ \lim_{x\to\infty} u_\varepsilon = 1 \\ u_\varepsilon = 0 \text{ on } \partial M \end{Bmatrix}$$

exists and satisfies $0 < u_\varepsilon < 1$. By standard elliptic theory and De Giorgi-Nash-Moser theory, $u_\varepsilon \in C^{2,\alpha}_{\text{loc}}(M \setminus \operatorname{sing}\mathcal{S}) \cap C^{0,\theta}(M)$, for some $\theta \in (0,1)$. Moreover,

$$\inf u_\varepsilon \geq c_1 \sup u_\varepsilon = c_1 = c_1(g, \mathcal{S}, \Lambda),$$

by Moser's Harnack inequality.

The metric $\widetilde{g}_\varepsilon = u_\varepsilon^4 \widehat{g}_\varepsilon$ is asymptotically flat with only isolated singularities and nonnegative scalar curvature away from $\operatorname{sing}\mathcal{S}$. Let $G(\cdot, \operatorname{sing}\mathcal{S})$ be the Green's function of $\Delta_{\widetilde{g}_\varepsilon} - \frac{1}{8}\phi(R(\widetilde{g}_\varepsilon; 1))$ with poles at $\operatorname{sing}\mathcal{S}$, and which decays to zero at infinity. Apply the excision lemma (Lemma 2.5.1) to the blown up metric

$$h_\varepsilon = (1 + \sigma(\varepsilon) G)^4 \widetilde{g}_\varepsilon$$

on $M \setminus \operatorname{sing}\mathcal{S}$, where $\sigma(\varepsilon)$ is the constant that appears in the proof of Proposition 2.5.2, and $\lim_{\varepsilon\to 0}\sigma(\varepsilon) = 0$. Excise (M, h_ε) along each area minimizing two-sphere in the asymptotically Euclidean end in (M, h_ε).

Notice that, since $\lim_{\varepsilon\to 0}\sigma(\varepsilon) = 0$,

$$m_{ADM}(g) = \lim_{\varepsilon\to 0} m_{ADM}(\widetilde{g}_\varepsilon) = \lim_{\varepsilon\to 0} m_{ADM}(h_\varepsilon)$$

on each asymptotically flat end of M (recall that we've excised one end). By the smooth positive mass theorem [45, 43], $m_{ADM}(h_\varepsilon) \geq 0$. Therefore $m_{ADM}(g) \geq 0$. (To see the above limits on the ADM masses, we first notice that the metrics h_ε and $\widetilde{g}_\varepsilon$ differ by a factor which converges to zero in C^2, as $x \to \infty$ and $\varepsilon \to 0$. Hence from the definition of ADM mass, $\lim_{\varepsilon\to 0} m(h_\varepsilon) = \lim_{\varepsilon\to 0} m(\widetilde{g}_\varepsilon)$. To see that $\lim_\varepsilon(\widetilde{g}_\varepsilon) = m(g)$, we just apply [35, Lemma 4.2] again on the family of conformal factors u_ε.)

Now we conclude the rigidity case. Assume $m_{ADM}(g) = 0$. If $R(g)$ is not identically zero on $\operatorname{reg}\mathcal{S}$, or the cone angle along $\operatorname{reg}\mathcal{S}$ is not identically 2π, then a similar concentration behavior as in the proof of Theorem 1.3.10, combined with [35, Proposition 4.2], show that

$$\liminf_{\varepsilon\to 0} m_{ADM}(h_\varepsilon) > 0;$$

this would contradict our rigidity assumption. Therefore g is scalar flat on $M \setminus \mathcal{S}$, and is $C^{2,\alpha}$ across $\operatorname{reg}\mathcal{S}$ locally away from $\operatorname{sing}\mathcal{S}$. Now we prove $\operatorname{Ric}(g) = 0$ away from sing $\mathcal{S}$. Consider the metrics $g_t = g - th$, where h is a $C^{2,\alpha}$ symmetric $(0,2)$ tensor, compactly supported away from sing $\mathcal{S}$. Let u_t be the weak solution to

$$\begin{Bmatrix} \Delta_{g_t} u_t + c_n R(g_t)_- u_t = 0 \\ \lim_{x\to\infty} u_t = 1 \\ u_t = 0 \text{ on } \partial M \end{Bmatrix}$$

Then (see, e.g., the proof of Theorem 1.3.9 for details) the metric $\widehat{g}_t = u_t^4 g_t$ has zero scalar curvature and isolated uniformly Euclidean point singularities. Therefore $m_{ADM}(\widehat{g}_t) \geq 0$ by the positive mass theorem for isolated L^∞ singularities established just above. On the other hand, by a similar calculation as in [45], we see that

$$\left.\frac{d}{dt}\right|_{t=0} m_{ADM}(M, \widehat{g}_t) = C_1(n) \int_M \langle \mathrm{Ric}(g), h \rangle \,.$$

Now if $\mathrm{Ric}(g) \neq 0$ in an open neighborhood of $M \setminus \operatorname{sing} \mathcal{S}$, we may pick $h = \xi(\mathrm{Ric}(g))_\sigma$, where ξ is a function compactly supported in U, $(\mathrm{Ric}(g))_\sigma$ is a $C^{2,\alpha}$ mollification of $\mathrm{Ric}(g)$, and make $m'_{ADM}(0) \neq 0$. (See the proof of Theorem 1.3.9.) This is a contradiction to the positive mass theorem for isolated L^∞ singularities, which would imply that $m_{ADM}(0) = 0$ is a global minimum of $t \mapsto m_{ADM}(M, \widehat{g}_t)$.

Finally, being in $n = 3$, we conclude that g is smooth and flat across sing $\mathcal{S}$ by the removable singularity theorem [51] of Einstein metrics. □

2.7 Examples, counterexamples, remarks

2.7.1 Codimension-1 singularities and mean curvature

Question 1.3.4 is *true* when $\mathcal{S} \subset M$ is a closed, embedded, two-sided submanifold, and:

1. g is smooth up to $\mathcal{S}$ from both sides,

2. g induces the same metric $g_{\mathcal{S}}$ on $\mathcal{S}$ from both sides, and

3. the sum of mean curvatures of $\mathcal{S}$ computed with respect to the two unit normals as outward unit normals is nonnegative.

We'd like to point out that condition (3) is imperative, as the following counterexample clearly shows: take a flat n-torus $\mathbf{R}^n/\mathbf{Z}^n$, remove a small geodesic ball, and replace it with a constant curvature half-sphere of the same radius. Indeed, here the sum of mean curvatures is negative (one negative, the other zero), and the resulting metric g does not have a removable singular set.

For some intuition on (3), one may use the first variation of mean curvature along a geodesic foliation of M about $\mathcal{S}$, and the Gauss equation on $\mathcal{S}$, to see that

$$R(g)|_{\mathcal{S}} = R(g_{\mathcal{S}}) - \left[\frac{d}{dt}H_t\right]_{t=0} - |A_{\mathcal{S}}|^2 - H_{\mathcal{S}}^2. \tag{2.7.1}$$

Heuristically, a positive sum of mean curvatures contributes a distributionally positive component to the scalar curvature $R(g)$ evaluated at $\mathcal{S}$.

2.7.2 Codimension-2 singularities and cone angles

Allowing edge metrics with cone angles larger than 2π invalidates Question 1.3.4. We illustrate this here with a counterexample:

The example in Figure 2.2 inspired by [17, Example 5.6-B']. For each integer $g \geq 2$, we describe a flat metric on a genus g Riemann surface with isolated conical points with cone angle $> 2\pi$.

Take a planar graph G with two nodes, p and q, and $g+1$ edges. The graph separates the plane into $g+1$ connected components. Excise one disk from each bounded face and, and the exterior of a disk from the unbounded face, as in Figure 2.2.

Each face component is diffeomorphic to $\mathbf{S}^1 \times [0,1]$. Endow each face with a flat product metric via this diffeomorphism. Note that the metric now is smooth away from p, q, and is conical at p, q, with cone angle $(g+1)\pi$. The $g+1$ boundary components of this manifold, namely the $\mathbf{S}^1 \times \{1\}$'s, are totally geodesic. Take the doubling of this manifold across its boundary to obtain a genus-g surface, Σ. Now

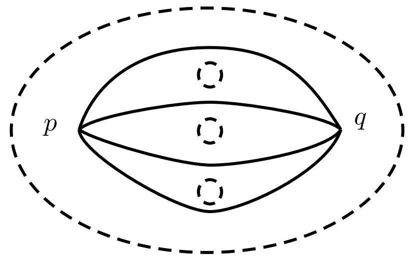

Figure 2.2: $g = 3$ construction.

Σ has a smooth flat metric with four isolated conical singularities, each of which has cone angle $(g+1)\pi$.

For $n \geq 3$, consider the manifold $M = \Sigma \times (\mathbf{S}^1)^{n-2}$ with the product metric. It's easy to see that this metric is an edge metric, flat on its regular part, with singularities with cone angles $(g+1)\pi$. However, since M trivially carries a smooth metric with nonpositive sectional curvature, its σ-invariant satisfies $\sigma(M) \leq 0$ by [18, Corollary A].

2.7.3 Codimension-3 singularities that are not uniformly Euclidean.

The "uniformly Euclidean" condition (i.e., that the metric be L^∞) is imperative for Conjecture 1.3.7 to hold true. Indeed, if g were allowed to blow up, then the doubled Riemannian Schwarzschild metric

$$g = \left(1 + \frac{m}{2r}\right)^4 \delta, \ m > 0,$$

on $\mathbf{R}^3 \setminus \{0\}$ would be a counterexample: it can be viewed as a non-Einstein, scalar-flat metric on a twice-punctured 3-sphere.

Likewise, if g were not bounded from below, then the negative-mass Riemannian Schwarzschild metric

$$g = \left(1 + \frac{m}{2r}\right)^4 \delta, \ m < 0,$$

on $\mathbf{R}^3 \setminus \overline{B}_{-m/2}(0)$ would yield a counterexample: it can be conformally truncated near infinity to match Euclidean space, where we then identity the opposide faces of a large cube to yield a topologically smooth 3-torus with positive scalar curvature, which would be a counterexample since tori are known to have $\sigma(T^3) = 0$. See [28, Section 6] for more details.

2.7.4 Examples of edge metrics

Orbifold metrics provide an important source of edge metrics (with angle $< 2\pi$); they can be obtained as the quotient metric under a $\mathbf{Z}_k$ isometry group with an $(n-2)$-dimensional fixed submanifold.

Generally speaking, the scalar curvature geometry of orbifolds can be substantially different from that of manifolds. For instance, Viaclovsky [61] showed that the Yamabe problem of finding constant scalar curvature metrics is *not* generally solvable on orbifolds. (On manifolds, the problem was shown to be completely solvable in [60, 8, 39, 44].) Theorem 1.3.6 nevertheless confirms that edge-type orbifold singularities along a codimension two submanifold cannot go so far as to change the Yamabe type from nonpositive to positive.

2.7.5 Sormani-Wenger intrinsic flat distance

For the reader's convenience, we recall the Sormani-Wenger definition:

Definition 2.7.1 ([54, Definition 1.1]). *Let (M_1^n, g_1), (M_2^n, g_2) be two closed Riemannian manifolds. Their intrinsic flat distance is defined as*

$$d_{\mathcal{F}}((M_1, g_1), (M_2, g_2)) := \inf\{d_F^Z((\varphi_1)_\# T_1, (\varphi_2)_\# T_2) : Z, \varphi_1, \varphi_2\};$$

the infimum is taken over all complete metric spaces (Z, d) and all possible isometric embeddings φ_i, $i = 1, 2$, of the metric spaces induced by (M_i, g_i) into (Z, d). The T_i, $i = 1, 2$ denote the integral n-currents $T_i(\omega) := \int_{M_i} \omega$, $(\varphi_i)_\# T_i$ denote their pushforwards to Z, and d_F^Z denotes the Ambrosio-Kirchheim metric space flat norm [5]:

$$d_F^Z(S, T) := \inf\{\mathbf{M}(U) + \mathbf{M}(V) : S - T = U + \partial V\};$$

this infimum is taken over integral n-currents U and integral $(n+1)$-currents V in (Z, d).

In this chapter we have shown that the following families of singular Riemannian manifolds (M^n, g) in Theorems 1.3.6, 1.3.9 will either have:

1. $\sigma(M) \leq 0$ and be everywhere smooth and Ricci-flat to begin with; or

2. $\sigma(M) > 0$ and carry smooth metrics of positive scalar curvature.

We conjecture that, in the second case, the desingularizations we have set up in this chapter give rise to $d_{\mathcal{F}}$-Cauchy sequences of smooth closed PSC manifolds, which, moreover, recover (M^n, g) as a metric $d_{\mathcal{F}}$-limit.

A more ambitious conjecture, that appears out of reach with today's state of the art, is to show that (M^n, g), $n \geq 4$, with singular sets of codimension ≥ 3 and positive scalar curvature everywhere else, arise as metric $d_{\mathcal{F}}$-limits of smooth closed PSC manifolds; cf. Conjecture 1.3.7.

Chapter 3

Dihedral rigidity conjecture in dimension three

In this chapter we study the variational problem (1.4.4) and prove Theorem 1.4.3 and Theorem 1.4.4. We start from the existence and regularity of the solution to problem (1.4.4).

