RESEARCH STATEMENT

CHAO LI

1. Introduction

My research lies in the interface of geometry, analysis and mathematical physics. A fundamental theme in differential geometry is to understand curvature conditions. Together with my collaborators, I have resolved a number of questions on the global and local geometric and topological affects of **scalar curvature**. A central technique in my research is the study of *geometric variational problems*. My most significant contributions can be organized as follows.

(A) Scalar curvature is a coarse measure of local bending, making its effects on global topology of a manifold rather delicate. In the last sixty years, there has been dramatic progress on understanding global topological obstructions of manifolds with positive scalar curvature. However, a list of fundamental conjectures have been open.

Recently, O. Chodosh and I proved the long-standing "$K(\pi, 1)$ conjecture" in dimension 4 and 5. This conjecture was proposed by R. Schoen and S.T. Yau [39] in 1987 and independently by M. Gromov [15] in 1986. Our new observations apply to other previously open problems. For example, in 1988, R. Schoen and S.T. Yau studied the structure of locally conformally flat manifolds with nonnegative scalar curvature, and established a Liouvelle type theorem for such manifolds with certain technical assumptions. In the same paper [9], O. Chodosh and I completed this Liouvelle type theorem for *all* locally conformally flat manifolds with nonnegative scalar curvature.

(B) In 2013, M. Gromov [16] proposed a geometric comparison theory for manifolds with positive scalar curvature using polyhedrons, aiming to study local geometry and the convergence of metrics with positive scalar curvature. This echoes the classical triangle comparison principle for sectional curvature lower bounds. Gromov illustrated the simplest case where comparison models are cubes in Euclidean spaces, and proposed, in a *dihedral rigidity conjecture*, that it should hold for general Euclidean polyhedrons and have a rigidity phenomenon.

In a sequence of work [30, 29, 31], I proved this conjecture for a large collection of polytopes, and extended the theory to allow negative scalar curvature lower bounds. These results localize the positive mass theorem and the Min-Oo rigidity theorem for scalar curvature lower bounds, and are deeply related to quasi-local mass in general relativity.

(C) Initiated by N. Hitchin, there have been extensive investigations of the modulic space of metrics with positive scalar curvature on a smooth closed manifold over the past forth years. Homotopy groups of such a moduli space are differential invariances of the manifold, and are studied by different mathematical areas such as the Ricci flow and K-theory. However, analogous questions for manifolds with boundary are not so well understood.

A. Carlotto and I [4] gave a complete topological characterization of 3-manifolds with boundary admitting metrics with positive scalar curvature and mean convex (or minimal) boundary. We also proved that the moduli space of such metrics, if non-empty, must be path-connected. Our theorems have applications to the moduli space of initial data sets in general relativity, where a blackhole is present.

(D) Minimal surfaces and soap bubbles are classical topics in calculus of variation, and have been a central tool in my research. For applications in scalar curvature problems, and as interestion problems of their own right, the existence and regularity properties of minimal surfaces in certain singular ambient spaces have caught my attentions.

Joint with N. Edelen, I [12] developed new techniques in geometric measure theory and obtained existence and regularity theorems for free boundary minimal surfaces in *locally polyhedral domains*. Our results extend classical results of W. Allard, M. Grüter and J. Jost in the smooth setting.

2. Topological properties of manifolds with positive scalar curvature

Scalar curvature is the simplest local invariant of a Riemannian metric. The condition of positive scalar curvature (PSC) on a Riemannian metric depicts both rigidity and flexibility phenomena: on one hand, it is known that the PSC condition places topological restrictions on the underlining manifold; on the other hand, the PSC condition is preserved under certain surgeries, giving the moduli space of such metrics a rich structure.