3.1 Existence and regularity

We discuss the existence and regularity of the minimizer for the variational problem (1.4.4). The goal of this section is:

Theorem 3.1.1. *Consider the variational problem* (1.4.4) *in a Riemannian polyhedron* (M^3, g) *of cone or prism type. Assume* $\mathcal{I} < 0$ *if* M *is of cone type. Then* $\mathcal{I}$ *is achieved by an open subset* E*. Moreover,* $\Sigma = E \cap \mathring{M}$ *is an area minimizing surface,* $C^{1,\alpha}$ *to its corners for some* $\alpha > 0$*, and meets* F_j *at constant angle* γ_j*.*

We first introduce some notations and basic geometric facts on capillary surfaces. Then we reduce the obstacle problem (1.4.4) equivalently to a variational problem without any obstacle. This is done via a varifold maximum principle. Hence the regularity theory develop in [58] is applicable, and we get regularity in $\mathring{\Sigma}$, and in $\partial\Sigma$ in $\mathring{F}_j$. The regularity at the corners of Σ is then studied with an idea of Simon [49].

At the corner, we prove that the surface is graphical over its planar tangent cone. Then we invoke the result of Lieberman [25], which showed that the unit normal vector field is Hölder continuous up to the corners.

3.1.1 Preliminaries

We start by discussing some geometric properties of capillary surfaces. In particular, we deduce the first and second variation formulas for the energy functional (1.4.3). Let us fix some notations.

Let K be an orientable Riemannian manifold of dimension n and M a closed compact polyhedron of cone or prism types in K. Let Σ^{n-1} be an orientable n dimensional compact manifold with non-empty boundary $\partial\Sigma$ and $\partial\Sigma \subset \partial M$. We denote the topological interior of a set U by $\mathring{U}$. Assume Σ separates $\mathring{M}$ into two connected components. Fix one component and call it E. Denote X the outward pointing unit normal vector field of ∂M in M, N the unit normal vector field of Σ in E pointing into E, ν the outward pointing unit normal vector field of $\partial\Sigma$ in Σ, $\overline{\nu}$ the unit normal vector field of $\partial\Sigma$ in ∂M pointing outward E. Let A denote the second fundamental form of $\Sigma \subset E$, II denote the second fundamental form of $\partial M \subset M$. We take the convention that $A(X_1, X_2) = \langle \nabla_{X_1} X_2, N\rangle$. Denote $H, \overline{H}$ the mean curvature of $\Sigma \subset E$, $\partial M \subset M$, respectively. Note that in our convention, the unit sphere in $\mathbf{R}^3$ has mean curvature 2.

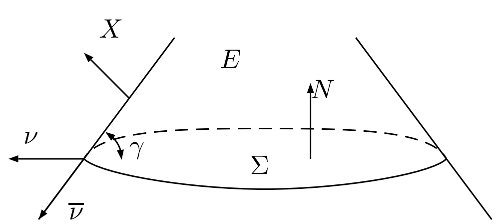

Figure 3.1: Capillary surfaces

By an admissible deformation we mean a diffeomorphism $\Psi : (-\varepsilon, \varepsilon) \times \Sigma \to M$ such that $\Psi_t : \Sigma \to M$, $t \in (-\varepsilon, \varepsilon)$, defined by $\Psi_t(q) = \Psi(t, q)$, $q \in \Sigma$, is an embedding satisfying $\Psi_t(\Sigma) \subset \mathring{M}$ and $\Psi_t(\partial\Sigma) \subset \partial M$, and $\Psi_0(x) = x$ for all $x \in \Sigma$. Denote $\Sigma_t = \Psi_t(\Sigma)$. Let E_t be the corresponding component separated by Σ_t. Denote

$Y = \frac{\partial \Psi(t,\cdot)}{\partial t}|_{t=0}$ the vector field generating Ψ. Then Y is tangential to ∂M along $\partial\Sigma$. Fix the angles $\gamma_1, \cdots, \gamma_k \in (0, \pi)$ on the faces $F_1, \cdots, F_k$ of M. Consider the energy functional

$$F(t) = \mathcal{H}^{n-1}(\Sigma_t) - \sum_{j=1}^{k} (\cos\gamma_j)\mathcal{H}^{n-1}(\partial E_t \cap F_j).$$

Then the first variation of $F(t)$ is given by

$$\left.\frac{d}{dt}\right|_{t=0} F(t) = -\int_\Sigma H f d\mathcal{H}^{n-1} + \sum_{j=1}^{k} \int_{\partial\Sigma \cap F_j} \langle Y, \nu - (\cos\gamma_j)\overline{\nu}\rangle \, d\mathcal{H}^{n-2}, \tag{3.1.1}$$

where $f = \langle Y, N\rangle$ is the normal component of the vector field Y. We defer the derivation of (3.1.1) later in (3.1.21), where we formulate it in the context of varifolds. The surface Σ is said to be minimal capillary if $F'(t) = 0$ for any admissible deformations. If follows from (3.1.1) that Σ is minimal capillary if and only if $H \equiv 0$ and $\nu - (\cos\gamma_j)\overline{\nu}$ is normal to F_j; that is, along F_j the angle between the normal vectors N and X, or equivalently, between ν and $\overline{\nu}$, is everywhere equal to γ_j.

Assume Σ is minimal capillary. We then have the second variational formula:

$$\begin{aligned}\left.\frac{d^2}{dt^2}\right|_{t=0} F(0) = &-\int_\Sigma (f\Delta f + (|A|^2 + \mathrm{Ric}(N,N))f^2) d\mathcal{H}^{n-1} \\ &+ \sum_{j=1}^{k} \int_{\partial\Sigma\cap F_j} f\left(\frac{\partial f}{\partial \nu} - Qf\right) d\mathcal{H}^{n-2},\end{aligned} \tag{3.1.2}$$

where on $\partial\Sigma \cap F_j$,

$$Q = \frac{1}{\sin\gamma_j}\,\mathrm{II}(\overline{\nu},\overline{\nu}) + \cot\gamma_j A(\nu,\nu).$$

Here Δ is the Laplace operator of the induced metric on Σ, and Ric is the Ricci curvature of M. For a proof of the second variation formula, we refer the readers to the appendix of [38].

3.1.2 Maximum principles

We first observe that (1.4.4) is a variational problem with *obstacles*: $E \cap B_1 = \emptyset$ if M is of cone type, and $B_2 \subset E$, $E \cap B_1 = \emptyset$ if M is of prism type. To apply the existence and the regularity theories of Taylor [58], we first prove that it suffices to consider a variational problem *without* any obstacles. Such reduction is usually achieved via varifold maximum principles, see e.g. [52, 62][24]. In our case, the maximum principles hinge upon the special structure of the obstacle: that B (or B_1, B_2) is mean convex, and that the dihedral angles along $\partial F_j \cap B$ are nowhere larger than γ_j. In fact, if $\Sigma = \partial E \cap \mathring{M}$ is a C^1 surface with piecewise smooth boundary, then it is not hard to see from the first variational formula (3.1.1) that

- Σ and B do not touch in the interior.
- $\partial\Sigma$ does not contain any point on $F_j \cap B$ where the dihedral angle is strictly less than γ_j.

Thus Σ is a minimal surface that meets each F_j at constant angle γ_j.

The interior maximum principle has been investigated in different scenarios [50, 52, 62, 63]. Notice that the energy minimizer of (1.4.4) is necessarily area minimizing in the interior. We apply the strong maximum principle by Solomon-White [52] and conclude that the surface $\Sigma = \partial E \cap \mathring{M}$ cannot touch the base face B, unless lies entirely in B.

Here we develop a new boundary maximum principle. For the purpose of this thesis, we only consider energy minimizing currents of codimension 1 associated to (1.4.4). However, we conjecture that a similar statement should hold for varifolds with boundary in general codimension. (See, for instance, the boundary maximum principle of Li-Zhou [24].)

Proposition 3.1.2. *Let M be a polyhedron of cone or prism types. Let $T \in \mathcal{D}^2(M)$, $E \in \mathcal{D}^3(U)$ be rectifiable currents with $T = \partial E \cap \mathring{M}$ and $\operatorname{spt}(\partial T) \subset F$. Assume E is an energy minimizer of* (1.4.4)*. Then $\operatorname{spt}(T)$ does not contain any point on the edge $F_j \cap B$ where the dihedral angle is less than γ_j.*

By a similar argument, in the case that M is of prism type, $\mathrm{spt}(T)$ does not contain any point on $F_j \cap B_2$ where the dihedral angle is less than $\pi - \gamma_j$. Combine this with the interior maximum principle, we conclude that the minimizer to (1.4.4) lies in the interior of M, and hence an energy-minimizer for the $\mathcal{F}$ *without* any barriers. Thus the existence and regularity theory developed in [58][14] concludes that the minimizer $T = \partial E \cap \mathring{M}$ exists, and is regular from the corners.

Proof. Assume, for the sake of contradiction, that a point $q \in F_j \cap B$ is also in $\mathrm{spt}\, T$, and that the dihedral angle between F_j and B at q is less than γ_j. In the rest of proof we use $\|T\|$ to denote the associated varifold. Since $T = \partial E \cap \mathring{M}$, it is a rectifiable current with multiplicity one, the first variational formula for the energy functional $\mathcal{F}$ applies:

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{F}(\Psi_t(E)) = -\int H f d\|T\| + \sum_j \int \langle Y, \nu - (\cos\gamma_j)\overline{\nu}\rangle \, d\|\partial T\|, \tag{3.1.3}$$

where $Y, f, \overline{\nu}$ are the geometric quantities defined as before, H is the generalized mean curvature of T, and ν is the generalized outward unit normal of $\|\partial T\|$. Since the dihedral angle between F_j and B at q is strictly less than γ_j, we have

$$\langle Y, \nu' - \cos\gamma_j \overline{\nu}\rangle > 0, \tag{3.1.4}$$

for any ν' at q which is the outward unit normal vector of some two-plane in $T_q M$. By the interior maximum principle, $H \equiv 0$.[1] Hence

$$\|\partial T\|(\mathrm{spt}(T) \cap \mathcal{B}) = 0, \tag{3.1.5}$$

where

$$\mathcal{B} = \bigcup_j \{q \in F_j \cap B : \text{the dihedral angle at } q \text{ is less than } \gamma_j.\}$$

The boundary regularity theorem of Taylor [58] implies that for any point $q' \in \partial T \setminus \mathcal{B}$, the current T is smooth up to q'. In particular, the density of T at q' is given

[1] The same argument here applies to the general case where the barrier B has bounded mean curvature, see Remark 3.1.3.

by $\Theta^2(\|T\|, q') = \frac{1}{2}$. Denote W the two dimensional varifold $v(\partial E \cap F_j)$ associated with $E \cap F_j$, $Z = \|T\| - \cos\gamma_j W$. Since the faces F_j and B intersects smoothly at q, we have the following monotonicity formula (we delay the derivation of a more general monotonicity formula in the next section, see (3.1.22)):

$$\exp(cr^\alpha)\frac{\|Z\|(B_r(q))}{r^2} \text{ is increasing in } r, \tag{3.1.6}$$

for r sufficiently small, where c and $\alpha > 0$ depends only on the geometry of F_j and B. It is then straightforward to check as in Theorem 3.5-(1) in [4] that the $\theta^2(\|T\|, \cdot)$ is an uppersemicontinuous function on $\operatorname{spt}(T) \cap \partial T$. By (3.1.5) we then conclude

$$\Theta^2(\|T\|, \cdot) \geq \frac{1}{2} > 0 \tag{3.1.7}$$

everywhere on $\operatorname{spt}(T) \cap \partial T$.

Consider a tangent cone T_∞ of T at q. Let E_∞ be the associated three dimensional current with $T_\infty = \partial E_\infty$. By the monotonicity (3.1.6) and the lower density bound (3.1.7), T_∞ is a nonempty cone in C through q_∞, where C is the region in $\mathbf{R}^3$ enclosed by the two planes $F_{j,\infty}$ and B_∞ intersecting at an angle $\gamma' < \gamma_j$, and where $q_\infty \in F_\infty \cap B_\infty$. By scaling, for any open set $U \subset\subset \mathbf{R}^3$, E_∞ minimizes the energy functional

$$\mathcal{F}_\infty(E') = \mathcal{H}^2(\partial E' \cap \mathring{C} \cap U) - (\cos\gamma_j)\mathcal{H}^2(\partial E' \cap F_\infty \cap U) \tag{3.1.8}$$

among open sets E' with $\partial E' \subset \overline{F_\infty}$. Since two-planes are the only minimal cones in $\mathbf{R}^3$, T_∞ is a two-plane through q_∞. However, since $\measuredangle(F_\infty, \mathring{B}_\infty) < \gamma_j$, no two-plane through q_∞ can be the minimizer of (3.1.8). Contradiction.

□

Remark 3.1.3. *The above proof only uses the fact that T is minimal in a very weak manner. In fact, the same argument holds with under the assumption that the generalized mean curvature H is bounded measurable. This is implied, for instance, by that the barrier $\overline{B}$ has bounded mean curvature (instead of being mean convex).*

Remark 3.1.4. *The fact that T is energy minimizing is only used to guarantee the* *existence of an area minimizing tangent cone. Motivated by [52], we speculate that a similar statement should hold for varifolds with boundary that are stationary for the energy functional* (1.4.4)*.*

3.1.3 Regularity at the corners

We proceed to study the regularity of the minimizer $T = \partial E \cap \mathring{M}$ at the corners. Since T is regular away from the corners, our idea is to adapt the argument of Simon [49], and prove $\operatorname{spt}(T)$ is graphical near a corner. We refer the readers to [49] for full details. Then we apply the theorem of Lieberman [25] to conclude that $\operatorname{spt}(T)$ has a Hölder continuous unit normal vector field to its corners.

Consider any two adjacent side faces F_j, F_{j+1} and let $L = F_j \cap F_{j+1}$. Without loss of generality let $j = 1$. Fix a point $q \in \operatorname{spt}(T) \cap L$. Let θ be the angle between F_1 and F_2 at q. Recall that we assume

$$|\pi - (\gamma_1 + \gamma_2)| < \theta < \pi - |\gamma_1 - \gamma_2|. \tag{3.1.9}$$

For $\rho > 0$, denote $C_\rho = \{x \in M : \operatorname{dist}_M(x, L) < \rho\}$, $B_\rho = \{x \in M : \operatorname{dist}_M(x, q) < \rho\}$. In this section, and subsequently, let C be a constant that may change from line to line, but only depend on the geometry of the polyhedron M. Our argument is parallel to that of [49]: we prove a uniform lower density bound around q, and consequently analyze the nontrivial the tangent cone at q.

Lower density bound

Our first task is to establish an upper bound for the area of T. Precisely, we prove:

Lemma 3.1.5. *For ρ small enough, $\mathcal{H}^2(T \cap C_\rho) \le c\rho$.*

Proof. This is straightforward consequence of the fact that $T = \partial E \cap \mathring{M}$ minimizes the energy $\mathcal{F}$. In fact, for any open subset $U \subset\subset M$, E minimizes the functional

$$\mathcal{H}^2(E' \cap \mathring{M} \cap U) - \sum_{j=1}^{k} (\cos \gamma_j) \mathcal{H}^2(E' \cap \partial M \cap U)$$

among all sets $E' \subset M$ with finite perimeter, p (or B_2)$\subset E'$, $E'\cap B = \emptyset$. In particular, choose E' to be a small open neighborhood of p when M is a (B,p)-cone, and a small tubular neighborhood of B_2 when M is a (B_1, B_2)-prism. Let $T' = \partial E' \cap \mathring{M}$. Choose $U = C_\rho$. We conclude that

$$\mathcal{H}^2(T \cap C_\rho) - \sum_{j=1}^{2} (\cos\gamma_j)\mathcal{H}^2(\partial E \cap C_\rho \cap F_j) \leq \mathcal{H}^2(T' \cap C_\rho) - \sum_{j=1}^{2} (\cos\gamma_j)\mathcal{H}^2(\partial E' \cap C_\rho \cap F_j). \quad (3.1.10)$$

By the rough estimate that

$$\mathcal{H}^2(C_\rho \cap F_j) \leq c\rho \quad \text{and} \quad \mathcal{H}^2(C_\rho \cap B) \leq c\rho^2,$$

we conclude the proof. □

Denote $\Sigma = \operatorname{spt}(T) \setminus L$. Since the mean curvature of T is zero in its interior, from the first variation formula for varifolds, we have that, for any C^1 vector field ϕ compactly supported in $M \setminus L$,

$$\int_\Sigma \operatorname{div}_\Sigma \phi d\mathcal{H}^2 = \int_{\partial\Sigma} \phi \cdot \nu d\mathcal{H}^1. \quad (3.1.11)$$

We first bound the length of $\partial\Sigma$. Precisely, let r be the radial distance function $r = \operatorname{dist}(\cdot, L)$, let ϕ be any vector field, supported in M with $\sup r|D\phi| < \infty$ and C^1 in $\mathring{M}$. (Note that we allow ϕ to have support on L.) By a standard approximation argument as in [49], we deduce that

$$\rho^{-1}\int_{\Sigma\cap C_\rho} \phi \cdot \nabla_\Sigma r d\mathcal{H}^2 - \int_{\partial\Sigma} \min\left\{\frac{r}{\rho}, 1\right\} \phi \cdot \nu d\mathcal{H}^1 = -\int_\Sigma \min\left\{\frac{r}{\rho}, 1\right\} \operatorname{div}_\Sigma \phi d\mathcal{H}^2. \quad (3.1.12)$$

We are going to use (3.1.12) in two different ways. By the angle assumption

(3.1.9), $|\pi - (\gamma_1 + \gamma_2)| < \theta$. Therefore in the 2-plane $(T_qL)^{\perp} \subset T_qM$, there is a unit vector τ such that

$$(-X)|_{F_j} \cdot \tau > \cos\gamma_j, \quad j = 1, 2, \tag{3.1.13}$$

where $(-X)|_{F_j}$ is the inward pointing unit normal vector of $\partial M \subset M$, restricted to the face F_j. Choose ϕ to be the constant vector τ defined in a small neighborhood of $q \in M$. (3.1.13) then implies that

$$(-\nu) \cdot \tau \geq c > 0, \tag{3.1.14}$$

in $\Sigma \cap C_{\rho_0}$, for sufficiently small ρ_0. Taking $\rho \to 0$ in (3.1.12) and using Lemma 3.1.5, we deduce that

$$\mathcal{H}^1(\partial\Sigma \cap C_\rho) < \infty. \tag{3.1.15}$$

The angle assumption $\theta < \pi$ then guarantees that the vector $\tau \in T_qM$ defined above also verifies that

$$\tau \cdot \nabla_M r \geq c > 0, \tag{3.1.16}$$

where r is the radial distance function. See Figure 3.2 for an illustration of the choice of τ.

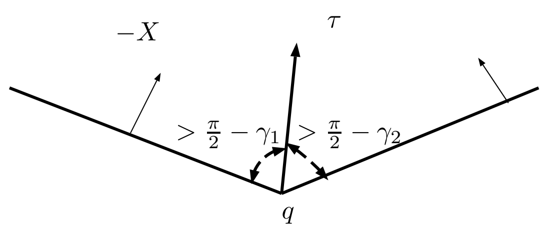

Figure 3.2: The choice of the vector τ.

In the first variational formula (3.1.12), we replace ϕ by $\psi\tau$, where by slight abuse of notation we here use τ to represent a parallel vector field in a small neighborhood of q. Then by an argument as (1.8)-(1.10) in [49],

$$\rho^{-1}\int_{\Sigma\cap C_\rho}\psi\tau\cdot\nabla_M r d\mathcal{H}^2-\int_{\partial\Sigma}\psi\tau\cdot\nu d\mathcal{H}^2 \leq c\int_\Sigma(\psi+|\nabla_M\psi|)d\mathcal{H}^2+o(1). \quad (3.1.17)$$

As a consequence of (3.1.14) and (3.1.16),

$$\|\delta T\|(\psi)\leq c(1+J)\int(\psi+|\nabla_M\psi|)d\|T\|, \quad (3.1.18)$$

where here we view T as the associated varifold. Then we apply the isoperimetric inequality (7.1 in [3]) and the Moser type iteration (7.5(6) in [3]) as in [49], and conclude that

$$\mathcal{H}^2(T\cap B_\rho(q))\geq c\rho^2. \quad (3.1.19)$$

Remark 3.1.6. *The argument above does not use the fact that Σ is a two dimensional surface in an essential way. The same argument should work for capillary surfaces in general dimensions.*

Remark 3.1.7. *Notice that we only require the weaker angle assumption $|\pi-(\gamma_1+\gamma_2)|<\theta<\pi$ for the lower density bound. We are going to see that the assumption $\theta<\pi-|\gamma_1-\gamma_2|$ is used to classify the tangent cone.*

Monotonicity and tangent plane

We proceed to derive the monotonicity formula and study the tangent cone at a point $q\in\operatorname{spt}(T)\cap L$. For $j=1,2$, denote $W_j=E_j\cap(F_j\setminus L)$. We also use W_j to denote the associated two dimensional varifold. The divergence theorem implies that

$$\delta W_j(\psi\phi)=\int_{\partial\Sigma}\psi\phi\cdot\overline{\nu}, \quad (3.1.20)$$

where recall $\overline{\nu}$ is the unit normal vector of $\partial\Sigma$ which is tangent to E and points outward E.

Since ϕ is tangential on $F_j \setminus L$, $\nu - (\cos\gamma_j)\overline{\nu}$ is normal to ϕ. We then multiply $-\cos\gamma_j$ to (3.1.20) and add the result to (3.1.12), thus obtaining

$$\left[\delta\|T\| - \sum_{j=1}^{2}(\cos\gamma_j)\delta W_j\right](\psi\phi) = 0. \tag{3.1.21}$$

Denote $Z = \|T\| - \sum_{j=1}^{2}(\cos\gamma_j)W_j$. Now since F_1, F_2 are smooth surfaces intersecting transversely on L, we may choose a vector field ϕ that is $C^{1,\alpha}$ close to the radial vector field $\nabla_M \operatorname{dist}(\cdot, q)$. For a complete argument, see (2.4) of [49]. Then by a minor modification of the argument of 5.1 in [3], we conclude that

$$\exp(c\rho^\alpha)\frac{\|Z\|(B_\rho(0))}{\rho^2} \text{ is increasing in } \rho, \text{ for } \rho < \rho_0. \tag{3.1.22}$$

We thus deduce from (3.1.19) and (3.1.22) that there is a nontrivial tangent cone Z_∞ of Z at q. Precisely, under the homothetic transformations μ_r defined by $x \mapsto r(x-q)$ $(r>0)$, $(\mu_{r_k\#}T, \mu_{r_k\#}W_1, \mu_{r_k\#}W_2, \mu_{r_k\#}Z)$ subsequentially converges to $Z_\infty = \|T_\infty\| - \sum_{j=1}^{2} W_{j,\infty}$ in $\mathbf{R}^3$. Let $F_{j,\infty}$, $j=1,2$, denote the corresponding limit planes in $\mathbf{R}^3$ of F_j, $L_\infty = F_{1,\infty} \cap F_{2,\infty}$. Denote $P_\infty = L^\perp$ the two-plane through 0 that is perpendicular to L. Denote D_r the open disk of radius r centered at 0 on the plane P_∞.

Proposition 3.1.8. *The tangent cone $T_\infty \subset \mathbf{R}^3$ is a single-sheeted planar domain that intersects $F_{j,\infty}$, $j = 1, 2$, at angle γ_j. Moreover, it is unique. Namely, T_∞ does not depend on the choice of subsequence for its construction.*

Proof. We first notice that the tangent cone $\|T_\infty\|$ is nontrivial by virtual of (3.1.19). Moreover, since (T, E) solves the variational problem (1.4.4), (T_∞, E_∞) minimizes the corresponding energy in $\mathbf{R}^3$. Precisely, let C be the open set in $\mathbf{R}^3$ enclosed by $F_{1,\infty}$ and $F_{2,\infty}$. Then for any open subset $U \subset\subset \mathbf{R}^3$, E_∞ minimizes the energy

$$\mathcal{F}(E'_\infty) = \mathcal{H}^2(\partial E'_\infty \cap \mathring{C} \cap U) - \sum_{j=1}^{2}(\cos\gamma_j)\mathcal{H}^2(\partial E'_\infty \cap \partial C \cap U). \tag{3.1.23}$$

It follows immediately that T_∞ is minimal in $\mathring{C}$. Therefore each connected component of T_∞ is part of a two-plane. We conclude that

$$T_\infty = \bigcup_{j=1}^{N} \pi_j \cap C, \tag{3.1.24}$$

where π_j are planes through the origin and $\pi_i \cap \pi_j \cap C = \emptyset$ for $i \neq j$. Therefore we conclude either

Case 1 $N = 1$ and $T_\infty = \pi_1 \cap C$ for some plane π_1 such that $\pi_1 \cap L_\infty = \{0\}$, or

Case 2 $N < \infty$ and $T_\infty = \cup_{j=1}^{N} \pi_j \cap C$, where $\pi_1, \cdots, \pi_N$ are planes with the line L in common.

Now we rule out case 2 by constructing proper competitors. Notice that in case 2, $E_\infty = E_\infty^{(1)} \times \mathbf{R}$ for some open $E_\infty^{(1)} \subset P_\infty$ with $\partial E_\infty^{(1)}$ a finite union of rays emanating from the origin. Define the functional

$$\mathcal{F}_\infty^{(1)}(E') = \mathcal{H}^1(\partial E' \cap C \cap D_1) - \sum_{j=1}^{2} (\cos\gamma_j)\mathcal{H}^1(\partial E' \cap F_{j,\infty} \cap D_1). \tag{3.1.25}$$

Notice that since E_∞ minimizes (3.1.23),

$$\mathcal{F}_\infty^{(1)}(E_\infty^{(1)}) \leq \mathcal{F}_\infty^{(1)}(E'),$$

for any competitor E'.

Observe that $C \setminus \overline{E_\infty^{(1)}}$ satisfies a variational principle similar to that satisfied by $E_\infty^{(1)}$ but with $\pi - \gamma_j$ in place of γ_j. In case $N > 1$, we may simply "smooth out" the vertex of $(\pi_1 \cap \pi_2) \cap \overline{D_1}$ to decrease the functional $E_\infty^{(1)}$. Thus $N = 1$. Without loss of generality assume that $\gamma_1 \leq \gamma_2$.