2.1. **The $K(\pi, 1)$ conjecture and structure of locally conformally flat manifolds with positive scalar curvature.** It has been a fundamental question what topological restrictions a manifold has, if it admits a metric with positive scalar curvature. There have been two classical approaches towards this problem: the historically earliest approach was via the Dirac operator and the index theorems, which was initiated by A. Lichnerowicz and developed by M. Gromov and B. Lawson; the second approach was invented by R. Schoen and S.T. Yau via minimal surfaces[1]. Despite the huge success achieved by both approaches (e.g. the solution to Geroch conjecture), the following conjecture, proposed by R. Schoen and S.T. Yau [39] and independently (in a slightly different form) by M. Gromov [15], has been open:

Conjecture 1. Any closed aspherical manifold of dimension at least 4 does not admit a Riemannian metric with positive scalar curvature.

Conjecture 1 is not only a challenge for the existing approaches, but is also deeply related to the Novikov conjecture on topological invariance of certain polynomials of Pontryagin classes (see [17, p. 25]). Contributions from M. Gromov, L. Guth, G. Yu, R. Schoen, S.T. Yau, and others have provided pieces to the puzzle. In particular, R. Schoen and S.T. Yau [39] gave an outline of Conjecture 1 in dimension 4. However, many essential parts of their outline have been missing since then.

Recently, O. Chodosh and I proved Conjecture 1 in dimensions 4 and 5[2]:

Theorem 2 ([9])**.** *Any closed aspherical manifold of dimension 4 or 5 does not admit a Riemannian metric with positive scalar curvature.*

Our approach broadly follows from the Schoen-Yau outline, but with several essential new observations. The most important new observation is to study certain soap bubbles (called μ-bubbles after M. Gromov) in such a manifold. This enables us to attack other previously open problems:

Theorem 3 ([9])**.** *Let $n \leq 7$ and X be an arbitrary n-manifold without boundary. Then $T^n \# X$ does not admit a complete Riemannian metric with positive scalar curvature.*

When X is compact, Theorem 3 was proved by R. Schoen and S.T. Yau [38] and independently by M. Gromov and B. Lawson [14] (when X is spin). However, the extra generality with noncompactness in Theorem 3 is the key to understanding the structure of locally conformally flat manifolds with nonnegative scalar curvature, due to [37]. In fact, when combined with the reduction procedure of R. Schoen and S.T. Yau [37] [3], we have the following Liouvelle type result:

Corollary 4 ([9])**.** *Suppose (M^n, g) is a complete Riemannian manifold with $R_g \geq 0$. If $\Phi : M \to S^n$ is a conformal map, then Φ is injective and $\partial\Phi(M)$ has zero Newtonian capacity.*

In [37], R. Schoen and S.T. Yau proved this result under additional assumptions on the geometry or the dimension of M (e.g. $n \geq 7$). We remark here that when $n = 3$, M. Lesourd, R. Under and S.T. Yau also claimed this result in a forthcoming paper.

[1]In dimension 3, D. Stern [40] recently has an intriguing new approach via the Bochner formula.

[2]On the same day O. Chodosh and I posted our updated paper, M. Gromov independently posted a paper on Conjecture 1 in dimension 5. Both papers extend ideas (in quite similar ways) in the earlier arXiv version of [9], where we proved the conjecture in dimension 4.

[3]This reduction procedure is also carefully carried out in a forthcoming paper of M. Lesourd, R. Unger and S.T. Yau.

Our long-term goal in this direction is to study Conjecture 1 in higher dimensions, as well as to apply our new techniques to study global structure of complete Riemannian manifolds with positive scalar curvature. Other directions include understanding analogous problems for manifolds with boundary (where, according to M. Gromov [17], surprisingly new difficulties occur).

2.2. **Moduli space structure for manifolds with boundary.** Assuming a manifold M admits a Riemannian metric with positive scalar curvature, a follow-up question is then what is the moduli space $\mathcal{M}$ of metrics with positive scalar curvature. Topological invariances of this moduli space provide invariances of the differential structure of the underline manifold. On a closed manifold, these questions have been extensively studied (cf. [26, 41, 13, 28, 19, 34]). For manifolds M with boundary, the analogous curvature assumptions are the scalar curvature is positive in M, and mean curvature is positive (or zero) on ∂M. In a join work with A. Carlotto, we provided a complete characterization of 3-manifolds admitting a metric with $R > 0$ and $H > 0$. Moreover, we showed

Theorem 5 ([4])**.** *If M^3 is orientable with $\partial M \neq \emptyset$, then the moduli space of metrics*

$$\mathcal{M} = \{g \in Met(M) : R_g > 0 \text{ in } M, H_g > 0 \text{ on } \partial M\}/\mathcal{D},$$

here D is the boundary preserving diffeomorphism groups, is path-connected whenever non-empty.

Our approach is an extension of previous fundamental contributions due to M. Gromov and B. Lawson, G. Perelman and F. Codá Marques. Our deformation approach is robust enough to prove path-connectedness of the space analogously defined by $R \geq 0$ and $H_g \equiv 0$, which has applications to the moduli space of asymptotically flat initial data sets with boundary, modelling blackhole solutions to the Einstein equation in general relativity (this was recently carried out by M. Lesourd and S. Hirsch [25]).

M. Gromov shared with us his speculation that analogous moduli spaces for manifolds with dimension at least 5 should not be path-connected. This would require a careful equivariant analysis on previously mentioned classical results. Other directions include investigating different curvature conditions, e.g. positive isotropic curvature on 4-manifolds with boundary, where a first intriguing question is to understand the compatible curvature conditions on the boundary.

3. Local geometry and convergence under scalar curvature lower bounds

3.1. **Geometric comparison theorems for scalar curvature lower bounds.** A fundamental question in differential geometry is to understand local geometry under curvature conditions, define weak notions of such conditions on spaces with low regularity, and study of convergence of such spaces. Usually this is done via geometric comparison theorems. For sectional curvature lower bounds, such question has been systematically treated by Alexandrov [1]. Similar questions for Ricci lower bounds have attracted a wide wealth of research in recent years; see, e.g. [5, 6, 7, 10, 8] and [33, 42, 43, 44]. The case of scalar curvature, however, is not as well developed, possibly due to a lack of satisfactory comparison theory.

In 2013, M. Gromov [16] suggested a geometric comparison theory for manifolds with positive scalar curvature using polyhedrons, and proposed a relevant dihedral rigidity conjecture:

Conjecture 6 (The dihedral rigidity conjecture)**.** Let M be a convex polyhedron in $\mathbf{R}^n$ and g_0 is the induced Euclidean metric on M. Suppose g is a Riemannian metric on M, such that (M, g) has nonnegative scalar curvature and weakly mean convex faces, and along the intersection of any two adjacent faces, the dihedral angle of (M, g) is not larger than the (constant) dihedral angle of (M, g_0). Then (M, g) is isometric to a flat Euclidean polyhedron.

The simplest case of this conjecture is when $M = [0, 1]^n$, a cube in Euclidean space. By reducing the question to the Geroch conjecture, M. Gromov was able to provide a proof (or a sketch of proof) of the following result without rigidity statement:

Theorem 7 ([16])**.** *Let $M = [0, 1]^n$ be a cube, and g be a Riemannian metric on M. Then (M, g) cannot simultaneously satisfy:*

(1) The scalar curvature of g is positive;
(2) Each face of M is strictly mean convex with respect to the outward normal vector field;
(3) Everywhere the dihedral angle between two faces of M is acute.