To show that $N = 1$ in case 2 cannot happen, we first observe that if β_0 is the angle formed by $E_\infty^{(1)}$ and $F_{1,\infty}$ at 0, then $\beta_0 \geq \gamma_1$. Otherwise we may construct a competitor E' that has strictly deceases (3.1.25) as follows. Let $q_1 \in \partial D_{1/2} \cap (\partial E_\infty^{(1)} \setminus \partial C)$ and let q_2 be the point on $\partial E_\infty^{(1)} \cap F_{1,\infty}$ at distance ε from 0. Then let $E' = E_\infty^{(1)} \setminus H$, where

H is the closed half plane with $0 \in H \setminus \partial H$ and $\{q_1, q_2\} \in \partial H$. If $\beta_0 < \gamma_1$, then it is easily checked (as illustrated in Figure 3.3) that

$$\mathcal{F}_\infty^{(1)}(E') < \mathcal{F}_\infty^{(1)}(E_\infty^{(1)}).$$

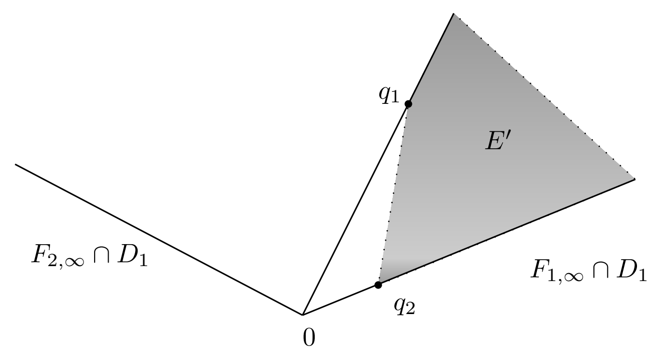

Figure 3.3: The construction of a competitor when $\beta_0 < \gamma_1$.

On the other hand, since $C \setminus \overline{E_\infty^{(1)}}$ satisfies a similar variational principle with angle $\pi - \gamma_j$ in place of γ_j, we deduce that

$$\theta - \beta_0 \geq \pi - \gamma_2.$$

We therefore conclude that $\theta \geq \pi + \gamma_1 - \gamma_2$, contradiction. Thus case 2 is impossible.

In case 1, T_∞ contains a single sheet of plane. Namely, there exists some plane $\pi_1 \subset \mathbf{R}^3$ such that $T_\infty = \pi_1 \cap C$.

Notice also that the plane $\pi_1 \subset \mathbf{R}^3$ should have constant contact angle along $F_{j,\infty}$, $j = 1, 2$:

$$\measuredangle(\pi_1, F_{1,\infty}) = \gamma_1, \quad \measuredangle(\pi_1, F_{2,\infty}) = \gamma_2, \tag{3.1.26}$$

since everywhere on $\partial\Sigma \cap (F_j \setminus L)$, Σ and F_j meet at constant contact angle γ_j. We point out that the angle assumption (3.1.9) is also a *necessary and sufficient condition* for the existence of $\pi_1 \subset \mathbf{R}^3$. As a consequence, $T_\infty = \pi_1 \cap \mathring{C}$ with π_1 uniquely determined by (3.1.26), independent of choice of the particular sequence of r_k chosen to construct T_∞. In other words, we have the strong property that the tangent cone is *unique* for T at q. $\square$

Remark 3.1.9. *This part of the argument relies heavily on the fact that T is two dimensional in two ways:*

- *The planes are the only minimal cones in $\mathbf{R}^3$.*
- *A plane is uniquely determined by its intersection angles with two fixed planes.*

Neither of these two statements is valid in higher dimensions.

Remark 3.1.10. *The proof suggests that without the upper bound $\theta < \pi - |\gamma_1 - \gamma_2|$, the tangent cone of T at the corners could be a half plane through L_∞. Moreover, T_∞ may depend on the choice of the sequences of r_k in its construction.*

Curvature estimates and consequences

We prove a curvature estimate for Σ near the corner q. Combined with the uniqueness of tangent cone, we deduce that Σ must be graphical over its tangent plane at q. Then we may apply the PDE theory from [25] to conclude that $\overline{\Sigma}$ is a $C^{1,\alpha}$ surface.

We begin with the following lemma, which is a consequence of the monotonicity formula.

Lemma 3.1.11. *Let $C \in \mathbf{R}^3$ be an open subset enclosed by two planes F_1, F_2 with $\measuredangle(F_1, F_2) = \theta$. Let Σ be an area minimizing surface in C such that Σ intersects F_j at constant angles γ_1, γ_2, and that $\mathcal{H}^2(\Sigma \cap B_0(R)) < C_0 R^2$ holds for large R and some $C_0 > 0$. Assume also that*

$$|\pi - (\gamma_1 + \gamma_2)| < \theta < \pi - |\gamma_1 - \gamma_2|.$$

Then there is a plane $\pi_1 \subset \mathbf{R}^3$ such that $\Sigma = \pi_1 \cap C$.

Proof. Without loss of generality assume $0 \in \overline{\Sigma}$. Consider the tangent cone of Σ at ∞ and at 0. Since Σ satisfies the angle assumption (3.1.9), its tangent cone at 0, denoted by Σ_0, exists and is planar. Now by the monotonicity formula (3.1.22) and the growth assumption $\mathcal{H}^2(\Sigma \cap B_0(R)) < C_0 R^2$, its tangent cone at infinity, denoted by Σ_∞, exists and is an area minimizing cone. Since Σ_0 and Σ_∞ are both minimal cones in $C \subset \mathbf{R}^3$, they are parts of planes. However, the same argument as in the proof of Proposition 3.1.8 implies that $\Sigma_0 = \Sigma_\infty = \pi \cap C$, where π is the unique plane intersecting F_j at angle γ_j. Therefore the equality in the monotonicity formula holds, and $\Sigma = \Sigma_0 = \Sigma_\infty$ is planar. □

We are ready to prove the curvature estimates:

Proposition 3.1.12. *Let $\Sigma = \mathrm{spt}(T) \cap \mathring{M}$ be a minimizer of the variational problem* (1.4.4)*. Let $L = F_1 \cap F_2$, $q \in \partial\Sigma \cap L$. Then the following curvature estimate holds:*

$$|A_\Sigma|(x) \cdot \mathrm{dist}(x, q) \to 0, \tag{3.1.27}$$

as $x \in \Sigma$ converges to q.

Proof. Assume, for the sake of contradiction, that there is $\delta > 0$ and a sequence of points $q_k \in \Sigma$ such that

$$\mathrm{dist}(q_k, q) = \varepsilon_k > 0, \quad \varepsilon_k |A_\Sigma|(q_k) = \delta_k > \delta.$$

By a standard point-picking argument, we could also assume that

$$|A_\Sigma|(x) < \frac{2\delta_k}{\varepsilon_k}, \quad x \in C_{2\varepsilon_k}. \tag{3.1.28}$$

Consider the rescaled surfaces

$$\Sigma_k = \frac{\delta_k}{\varepsilon_k}(\Sigma - q_k) \subset \frac{\delta_k}{\varepsilon_k}(M - q_k).$$

Since $\delta_k > \delta$, $\varepsilon_k \to 0$, the ambient manifold M converges, in the sense of Gromov-Hausdorff, to (C, g_{Euclid}). Since Σ_k is area minimizing, a subsequence (which we still denote by Σ_k) converges to an area minimizing surface Σ_∞. By (3.1.19), Σ_∞ is nontrivial. We consider two different cases

- If $\limsup_k \delta_k = \infty$, then by taking a further subsequence (which we still denote by Σ_k), Σ_k converges to an area minimizing surface in $\mathbf{R}^3$. Moreover, (3.1.28) guarantees that the $|A_{\Sigma_\infty}|(x) < 2$ for all $x \in \mathbf{R}^3$. Therefore the convergence

$\Sigma_k \to \Sigma_\infty$ is, in fact, in C^∞. This produces a contradiction, since $|A_{\Sigma_k}|(0) = 1$ for all k, and Σ_∞ is a plane through the origin.

- If $\limsup_k \delta_k < C < \infty$, then the sequence Σ_k converges to an area minimizing surface in the open set $C \subset \mathbf{R}^3$ enclosed by the two limit planes. This produces a similar contradiction, because $|A_{\Sigma_k}|(0) = 1$, $\Sigma_k \to \Sigma_\infty$ smoothly, and by Lemma 3.1.11, Σ_∞ is flat in its interior.

□

With the curvature estimate, we may conclude the regularity discussion by concluding that Σ is graphical near the corner q:

Proposition 3.1.13. *Let Σ be an energy minimizer of* (1.4.4)*, $q \in \overline{\Sigma} \cap L$. Then Σ is a graph over the tangent plane at q, and its normal vector extends Hölder continuously to q; thus Σ is a $C^{1,\alpha}$ surface with corners.*

Proof. We first prove that Σ is graphical near q. Embed a neighborhood of q isometrically into some Euclidean space $\mathbf{R}^N$. Take the unique plane $\pi_1 \subset T_qM$ obtained above such that the tangent cone of Σ at q is $\pi_1 \cap C$. Assume, for the sake of contradiction, that there is a sequence of points $q_k \in \Sigma$, $\mathrm{dist}_M(q_k, q) \to 0$, and that the normal vectors N_k of $\Sigma \subset M$ at q_k is parallel to π_1. Denote $\varepsilon_k = \mathrm{dist}_M(q_k, q)$. Consider the rescaled surfaces $\Sigma_k = \varepsilon_k^{-1}(\Sigma - q)$. By the monotonicity formula (3.1.22) and the lower density bound (3.1.19), a subsequence of $\{\Sigma_k\}$ converges to the unique tangent cone $\pi_1 \cap C$ in the sense of varifolds. Notice that on Σ_k, the image of q_k under the homothety has unit distance to the origin. By taking a further subsequence (which we still denote by $\{(\Sigma_k, q_k, N_k)\}$), we may assume that $q_k \to q_\infty$, $N_k \to N_\infty$, and $\mathrm{dist}_{\mathbf{R}^3}(q_\infty, 0) = 1$. Now the curvature estimate (3.1.27) implies that,

$$|A_{\Sigma_\infty}|(x) < 2, \quad \text{for all points } x \in \Sigma \cap B_{1/2}(q_\infty).$$

For any point $x \in \Sigma_\infty$, and any curve l connecting q_∞ and x, we have

$$|N_\infty(x) - N_\infty(q_\infty)| < \int_l |A_{\Sigma_\infty}|(y)dy.$$

Therefore we conclude that, for points x on a neighborhood V of q_∞ on Σ_∞,

$$|N_\infty(x) - N_{\pi_1}| > \frac{1}{2},$$

where N_{π_1} is the unit normal vector of π_1. This contradicts the fact that Σ_k converges to Σ_∞ as varifolds.

Once we know that Σ is a graph over $T_q\Sigma$ near q, the result of [25] directly applies, and we conclude that Σ has a Hölder continuous unit normal vector field up to q. □

3.2 Proof for Theorem 1.4.3

We prove Theorem 1.4.3 in this section. Let P be a polyhedron in $\mathbf{R}^3$ of cone or prism types. Assume, for the sake of contradiction, that there exists a P-type polyhedron (M^3, g) with $R(g) \geq 0$, $\overline{H} \geq 0$ and $\measuredangle_{ij}(M) < \measuredangle_{ij}(P)$. The strategy is to take the minimizer $\Sigma = \partial E$ of the (1.4.4). When M is of prism type, the existence and regularity of Σ follows from the maximum principle in Proposition 3.1.2. When M is of cone type, we need the extra assumption that $\mathcal{I} < 0$ to guarantee that $E \neq \emptyset$. Hence we prove the following:

Lemma 3.2.1. *Let $P \subset \mathbf{R}^3$ be polyhedron of cone type, (M, g) be of P-type. Assume $\measuredangle_{ij}(M) < \measuredangle_{ij}(P)$, then the infimum $\mathcal{I}$ appeared in* (1.4.4) *is negative.*

Proof. As before let F_j, F_j' denote the side faces of M, P, respectively; B, B' denote their base faces. Assume, for the sake of contradiction, that

$$\mathcal{H}^2(\partial E \cap \mathring{M}) - \sum_{j=1}^{k} (\cos\gamma_j)\mathcal{H}^2(\partial E \cap F_j) \geq 0. \tag{3.2.1}$$

Notice that the inequality (3.2.1) is scaling invariant. Precisely, if $E \subset M$ satisfies (3.2.1), then under the homothety μ_r defined by $x \mapsto r(x - p)$, the set $(\mu_r)_\#(E) \subset (\mu_r)_\#(M)$ satisfies (3.2.1). Letting $r \to \infty$, the tangent cone $T_pM$of M at p should share the same property. Let $F_{j,\infty}$ denote the corresponding faces in T_pM. By assumption, $\measuredangle(F_{j,\infty}, F_{j+1,\infty}) < \measuredangle(F_j', F_{j+1}')$. Therefore T_pM can be placed strictly inside the tangent cone of P at its vertex. By elementary Euclidean geometry, there exists a plane $\pi \subset \mathbf{R}^3$ such that π meets $F_{j,\infty}$ with angle $\gamma_j' > \gamma_j$. See Figure 3.4 for an illustration, where the dashed polyhedral cone is T_pM.

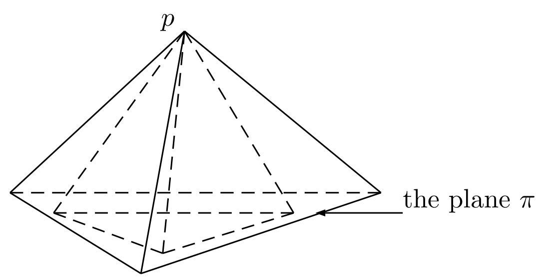

Figure 3.4: The tangent cone T_pM contained in P.

Let proj_π denote the projection $\mathbf{R}^3 \to \pi$. Then the Jacobian of proj_π, restricted to each $F_{j,\infty}$, is $\cos\gamma_j'$. Denote E_∞ the open domain enclosed by π and $F_{j,\infty}$, $j = 1, \cdots, k$. By the area formula,

$$\mathcal{H}^2(\pi \cap \partial E_\infty) - \sum_{j=1}^{k} (\cos\gamma_j')\mathcal{H}^2(F_{j,\infty} \cap \partial E_\infty) = 0.$$

Since $\gamma_j' > \gamma_j$, we conclude

$$\mathcal{H}^2(\pi \cap \partial E_\infty) - \sum_{j=1}^{k} (\cos\gamma_j)\mathcal{H}^2(F_{j,\infty} \cap \partial E_\infty) < 0,$$

contradiction. □

In the proof we are going to need another simple fact from Euclidean geometry. We leave its proof to the readers.

Lemma 3.2.2. *Let P_i, Q_i, R_i, $i = 1, 2$, be six planes in $\mathbf{R}^3$ with the property that $\measuredangle(P_1, R_1) = \measuredangle(P_2, R_2)$, $\measuredangle(Q_1, R_1) = \measuredangle(Q_2, R_2)$ and $\measuredangle(P_1, Q_1) \le \measuredangle(P_2, Q_2)$. Let $L_i = P_i \cap R_i$, $L_i' = Q_i \cap R_i$, $i = 1, 2$. Then $\measuredangle(L_1, L_1') \le \measuredangle(L_2, L_2')$.*

Now we prove Theorem 1.4.3.

Proof. Assume, for the sake of contradiction, that $\measuredangle_{ij}(M) < \measuredangle_{ij}(P)$. By Theorem 3.1.1 and Lemma 3.2.1, the infimum in (1.4.4) is achieved by an open set E, with $\Sigma = \partial E \cap \mathring{M}$ a smooth surface which is $C^{1,\alpha}$ up to its corners for some $\alpha \in (0,1)$. By the first variation formula (3.1.1), Σ is capillary minimal. We apply the second variational formula (3.1.2) and conclude

$$\int_\Sigma [|\nabla f|^2 - (|A|^2 + \mathrm{Ric}(N,N))f^2] d\mathcal{H}^2 - \int_{\partial\Sigma} Q f^2 d\mathcal{H}^1 \geq 0, \tag{3.2.2}$$

for any C^2 function f compactly supported away from the corners, where on $\partial\Sigma \cap F_j$,

$$Q = \frac{1}{\sin\gamma_j} \mathrm{II}(\overline{\nu}, \overline{\nu}) + (\cot\gamma_j) A(\nu,\nu).$$

Since the surface Σ is $C^{1,\alpha}$ to its corners, its curvature $|A|$ is square integrable. Hence by a standard approximation argument we conclude that the about inequality holds for the constant function $f = 1$. We have

$$\begin{aligned} -\int_\Sigma (|A|^2 + \mathrm{Ric}(N,N)) - \\ \sum_{j=1}^n \int_{\partial\Sigma\cap F_j} \left[\frac{1}{\sin\gamma_j} \mathrm{II}(\overline{\nu},\overline{\nu}) + \cot\gamma_j A(\nu,\nu)\right] \geq 0. \end{aligned} \tag{3.2.3}$$

Applying the Gauss equation on Σ, we have

$$|A|^2 + \mathrm{Ric}(N,N) = \frac{1}{2}(R - 2K_\Sigma + |A|^2), \tag{3.2.4}$$

where R is the scalar curvature of M, K_Σ is the Gauss curvature of Σ.

By the Gauss-Bonnet formula for $C^{1,\alpha}$ surfaces with piecewise smooth boundary components, we have that

$$\int_\Sigma K_\Sigma d\mathcal{H}^2 + \int_{\partial\Sigma} k_g d\mathcal{H}^1 + \sum_{j=1}^n (\pi - \alpha_j) = 2\pi\chi(\Sigma) \leq 2\pi, \tag{3.2.5}$$

here k_g is the geodesic curvature of $\partial\Sigma \subset \Sigma$, and α_j are the interior angles of Σ at the corners. By Lemma 3.2.2, $\alpha_j < \alpha_j'$, where α_j' is the corresponding interior angle of the base face of the Euclidean polyhedron P. Since $\sum_{j=1}^k(\pi - \alpha_j') = 2\pi$, we conclude $\sum_{j=1}^k(\pi - \alpha_j) > 2\pi$. As a result, we have that

$$-\int_\Sigma K_\Sigma d\mathcal{H}^2 > \int_{\partial\Sigma} k_g d\mathcal{H}^1. \tag{3.2.6}$$

Combining (3.2.3), (3.2.4) and (3.2.6) we conclude that

$$\int_\Sigma \frac{1}{2}\left(R + |A|^2\right) d\mathcal{H}^2 + \sum_{j=1}^n \int_{\partial\Sigma\cap F_j}\left[\frac{1}{\sin\gamma_j}\mathrm{II}(\overline{\nu},\overline{\nu}) + \cot\gamma_j A(\nu,\nu) + k_g\right] d\mathcal{H}^1 < 0. \tag{3.2.7}$$

To finish the proof, let us analyze the last integrand in (3.2.7). Fix one j and consider $\partial\Sigma \cap F_j$. For convenience let $\gamma = \gamma_j$. We make the following

claim 3.2.3.

$$\mathrm{II}(\overline{\nu},\overline{\nu}) + \cos\gamma A(\nu,\nu) + \sin\gamma k_g = \overline{H}, \tag{3.2.8}$$

where $\overline{H}$ is the mean curvature of ∂M in M.

Let T be the unit tangential vector of $\partial\Sigma$. Since Σ is minimal, $A(\nu,\nu) = -A(T,T)$. Therefore

$$\begin{aligned}
\cos\gamma A(\nu,\nu) + \sin\gamma k_g &= -\cos\gamma A(T,T) + \sin\gamma k_g \\
&= -\left\langle \nabla_T T, \cos\gamma N + \sin\gamma\nu\right\rangle \\
&= -\left\langle \nabla_T T, X\right\rangle \\
&= \mathrm{II}(T,T).
\end{aligned}$$

Since T and $\overline{\nu}$ form an orthonormal basis of ∂M, we have

$$\mathrm{II}(\overline{\nu},\overline{\nu}) + \cos\gamma A(\nu,\nu) + \sin\gamma k_g = \mathrm{II}(T,T) + \mathrm{II}(\overline{\nu},\overline{\nu}) = \overline{H}.$$

The claim is proved.

To finish the proof, we note that (3.2.7) implies that

$$\int_{\Sigma} \frac{1}{2}\left(R+|A|^2\right) d\mathcal{H}^2 + \sum_{j=1}^{n} \int_{\partial\Sigma\cap F_j} \frac{1}{\sin\gamma_j} \overline{H} d\mathcal{H}^1 < 0, \tag{3.2.9}$$

contradicting the fact that the scalar curvature R of M and the surface mean curvature $\overline{H}$ of $\partial M \subset M$ are nonnegative. □

3.3 Rigidity

In this section we prove Theorem 1.4.4. Rigidity properties of minimal and area-minimizing surfaces have attracted lots of interests in recent years. Following the Schoen-Yau proof of the positive mass theorem, Cai-Galloway [10] studied the rigidity of area-minimizing tori in three-manifolds in nonnegative scalar curvature. The case of area-minimizing spheres was carried out by Bray-Brendle-Neves [9]. Their idea is to study constant mean curvature (CMC) foliation around an infinitesimally rigid area-minimizing surface, and obtain a local splitting result for the manifold. It is very robust and applies to a wide variety of rigidity analysis: in the case of negative [37] scalar curvature, and for area-minimizing surfaces with boundary [6] (see also [36]). We adapt their idea for our rigidity analysis, and perform a dynamical analysis for foliations with constant mean curvature capillary surfaces. The new challenge here is that, when M is of cube type, the energy minimizer of (1.4.4) may be empty. In this case the tangent cone T_pM coincides with that of the Euclidean model P, and $\mathcal{I} = 0$. Our strategy, motivated by the earlier work of Ye [65], is to construct a constant mean curvature foliation near the vertex p, such that the mean curvature on each leaf converges to zero when approaching p.

3.3.1 Infinitesimally rigid minimal capillary surfaces

Assume the energy minimizer $\Sigma = \partial E \cap \mathring{M}$ exists for (1.4.4). Tracing equality in the proof in Section 3, we conclude that

$$\begin{gathered} \chi(\Sigma) = 0, \quad R_M = 0, \quad |A| = 0 \quad \text{on } \Sigma \\ \overline{H} = 0 \quad \text{on } \partial\Sigma, \qquad \alpha_j = \alpha_j' \quad \text{at the corners of } \Sigma. \end{gathered} \tag{3.3.1}$$

Moreover, by the second variation formula (3.1.2),

$$Q(f,f) = -\int_\Sigma (f\Delta f + (|A|^2 + \mathrm{Ric}(N,N))f^2) d\mathcal{H}^{n-1} + \sum_{j=1}^k \int_{\partial\Sigma\cap F_j} f\left(\frac{\partial f}{\partial \nu} - Qf\right) d\mathcal{H}^{n-2} \geq 0,$$

with $Q(1,1) = 0$. We then conclude that for any C^2 function f compactly supported away from the vertices of Σ, $Q(1,f) = 0$. By choosing appropriate g, we further conclude that

$$\mathrm{Ric}(N,N) = 0 \quad \text{on } \Sigma, \qquad \frac{1}{\sin\gamma_j}\mathrm{II}(\overline{\nu},\overline{\nu}) + \cot\gamma_j A(\nu,\nu) = 0 \quad \text{on } \partial\Sigma \cap F_j.$$

Combining with (3.2.4) and (3.2.8), we conclude that

$$K_\Sigma = 0 \quad \text{on } \Sigma, \quad k_g = 0 \quad \text{on } \partial\Sigma. \tag{3.3.2}$$

Call a surface Σ satisfying (3.3.1) and (3.3.2) *infinitesimally rigid*. Notice that such a surface is isometric to an flat k-polygon in $\mathbf{R}^2$.

Next, we construct a local foliation by CMC capillary surfaces Σ_t. Take a vector field Y defined in a neighborhood of Σ, such that Y is tangential when restricted to ∂M. Let $\phi(x,t)$ be the flow of Y. Precisely, we have:

Proposition 3.3.1. *Let Σ^2 be a properly embedded, two-sided, minimal capillary surface in M^3. If Σ is infinitesimally rigid, then there exists $\varepsilon > 0$ and a function*

$w : \Sigma \times (-\varepsilon, \varepsilon) \to \mathbf{R}$ such that, for every $t \in (-\varepsilon, \varepsilon)$, the set

$$\Sigma_t = \{\phi(x, w(x,t) : x \in \Sigma)\}$$

is a capillary surface with constant mean curvature $H(t)$ that meets F_j at constant angle γ_j. Moreover, for every $x \in \Sigma$ and every $t \in (-\varepsilon, \varepsilon)$,

$$w(x,0) = 0, \quad \int_\Sigma (w(x,t) - t) d\mathcal{H}^2 = 0 \quad \text{and} \quad \frac{\partial}{\partial t} w(x,t)\bigg|_{t=0} = 1.$$

Thus, by possible choosing a smaller ε, $\{\Sigma_t\}_{t \in (-\varepsilon,\varepsilon)}$ is a foliation of a neighborhood of $\Sigma_0 = \Sigma$ in M.

Our proof goes by an argument involving the inverse function theorem, and is essentially taken from [9] and [6]. We do, however, need an elliptic theory on cornered domains. This is done by Lieberman [26]. The following Schauder estimate is what we need:

Theorem 3.3.2 (Lieberman,[26])**.** *Let $\Sigma^2 \subset \mathbf{R}^3$ be an open polygon with interior angles less than π. Let $L_1, \cdots, L_k$ be the edges of Σ. Then there exists some $\alpha > 0$ depending only on the interior angles of Σ, such that if $f \in C^{0,\alpha}(\overline{\Sigma})$, $g|_{\overline{L_j}} \in C^{0,\alpha}(\overline{L_j})$, then the Neumann boundary problem*

$$\begin{cases} \Delta u = f & \text{in } \Sigma \\ \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Sigma \end{cases} \tag{3.3.3}$$

has a solution u with $\int_\Sigma u = 0$, and $u \in C^{2,\alpha}(\Sigma) \cap C^{1,\alpha}(\overline{\Sigma})$. Moreover, the Schauder estimate holds:

$$|u|_{2,\alpha,\Sigma} + |u|_{1,\alpha,\overline{\Sigma}} \le C(|f|_{0,\alpha,\overline{\Sigma}} + \sum_{j=1}^{k} |g|_{0,\alpha,L_j}).$$

We now prove Proposition 3.3.1.

Proof. For a function $u \in C^{2,\alpha}(\Sigma) \cap C^{1,\alpha}(\overline{\Sigma})$, consider the surface $\Sigma_u = \{\phi(x, u(x)) : x \in \Sigma\}$, which is properly embedded if $|u|_0$ is small enough. We use the subscript u to denote the quantities associated to Σ_u. For instance, H_u denotes the mean curvature of Σ_u, N_u denotes the unit normal vector field of Σ_u, and X_u denotes the restriction of X onto Σ_u. Then $\Sigma_0 = \Sigma$, $H_0 = 0$, and $\langle N_u, X_u\rangle = \cos\gamma_j$ along $\partial\Sigma \cap F_j$.

Consider the Banach spaces

$$F = \left\{ u \in C^{2,\alpha}(\Sigma) \cap C^{1,\alpha}(\overline{\Sigma}) : \int_\Sigma u = 0 \right\},$$
$$G = \left\{ u \in C^{0,\alpha}(\Sigma) : \int_\Sigma u = 0 \right\}, \quad H = \left\{ u \in L^\infty(\partial\Sigma) : u|_{\overline{L_j}} \in C^{0,\alpha}(\overline{L_j}) \right\}.$$

Given small $\delta > 0$ and $\varepsilon > 0$, define the map $\Psi : (-\varepsilon, \varepsilon) \times (B_0(\delta) \subset F) \to G \times H$ given by

$$\Psi(t, u) = \left(H_{t+u} - \frac{1}{|\Sigma|} \int_\Sigma H_{t+u}, \langle N_{t+u}, X_{t+u}\rangle - \cos\gamma \right),$$

where $\gamma = \gamma_j$ on $\partial\Sigma \cap \mathring{F}_j$.

In order to apply the inverse function theorem, we need to prove that $D_u\Psi|_{(0,0)}$ is an isomorphism when restricted to $\{0\} \times F$. In fact, for any $v \in F$,

$$D_u\Psi|_{(0,0)}(0, v) = \left.\frac{d}{ds}\right|_{s=0} \psi(0, sv) = \left(\Delta v - \frac{1}{|\Sigma|} \int_{\partial\Sigma} \frac{\partial v}{\partial\nu}, -\frac{\partial v}{\partial\nu} \right).$$

The calculation is given in Lemma A.0.2 and Lemma A.0.3 in the appendix. Now the fact that $D_u\Psi|_{(0,0)}$ is an isomorphism follows from Theorem 3.3.2. The rest of the proof is the same as Proposition 10 in [6], which we will omit here. $\square$

3.3.2 CMC capillary foliation near the vertex

When (M^3, g) is of cone type with vertex p, we have proved that $\mathcal{I}$ is realized by a minimizer $\partial E \neq \emptyset$ when $\mathcal{I} < 0$. Now it is obvious from the definition that $\mathcal{I} \leq 0$. However, in the case that $\mathcal{I} = 0$, it is a priori possible that the minimizer $E = \emptyset$. Assume $\mathcal{I} = 0$. We investigate this case with a different approach.

Notice that, as a consequence of Lemma 3.2.1, $\mathcal{I} = 0$ implies that

$$\measuredangle(F_j, F_{j+1})|_p = \measuredangle(F_j', F_{j+1}'),$$

where F_j' is the corresponding face of the Euclidean model P. Recall that in the Euclidean model P', its base face B' intersects F_j' at angle γ_j. Thus P is foliated by a family of planes parallel to B', where each leaf is minimal, and meets F_j' at constant angle γ_j. We generalize this observation to arbitrary Riemannian polyhedra, and obtain:

Theorem 3.3.3. *Let (M^3, g) be a cone type Riemannian polyhedron with vertex p. Let $P \subset \mathbf{R}^3$ be a polyhedron with vertex p', such that the tangent cones (T_pM, g_p) and $(T_{p'}P, g_{Euclid})$ are isometric. Denote $\gamma_1, \cdots, \gamma_k$ the angles between the base face and the side faces of P. Then there exists a small neighborhood U of p in M, such that U is foliated by surfaces $\{\Sigma_\rho\}_{\rho\in(0,\varepsilon)}$ with the properties that:*

1. *for each $\rho \in (0, \varepsilon)$, Σ_ρ meet the side face F_j at constant angle γ_j;*
2. *each Σ_ρ has constant mean curvature λ_ρ, and $\lambda_\rho \to 0$ as $\rho \to 0$.*

Remark 3.3.4. *Before proceeding to the proof, let us remark that the local foliation structure of Riemannian manifolds has been a thematic program in geometric analysis, and has deep applications to mathematical general relativity. See: Ye [65] for spherical foliations around a point; Huisken-Yau [20] for foliations in asymptotically flat spaces; Mahmoudi-Mazzeo-Pacard [34][30] for foliations around general minimal submanifolds.*

Remark 3.3.5. *As a technical remark, let us recall that in all of the aforementioned foliation results, some extra conditions are necessary (e.g. Ye's result required the center point to be a non-degenerate critical point of scalar curvature; Mahmoudi-Mazzeo-Pacard needed the minimal submanifold to be non-degenerate critical point for the volume functional). However, in our result, no extra condition is needed. Geometrically, this is because in the tangent cone $T_pM \subset \mathbf{R}^3$, the desired foliation is unique.*

Proof. Let U be a small neighborhood of p in M. Take a local diffeomorphism $\varphi : P \to U$, such that the pull back metric $\varphi^* g$ and g_{Euclid} are C^1 close. Place the vertex

p' of P at the origin of $\mathbf{R}^3$. In local coordinates on $\mathbf{R}^3$, the above requirement is then equivalent to

$$g_{ij}(0) = g_{ij,k} = 0, \quad g_{ij}(x) = o(|x|), g_{ij,k}(x) = o(1) \text{ for } |x| < \rho_0.$$

φ may be constructed, for instance, via geodesic normal coordinates.

Denote $\overline{M} \subset \mathbf{R}^3$ the tangent cone of M at p. By assumption, the dihedral angles $\measuredangle(F_i, F_j)|_p = \measuredangle(F_i', F_j')$. Let π be the plane in $\mathbf{R}^3$ such that in Euclidean metric, π and F_j meet at constant angle γ_j. For $\rho \in (0, 1]$, let π_ρ be the plane that is parallel to π and has distance ρ to 0. Let Σ_ρ be the intersection of π_ρ with the interior of the cone T_pM. Denote X the outward pointing unit normal vector field on $\partial\overline{M}$, N_ρ the unit vector field of $\Sigma_\rho \subset \overline{M}$ pointing towards 0. Denote Y the vector field such that for each $x \in \Sigma_\rho$, $Y(x)$ is parallel to $\vec{x}$. Moreover, we require that the flow of Y parallel translates $\{\Sigma_\rho\}$, and $Y(x)$ is tangent to $\partial\overline{M}$ when $x \in \partial\overline{M}$. Let $\phi(x, t)$ be the flow of Y. For a function $u \in C^{2,\alpha}(\Sigma_1) \cap C^{1,\alpha}(\overline{\Sigma_1})$ (Σ_1 is parallel to π, and of distance 1 to the origin), define the perturbed surface

$$\Sigma_{\rho,u} = \{\phi(\rho x, u(\rho x)) : x \in \Sigma_1\}.$$

Since $\Sigma_\rho = \rho\Sigma_1$, the surface $\Sigma_{\rho,u}$ is a small perturbation of Σ_ρ, and is properly embedded, if $|u|_0$ is sufficiently small.

We use the subscript ρ to denote geometric quantities related to Σ_ρ, and the subscript (ρ, u) to denote geometric quantities related to the perturbed surfaces $\Sigma_{\rho,u}$, both in the metric $\varphi^* g$. In particular, $H_{\rho,u}$ denotes the mean curvature of $\Sigma_{\rho,u}$, and $N_{\rho,u}$ denotes the unit normal vector field of $\Sigma_{\rho,u}$ pointing towards 0. It follows from Lemma A.0.1 and Lemma A.0.2 that we have the following Taylor expansion of geometric quantities.

$$\begin{aligned} H_{\rho,u} &= H_\rho + \frac{1}{\rho^2}\Delta_\rho u + (\mathrm{Ric}(N_\rho, N_\rho) + |A_\rho|^2)u + L_1 u + Q_1(u) \\ \langle X_{\rho,u}, N_{\rho,u}\rangle &= \langle X_\rho, u_\rho\rangle - \frac{\sin\gamma_j}{\rho}\frac{\partial u}{\partial \nu_\rho} \\ &\quad + (\cos\gamma_j A(\nu_\rho, \nu_\rho) + \mathrm{II}(\overline{\nu_\rho}, \overline{\nu_\rho}))u + L_2 u + Q_2(u). \end{aligned} \tag{3.3.4}$$

Let us explain (3.3.4) a bit more. Q_1, Q_2 are terms that are at least quadratic in u. The functions L_1, L_2 exhibit how the mean curvature H_ρ and the contact angle γ_j deviate from being constant. In particular, they are bounded in the following manner:

$$L_1 \leq C|\nabla_\rho H_\rho||Y| \leq C|g|_{C^2} < C, \quad L_2 \leq C|\nabla_\rho \langle X_\rho, N_\rho\rangle||Y| < C|g|_{C^1} < C.$$

The operator Δ_ρ is the Laplace operator on Σ_ρ. At $x \in \Sigma_\rho$,

$$\Delta_\rho = \frac{1}{\sqrt{\det(g)}}\partial_i\left(\sqrt{\det(g)}g^{ij}\partial_j\right).$$

In particular, Δ_ρ converges to the Laplace operator on $\mathbf{R}^2$ as $\rho \to 0$. In local coordinates, it is not hard to see that

$$|H_\rho| \leq C|g|_{C^1} = o(1), \quad |\langle X_\rho, N_\rho\rangle - \cos\gamma_j| \leq |g|_{C^0} = o(\rho).$$

Denote $D_\rho = \langle X_\rho, N_\rho\rangle - \cos\gamma_j$. Letting $H_{\rho,u} \equiv \lambda$, we deduce from (3.3.4) that we need to solve for u from

$$\begin{cases} \Delta_\rho u + \rho^2 L_1 u + \rho^2 Q_1(u) = \rho^2(\lambda - H_\rho) & \text{in } \Sigma_1, \\ \frac{\partial u}{\partial \nu_\rho} = \rho D_\rho + \rho L_2 u + \rho Q_2(u) & \text{on } \partial\Sigma_1. \end{cases} \tag{3.3.5}$$

We use inverse function theorem as in the proof of Proposition 3.3.1. Precisely, denote the operator

$$\begin{cases} \mathcal{L}_\rho(u) = \Delta_\rho u - \rho^2 L_1 u - \rho^2 Q_1(u) + \rho^2 H_\rho, \\ \mathcal{B}_\rho(u) = \frac{\partial u}{\partial \nu_\rho} - \rho D_\rho - \rho L_2 u - \rho Q_2(u), \end{cases}$$

and consider the Banach spaces

$$F = \left\{u \in C^{2,\alpha}(\Sigma_1) \cap C^{1,\alpha}(\overline{\Sigma_1}) : \int_{\Sigma_1} u = 0\right\},$$

$$G = \left\{u \in C^{0,\alpha}(\Sigma_1) : \int_{\Sigma_1} u = 0\right\}, H = \left\{u \in L^\infty(\partial\Sigma_1) : u|_{\overline{L_j}} \in C^{0,\alpha}(\overline{L_j})\right\}.$$

Again we use $L_1, \cdots, L_j$ to denote the edges of Σ_1.

For a small $\delta > 0$, let $\Psi : (-\varepsilon, \varepsilon) \times (B_\delta(0) \subset F) \to G \times H$ given by

$$\Psi(\rho, u) = \left(\mathcal{L}_\rho(u) - \frac{1}{|\Sigma_1|} \int_{\Sigma_1} \mathcal{L}_\rho(u) d\mathcal{H}^2, \mathcal{B}_\rho(u) \right).$$

By the asymptotic behavior as $\rho \to 0$ discussed above, the linearized operator $D_u\Psi|_{(0,0)}$, when restrited to $\{0\} \times F$, is given by

$$D_u\Psi|_{(0,0)}(0, v) = \left.\frac{d}{ds}\right|_{s=0} \Psi(0, sv) = \left(\Delta v - \int_{\Sigma_1} \Delta v, \frac{\partial v}{\partial \nu} \right).$$

By Theorem 3.3.2, for some $\alpha \in (0, 1)$, $D_u\Psi|_{(0,0)}$ is an isomorphism when restricted to $\{0\} \times F$. We therefore apply the inverse function theorem and conclude that, for small $\varepsilon > 0$, there exists a C^1 map between Banach spaces $\rho \in (-\varepsilon, \varepsilon) \mapsto u(\rho) \in B_\delta(0) \subset F$ for every $\rho \in (-\varepsilon, \varepsilon)$, such that $\Psi(\rho, u(\rho)) = (0, 0)$. Thus the surface $\Sigma_{\rho, u(\rho)}$ is minimal, and meets F_j at constant angle γ_j.

By definition, $u(0)$ is the zero function. Denote $v = \frac{\partial u(\rho)}{\partial \rho}$. Differentiating (3.3.5) with respect to ρ and evaluating at $\rho = 0$, we deduce

$$\begin{cases} \Delta v = 0 & \text{in } \Sigma_1, \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Sigma_1. \end{cases} \tag{3.3.6}$$

Therefore v is also the zero function. Thus we conclude that

$$|u|_{1,\alpha,\overline{\Sigma_1}} = o(\rho),$$

for $|\rho| < \rho_0$.

Therefore the surfaces $\Sigma_{\rho, u(\rho)}$ is a foliation of a small neighborhood of p. Moreover, integrating (3.3.5) over Σ_1, we find that the constant mean curvature of $\Sigma_{\rho,u(\rho)}$ satisfies

$$\begin{aligned}\lambda_\rho &= \frac{1}{\rho^2}\int_{\Sigma_1}\Delta u + \int_{\Sigma_1}(L_1 u + Q_1(u) + H_\rho)\\ &= \frac{1}{\rho^2}\int_{\partial\Sigma_1}\frac{\partial u}{\partial\nu} + \int_{\Sigma_1}(L_1 u + Q_1(u) + H_\rho) + o(1)\\ &= \frac{1}{\rho}\int_{\partial\Sigma_1}(D_\rho + L_2 u + Q_2(u)) + \int_{\Sigma_1}(L_1 u + Q_1(u) + H_\rho) + o(1).\end{aligned} \tag{3.3.7}$$

Since

$$D_\rho = o(\rho), \quad |u|_{1,\alpha,\overline{\Sigma_1}} = o(\rho), \quad H_\rho = o(1),$$

we conclude that $\lambda_\rho \to 0$, as $\rho \to 0$. □

3.3.3 Local splitting

We analyze the CMC capillary foliations developed above to prove a local splitting theorem, thus prove Theorem 1.4.4. We need the extra assumption (1.4.2) that

$$\gamma_j \leq \pi/2, j = 1, \cdots, k \quad \text{or} \quad \gamma_j \geq \pi/2, j = 1, \cdots, k.$$

First notice that, if $P \subset \mathbf{R}^3$ is a cone, then (1.4.2) is possible only when $\gamma_j \leq \pi/2$, $j = 1, \cdots, k$; if P is a prism and $\gamma_j > \pi/2$, then instead of (1.4.4), we consider, for $E_1 = M \setminus \overline{E}$,

$$\mathcal{F}(E_1) = \mathcal{H}^2(\partial E_1 \cap \mathring{M}) - \sum_{j=1}^k (\cos\gamma_j)\mathcal{H}^2(\partial E_1 \cap F_j), \tag{3.3.8}$$

and reduce the problem to the case where $\gamma_j \leq \pi/2$. Thus we always assume $\gamma_j \leq \pi/2$, $j = 1, \cdots, k$.

Under the conventions as before, assume we have a local CMC capillary foliation $\{\Sigma_\rho\}_{\rho\in I}$, where as ρ increase, Σ_ρ moves in the direction of N_ρ. We will take I to be $(-\varepsilon, \varepsilon)$, $(-\varepsilon, 0)$ or $(0, \varepsilon)$, according to the location of the foliation. We prove the following differential inequality for the mean curvature $H(\rho)$.

Proposition 3.3.6. *There exists a nonnegative continuous function $C(\rho) \geq 0$ such that*

$$H'(\rho) \geq C(\rho)H(\rho).$$

Proof. Let $\psi : \Sigma \times I \to M$ parametrizes the foliation. Denote $Y = \frac{\partial \psi}{\partial t}$. Let $v_\rho = \langle Y, N_\rho \rangle$ be the lapse function. Then by Lemma A.0.1 and Lemma A.0.2, we have

$$\frac{d}{d\rho}H(\rho) = \Delta_\rho v_\rho + (\mathrm{Ric}(N_\rho, N_\rho) + |A_\rho|^2)v_\rho \quad \text{in } \Sigma_\rho, \tag{3.3.9}$$

$$\frac{\partial v_\rho}{\partial \nu_\rho} = \left[(\cot\gamma_j)A_\rho(\nu_\rho, \nu_\rho) + \frac{1}{\sin\gamma_j}\mathrm{II}(\overline{\nu_\rho}, \overline{\nu_\rho})\right] v_\rho \quad \text{on } \partial\Sigma_\rho \cap F_j. \tag{3.3.10}$$

By shrinking the interval I if possible, we may assume $v_\rho > 0$ for $\rho \in I$. Multiplying $\frac{1}{v_\rho}$ on both sides of (3.3.9) and integrating on Σ_ρ, we deduce that

$$\begin{aligned} H'(\rho)\int_{\Sigma_\rho}\frac{1}{v_\rho} &= \int_{\Sigma_\rho}\frac{|\nabla v_\rho|^2}{v_\rho^2}d\mathcal{H}^2 + \frac{1}{2}\int_{\Sigma_\rho}(R + |A|^2 + H^2)d\mathcal{H}^2 - \int_{\Sigma_\rho}K_{\Sigma_\rho}d\mathcal{H}^2 \\ &\quad + \sum_{j=1}^{k}\int_{\partial\Sigma_\rho\cap F_j}\left[\cot\gamma_j A_\rho(\nu_\rho, \nu_\rho) + \frac{1}{\sin\gamma_j}\mathrm{II}(\overline{\nu_\rho}, \overline{\nu_\rho})\right]d\mathcal{H}^1 \\ &\geq -\int_{\Sigma_\rho}K_{\Sigma_\rho}d\mathcal{H}^2 + \sum_{j=1}^{k}\int_{\partial\Sigma_\rho\cap F_j}\left[\cot\gamma_j A_\rho(\nu_\rho, \nu_\rho) + \frac{1}{\sin\gamma_j}\mathrm{II}(\overline{\nu_\rho}, \overline{\nu_\rho})\right]d\mathcal{H}^1. \end{aligned} \tag{3.3.11}$$

Using the Gauss-Bonnet formula and Lemma 3.2.2,

$$-\int_{\Sigma_\rho}K_{\Sigma_\rho}d\mathcal{H}^2 \geq \int_{\partial\Sigma_\rho}k_g d\mathcal{H}^1. \tag{3.3.12}$$

As in (3.2.8), we also have

$$k_g + \cot\gamma_j A(\nu_\rho, \nu_\rho) + \frac{1}{\sin\gamma_j}\mathrm{II}(\overline{\nu_\rho}, \overline{\nu_\rho}) = (\cot\gamma_j)H(\rho) + \frac{1}{\sin\gamma_j}\overline{H}, \tag{3.3.13}$$

on $\partial\Sigma_\rho \cap F_j$. Combining these, we deduce

$$\begin{aligned} H'(\rho)\int_{\Sigma_\rho}\frac{1}{v_\rho} &\geq \sum_{j=1}^{k}\int_{\partial\Sigma_\rho\cap F_j}\left[(\cot\gamma_j)H(\rho)+\frac{1}{\sin\gamma_j}\overline{H}\right]d\mathcal{H}^1 \\ &\geq \left[\sum_{j=1}^{k}(\cot\gamma_j)\mathcal{H}^1(\partial\Sigma_\rho\cap F_j)\right]H(\rho). \end{aligned} \tag{3.3.14}$$

Take $C(\rho)=\sum_{j=1}^{k}(\cot\gamma_j)\mathcal{H}^1(\partial\Sigma_\rho\cap F_j)$. The proposition is proved.

□

We are now ready to prove Theorem 1.4.4.

Proof. If (M^3,g) is of prism type, or if (M^3,g) is of cone type with $\mathcal{I}<0$, then the variational problem (1.4.4) has a nontrivial solution E with a $C^{1,\alpha}$ boundary Σ. Therefore Σ is infinitesimally rigid minimal capillary, and there is a CMC capillary foliation $\{\Sigma_\rho\}_I$ around Σ, where $I=(-\varepsilon,\varepsilon)$ if $\Sigma\subset \mathring{M}$, $I=[0,\varepsilon)$ if $\Sigma=B_1$, and $I=(-\varepsilon,0]$ if $\Sigma=B_2$. By Proposition 3.3.6, the mean curvature $H(\rho)$ of Σ_ρ satisfies

$$\begin{cases} H(0)=0 \\ H'(\rho)\geq C(\rho)H(\rho), \end{cases}$$

where $C(\rho)\geq 0$. By standard ordinary differential equation theory,

$$H(\rho)\geq 0 \text{ when } \rho\geq 0,\quad H(\rho)\leq 0 \text{ when } \rho\leq 0.$$

Denote E_ρ the corresponding open domain in M. Since each Σ_ρ meets F_j at constant angle γ_j, the first variation formula (3.1.1) implies that

$$F(\rho_1)-F(\rho_2)=-\int_{\rho_2}^{\rho_1}d\rho\int_{\Sigma_\rho}H(\rho)v_\rho d\mathcal{H}^2.$$

We then conclude that for $\delta>0$,

$$F(\delta)\leq F(0),\qquad F(-\delta)\leq F(0).$$

However, $\Sigma_0 = \Sigma$ minimizes the functional (1.4.4). Therefore in a neighborhood of Σ, $F(\rho) = F(0)$, $H(\rho) \equiv 0$. Tracing back the equality conditions, we find that

$$v_\rho \equiv \text{constant}, \quad \text{each } \Sigma_\rho \text{ is infinitesimally rigid.}$$

It is then straightforward to check that the normal vector fields of Σ_ρ is parallel (see [9] or [36]). In particular, its flow is a flow by isometries and therefore provides the local splitting. Since M is connected, this splitting is also global, and we conclude that (M^3, g) is isometric to a flat polyhedron in $\mathbf{R}^3$.

If (M^3, g) is of cone type with $\mathcal{I} = 0$, then by Theorem 3.3.3, there is a CMC capillary foliation $\{\Sigma_\rho\}_{\rho \in (-\varepsilon, 0)}$ near the vertex, with $H(\rho) \to 0$ as $\rho \to 0$. By Proposition 3.3.6, the mean curvature $H(\rho)$ satisfies

$$\begin{cases} H'(\rho) \geq C(\rho) H(\rho) & \rho \in (-\varepsilon, 0) \\ H(\rho) \to 0 & \rho \to 0. \end{cases}$$

Since $C(\rho) \geq 0$, we conclude that $H(\rho) \leq 0$, $\rho \in (-\varepsilon, 0)$. Let E_ρ be the open subset bounded by Σ_ρ. Take $0 < \eta < \delta$, then

$$F(-\eta) - F(-\delta) = -\int_{-\delta}^{-\eta} d\rho \int_{\Sigma_\rho} H v_\rho d\mathcal{H}^2 \geq 0 \quad \Rightarrow \quad F(-\delta) \leq F(-\eta).$$

Letting $\eta \to 0$, we have

$$F(-\delta) \leq 0.$$

As before, we conclude that $F(\rho) \equiv 0$ for $\rho \in (-\varepsilon, 0)$, and that each leaf Σ_ρ is infinitesimally rigid. Thus (M^3, g) admits a global splitting of flat k-polygon in $\mathbf{R}^2$, and hence is isometric to a flat polyhedron in $\mathbf{R}^3$. $\square$

Appendix A

Deformations of capillary surfaces

We provide some general calculation for infinitesimal variations of geometric quantities of properly immersed hypersurfaces under variations of the ambient manifold (M^{n+1}, g) that leave the boundary of the hypersurface inside ∂M. We also refer the readers to the thorough treatment in [38] and [6] (warning: the choice of orientation for the unit normal vector field N in [6] is the opposite to ours).

We keep the notations used in Section 3.1 and for each $t \in (-\varepsilon, \varepsilon)$, we use the subscript t for the terms related to Σ_t. Recall that $Y = \frac{\partial \Psi(t,\cdot)}{\partial t}$ is the deformation vector field. Denote Y_0 the tangent part of Y on Σ, Y_0 the tangent part of Y on $\partial\Sigma$. Let $v = \langle Y, N\rangle$. For $q \in \Sigma$, let $e_1, \cdots, e_n$ be an orthonormal basis of $T_q\Sigma$, and let $e_i(t) = d\Psi_t(e_i)$. Let S_0, S_1 be the shape operators of $\Sigma \subset M$ and $\partial M \subset M$. Precisely, $S_0(Z_1) = -\nabla_{Z_1} N$, $S_1(Z_2) = \nabla_{Z_2} X$. We have:

Lemma A.0.1 (Lemma 4.1(1) of [38], Proposition 15 of [6])**.**

$$\nabla_Y N = -\nabla^\Sigma v - S_0(Y_0). \tag{A.0.1}$$

We use Lemma A.0.1 to calculate the evolution of the contact angle along the boundary.

Lemma A.0.2. *Let γ denote the contact angle between Σ and F_j. Then*

$$\left.\frac{d}{dt}\right|_{t=0} \langle N_t, X_t\rangle = -\sin\gamma \frac{\partial v}{\partial \nu} + (\cos\gamma) A(\nu,\nu) v + \mathrm{II}(\overline{\nu}, \overline{\nu}) v + \left\langle L, \nabla^{\partial\Sigma}\gamma_j\right\rangle v, \tag{A.0.2}$$

where L is a bounded vector field on $\partial\Sigma$.

In particular, if each Σ_t meets F_j at constant angle γ_j, then on F_j,

$$\frac{\partial v_t}{\partial \nu_t} = \left[(\cot\gamma_j)A_t(\nu_t,\nu_t) + \frac{1}{\sin\gamma_j}\,\mathrm{II}(\overline{\nu_t},\overline{\nu_t})\right] v_t.$$

Proof. Let us fix one boundary face F_j and denote γ_j by γ. By Lemma A.0.1,

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0}\langle N_t, X_t\rangle &= \langle \nabla_Y N, X\rangle + \langle N, \nabla_Y X\rangle \\ &= -\left\langle \nabla^\Sigma v, X\right\rangle - \langle S_0(Y_0), X\rangle + \langle N, \nabla_Y X\rangle .\end{aligned}$$

On ∂M, Y decomposes into $Y = Y_1 - \frac{v}{\sin\gamma}\overline{\nu}$. Notice that since $X = \cos\gamma N + \sin\gamma N$,

$$\langle S_0(Y_0), X\rangle = \langle S_0(Y_0), \cos\gamma N + \sin\gamma\nu\rangle = \sin\gamma A(Y_0,\nu).$$

We also have the vector decomposition on ∂M with respect to the orthonormal basis $\overline{\nu}, X$:

$$N = \cos\gamma X - \sin\gamma\overline{\nu}, \qquad \nu = \cos\gamma\overline{\nu} + \sin\gamma X. \tag{A.0.3}$$

Since $\langle X, X\rangle = 1$ along ∂M, we have $\langle X, \nabla_Z X\rangle = 0$ for any vector Z on ∂M. We have

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0}\langle N_t, X_t\rangle &= -\sin\gamma\frac{\partial v}{\partial \nu} - \langle S_0(Y_0), X\rangle \\ &\quad + \left\langle \cos\gamma X - \sin\gamma\overline{\nu}, \nabla_{Y_1 - \frac{v}{\sin\gamma}\overline{\nu}} X\right\rangle \\ &= -\sin\gamma\frac{\partial v}{\partial \nu} - \sin\gamma A(Y_0,\nu) - \sin\gamma\,\langle\overline{\nu}, \nabla_{Y_1}X\rangle + \langle \overline{\nu}, \nabla_{\overline{\nu}}X\rangle\, v.\end{aligned}$$

Now we deal with the second and the third terms above. Notice that on $\partial\Sigma\cap F_j$,

$$Y_0 = Y_1 - (\cot\gamma)v\nu.$$

Thus $A(Y_0,\nu) = A(Y_1,\nu) - (\cot\gamma)vA(\nu,\nu) = -\langle\nabla_{Y_1}N,\nu\rangle - (\cot\gamma)A(\nu,\nu)v$. On the other hand, using the vector decomposition (A.0.3), we find

$$\begin{aligned}\langle \nabla_{Y_1} N, \nu \rangle &= \langle \nabla_{Y_1} (\cos\gamma X - \sin\gamma \overline{\nu}), \cos\gamma \overline{\nu} + \sin\gamma X \rangle \\ &= \cos^2\gamma \langle \nabla_{Y_1} X, \overline{\nu} \rangle - \sin^2\gamma \langle \nabla_{Y_1} \overline{\nu}, X \rangle + \left\langle L, \nabla^{\partial\Sigma}\gamma \right\rangle . \\ &= \langle \nabla_{Y_1} X, \overline{\nu} \rangle + \left\langle L, \nabla^{\partial\Sigma}\gamma \right\rangle .\end{aligned}$$

Here L is a vector field along $\partial\Sigma$, and $|L| \leq C = C(Y, X, \nu)$. Thus we conclude that

$$\left.\frac{d}{dt}\right|_{t=0} \langle N_t, X_t \rangle = -\sin\gamma \frac{\partial v}{\partial \nu} + (\cos\gamma) A(\nu,\nu) v + \mathrm{II}(\overline{\nu}, \overline{\nu}) v + \left\langle L, \nabla^{\partial\Sigma}\gamma \right\rangle ,$$

as desired.

□

The evolution equation of the mean curvature has been studied in many circumstances. We refer the readers to the thorough calculation in Proposition 16, [6]:

Lemma A.0.3 (Proposition 16 of [6])**.** *Let H_t be the mean curvature of Σ_t. Then*

$$\left.\frac{d}{dt}\right|_{t=0} H_t = \Delta_\Sigma v + (\mathrm{Ric}(N,N) + |A|^2) v - \langle \nabla_\Sigma H, Y_0 \rangle .$$

In particular, if each Σ_t has constant mean curvature, then

$$\frac{d}{dt} H_t = \Delta_{\Sigma_t} v_t + (\mathrm{Ric}(N_t, N_t) + |A_t|^2) v_t.$$

Bibliography

[1] K. Akutagawa, G. Carron, and R. Mazzeo. The Yamabe problem on stratified spaces. *Geom. Funct. Anal.*, 24(4):1039–1079, 2014.

[2] A. D. Aleksandrov. A theorem on triangles in a metric space and some of its applications. In *Trudy Mat. Inst. Steklov., v 38*, Trudy Mat. Inst. Steklov., v 38, pages 5–23. Izdat. Akad. Nauk SSSR, Moscow, 1951.

[3] William K. Allard. On the first variation of a varifold. *Ann. of Math. (2)*, 95:417–491, 1972.

[4] William K. Allard. On the first variation of a varifold: boundary behavior. *Ann. of Math. (2)*, 101:418–446, 1975.

[5] L. Ambrosio and B. Kirchheim. Currents in metric spaces. *Acta Math.*, 185(1):1–80, 2000.

[6] Lucas C. Ambrozio. Rigidity of area-minimizing free boundary surfaces in mean convex three-manifolds. *J. Geom. Anal.*, 25(2):1001–1017, 2015.

[7] M. Atiyah and C. LeBrun. Curvature, cones and characteristic numbers. *Mathematical Proceedings of the Cambridge Philosophical Society*, 155(1):1337, 2013.

[8] T. Aubin. The scalar curvature. pages 5–18. Mathematical Phys. and Appl. Math., Vol. 3, 1976.

[9] Hubert Bray, Simon Brendle, and Andre Neves. Rigidity of area-minimizing two-spheres in three-manifolds. *Comm. Anal. Geom.*, 18(4):821–830, 2010.

[10] Mingliang Cai and Gregory J. Galloway. Rigidity of area minimizing tori in 3-manifolds of nonnegative scalar curvature. *Comm. Anal. Geom.*, 8(3):565–573, 2000.

[11] X. Chen, S. Donaldson, and S. Sun. Kähler-Einstein metrics on Fano manifolds. I: Approximation of metrics with cone singularities. *J. Amer. Math. Soc.*, 28(1):183–197, 2015.

[12] X. Chen, S. Donaldson, and S. Sun. Kähler-Einstein metrics on Fano manifolds. II: Limits with cone angle less than 2π. *J. Amer. Math. Soc.*, 28(1):199–234, 2015.

[13] X. Chen, S. Donaldson, and S. Sun. Kähler-Einstein metrics on Fano manifolds. III: Limits as cone angle approaches 2π and completion of the main proof. *J. Amer. Math. Soc.*, 28(1):235–278, 2015.

[14] G. De Philippis and F. Maggi. Regularity of free boundaries in anisotropic capillarity problems and the validity of Young's law. *Arch. Ration. Mech. Anal.*, 216(2):473–568, 2015.

[15] R. Finn. *Equilibrium Capillary Surfaces*. Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen. Springer, 1986.

[16] D. Fischer-Colbrie and R. Schoen. The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature. *Comm. Pure Appl. Math.*, 33(2):199–211, 1980.

[17] M. Gromov. Filling Riemannian manifolds. *J. Differential Geom.*, 18(1):1–147, 1983.

[18] M. Gromov and H. B. Lawson, Jr. Positive scalar curvature and the Dirac operator on complete Riemannian manifolds. *Inst. Hautes Études Sci. Publ. Math.*, (58):83–196 (1984), 1983.

[19] Misha Gromov. Dirac and Plateau billiards in domains with corners. *Cent. Eur. J. Math.*, 12(8):1109–1156, 2014.

[20] G. Huisken and S.-T. Yau. Definition of center of mass for isolated physical systems and unique foliations by stable spheres with constant mean curvature. *Inventiones Mathematicae*, 124:281–311, January 1996.

[21] T. Jeffres, R. Mazzeo, and Y. A. Rubinstein. Kähler-Einstein metrics with edge singularities. *Ann. of Math. (2)*, 183(1):95–176, 2016.

[22] J. L. Kazdan and F. W. Warner. Scalar curvature and conformal deformation of Riemannian structure. *J. Differential Geometry*, 10:113–134, 1975.

[23] D. A. Lee and P. G. LeFloch. The positive mass theorem for manifolds with distributional curvature. *Comm. Math. Phys.*, 339(1):99–120, 2015.

[24] M. Li and X. Zhou. A maximum principle for free boundary minimal varieties of arbitrary codimension. *ArXiv e-prints*, August 2017.

[25] Gary M. Lieberman. Hölder continuity of the gradient at a corner for the capillary problem and related results. *Pacific J. Math.*, 133(1):115–135, 1988.

[26] Gary M. Lieberman. Optimal Hölder regularity for mixed boundary value problems. *J. Math. Anal. Appl.*, 143(2):572–586, 1989.

[27] W. Littman, G. Stampacchia, and H. F. Weinberger. Regular points for elliptic equations with discontinuous coefficients. *Ann. Scuola Norm. Sup. Pisa (3)*, 17:43–77, 1963.

[28] J. Lohkamp. Scalar curvature and hammocks. *Math. Ann.*, 313(3):385–407, 1999.

[29] J. Lott and C. Villani. Ricci curvature for metric-measure spaces via optimal transport. *Ann. of Math. (2)*, 169(3):903–991, 2009.

[30] F. Mahmoudi, R. Mazzeo, and F. Pacard. Constant mean curvature hypersurfaces condensing on a submanifold. *Geom. Funct. Anal.*, 16(4):924–958, 2006.

[31] C. Mantoulidis. *Geometric variational problems in mathematical physics*. PhD thesis, Stanford University, 2017.

[32] C. Mantoulidis and P. Miao. Total mean curvature, scalar curvature, and a variational analog of Brown-York mass. *Comm. Math. Phys.*, 352(2):703–718, 2017.

[33] Christos Mantoulidis and Richard Schoen. On the Bartnik mass of apparent horizons. *Classical Quantum Gravity*, 32(20):205002, 16, 2015.

[34] Rafe Mazzeo and Frank Pacard. Foliations by constant mean curvature tubes. *Comm. Anal. Geom.*, 13(4):633–670, 2005.

[35] P. Miao. Positive mass theorem on manifolds admitting corners along a hypersurface. *Adv. Theor. Math. Phys.*, 6(6):1163–1182 (2003), 2002.

[36] Mario Micallef and Vlad Moraru. Splitting of 3-manifolds and rigidity of area-minimising surfaces. *Proc. Amer. Math. Soc.*, 143(7):2865–2872, 2015.

[37] Ivaldo Nunes. Rigidity of area-minimizing hyperbolic surfaces in three-manifolds. *J. Geom. Anal.*, 23(3):1290–1302, 2013.

[38] Antonio Ros and Rabah Souam. On stability of capillary surfaces in a ball. *Pacific J. Math.*, 178(2):345–361, 1997.

[39] R. Schoen. Conformal deformation of a Riemannian metric to constant scalar curvature. *J. Differential Geom.*, 20(2):479–495, 1984.

[40] R. Schoen. Variational theory for the total scalar curvature functional for Riemannian metrics and related topics. In *Topics in calculus of variations (Montecatini Terme, 1987)*, volume 1365 of *Lecture Notes in Math.*, pages 120–154. Springer, Berlin, 1989.

[41] R. Schoen and S.-T. Yau. Positive Scalar Curvature and Minimal Hypersurface Singularities. `https://arxiv.org/abs/1704.05490`. Accessed: 2018-01-17.

[42] R. Schoen and S. T. Yau. On the structure of manifolds with positive scalar curvature. *Manuscripta Math.*, 28(1-3):159–183, 1979.

[43] R. Schoen and S.-T. Yau. Proof of the positive mass theorem. II. *Comm. Math. Phys.*, 79(2):231–260, 1981.

[44] R. Schoen and S.-T. Yau. Conformally flat manifolds, Kleinian groups and scalar curvature. *Invent. Math.*, 92(1):47–71, 1988.

[45] Richard Schoen and Shing Tung Yau. On the proof of the positive mass conjecture in general relativity. *Comm. Math. Phys.*, 65(1):45–76, 1979.

[46] Y. Shi and L.-F. Tam. Scalar curvature and singular metrics. *ArXiv e-prints*, November 2016.

[47] Yuguang Shi and Luen-Fai Tam. Positive mass theorem and the boundary behaviors of compact manifolds with nonnegative scalar curvature. *J. Differential Geom.*, 62(1):79–125, 2002.

[48] L. Simon. *Lectures on geometric measure theory*, volume 3 of *Proceedings of the Centre for Mathematical Analysis, Australian National University*. Australian National University, Centre for Mathematical Analysis, Canberra, 1983.

[49] Leon Simon. Regularity of capillary surfaces over domains with corners. *Pacific J. Math.*, 88(2):363–377, 1980.

[50] Leon Simon. A strict maximum principle for area minimizing hypersurfaces. *J. Differential Geom.*, 26(2):327–335, 1987.

[51] P. D. Smith and D. Yang. Removing point singularities of Riemannian manifolds. *Trans. Amer. Math. Soc.*, 333(1):203–219, 1992.

[52] Bruce Solomon and Brian White. A strong maximum principle for varifolds that are stationary with respect to even parametric elliptic functionals. *Indiana Univ. Math. J.*, 38(3):683–691, 1989.

[53] C. Sormani. Scalar curvature and intrinsic flat convergence. `https://www.degruyter.com/downloadpdf/books/9783110550832/9783110550832-008/9783110550832-008.pdf`. Accessed: 2017-12-21.

[54] C. Sormani and S. Wenger. The intrinsic flat distance between Riemannian manifolds and other integral current spaces. *J. Differential Geom.*, 87(1):117–199, 2011.

[55] K.-T. Sturm. A curvature-dimension condition for metric measure spaces. *C. R. Math. Acad. Sci. Paris*, 342(3):197–200, 2006.

[56] K.-T. Sturm. On the geometry of metric measure spaces. I. *Acta Math.*, 196(1):65–131, 2006.

[57] K.-T. Sturm. On the geometry of metric measure spaces. II. *Acta Math.*, 196(1):133–17 7, 2006.

[58] Jean E. Taylor. Boundary regularity for solutions to various capillarity and free boundary problems. *Comm. Partial Differential Equations*, 2(4):323–357, 1977.

[59] G. Tian. K-stability and Kähler-Einstein metrics. *Comm. Pure Appl. Math.*, 68(7):1085–1156, 2015.

[60] N. S. Trudinger. Remarks concerning the conformal deformation of Riemannian structures on compact manifolds. *Ann. Scuola Norm. Sup. Pisa (3)*, 22:265–274, 1968.

[61] J. A. Viaclovsky. Monopole metrics and the orbifold Yamabe problem. *Ann. Inst. Fourier (Grenoble)*, 60(7):2503–2543 (2011), 2010.

[62] Brian White. The maximum principle for minimal varieties of arbitrary codimension. *Comm. Anal. Geom.*, 18(3):421–432, 2010.

[63] Neshan Wickramasekera. A sharp strong maximum principle and a sharp unique continuation theorem for singular minimal hypersurfaces. *Calc. Var. Partial Differential Equations*, 51(3-4):799–812, 2014.

[64] E. Witten. A new proof of the positive energy theorem. *Comm. Math. Phys.*, 80(3):381–402, 1981.

[65] Rugang Ye. Foliation by constant mean curvature spheres. *Pacific J. Math.*, 147(2):381–396, 1991.