The crucial observation is that conditions (2) and (3) above may be interpreted as C^0 properties of the metric g. Thus, M. Gromov proposed a possible of definition of '$R \geq 0$' for C^0 metrics:

$$R(g) \geq 0 \Leftrightarrow \text{there exists no cube } M \text{ with mean convex faces and everywhere acute dihedral angle.} \quad (3.1)$$

The strategy of M. Gromov relies on the fact that a cube is the fundamental domain of the $\mathbf{Z}^n$ action on $\mathbf{R}^n$, and reduces the problem to Geroch conjecture. In a sequence of papers [30, 29], I investigated Conjecture 6 with a different approach, via minimal surfaces with free boundary or capillary boundaries. This new observation enables me to prove Conjecture 6 for a large collection of polyhedrons. For example, I proved:

Theorem 8 ([30, 29])**.** *Conjecture 6 holds when M is any 3-dimensional simplex, or M is any n-dimensional prism with $3 \leq n \leq 7$.*

Here a prism is a polyhedron in the form of $P^2 \times [0,1]^{n-2}$, and $P^2 \subset \mathbf{R}^2$ is a polygon with non-obtuse interior angles. Theorem 8 is also connected with mass and quasi-local mass in Riemannian generality. Indeed, if Conjecture 6 holds for a *single* polyhedron M in $\mathbf{R}^n$, then the positive mass theorem for asymptotically flat n-manifolds holds.

Recently, I have been able to extend Theorem 8 to allow negative scalar curvature lower bounds, where the model comparison objects are parabolic prisms in the hyperbolic space. Denote by $\mathbf{H}^n$ the hyperbolic space with sectional curvature -1. A simple case of the result is:

Theorem 9 ([31])**.** *Let $n \leq 7$, $M = [0,1]^n$ be a parabolic rectangle in $\mathbf{H}^n$, g_H be the hyperbolic metric on M. Denote the face $\partial M \cap \{x_1 = 1\}$ by F_T, the face $\partial M \cap \{x_1 = 0\}$ by F_B. Assume g is a Riemannian metric on M such that:*
(1) $R(g) \geq -n(n-1)$ in M;
(2) $H(g) \geq n-1$ on F_T, $H(g) \geq -(n-1)$ on F_B, and $H(g) \geq 0$ on $\partial M \setminus (F_T \cup F_B)$;
(3) the dihedral angles between adjacent faces of M are everywhere not larger than $\pi/2$.
Then (M, g) is isometric to a parabolic rectangle in $\mathbf{H}^n$.

By applying the obvious scaling, Theorem 9 suggested a possible definition of '$R \geq -k$' ($k > 0$) for C^0 metrics, analogous to (3.1). Incidentally, Theorem 9 can be applied to prove the following convergence result:

Corollary 10 ([16],[31])**.** *Let M^n, $n \leq 7$, be a smooth manifold. Suppose g_k a sequence of C^2 Riemannian metrics with $R(g_k) \geq \kappa$, $\kappa \leq 0$, and that g_k converges to g in C^0. Then $R(g) \geq \kappa$.*

We emphasize that Corollary 10 was first proven by M. Gromov in [16], but its proof of the case $\kappa \leq 0$ was, according to M. Gromov, "artificial" (see [16, p. 15]). On the other hand, the proof using Theorem 9 is intrinsic. We also note here that Corollary 10 was also proven by R. Bamler with the Ricci flow.

A natural follow-up question to investigate is the extension of Conjecture 6 to allow positive scalar curvature lower bounds. Given the counter example to Min-Oo conjecture by S. Brendle, F. Codá Marques and A. Neves [3]), this extension is likely to involve delicate analysis on spherical polyhedrons. More speculatively, one might hope to extend Theorem 8 to allow more general Euclidean polytopes, by triangulating a general polyhedron in a smart way and applying Theorem 8 on each piese.

3.2. **Singular limit spaces of metrics with positive scalar curvature.** The comparison theory above motivates the following natural question: given a smooth manifold, how can a sequence of metrics g_k with $R(g_k) > 0$ degenerate? This question, in this form of generality, seems entirely out of reach. As a first step towards understanding general degenerations, it is necessary to first investigate the following:

Question 11. Given a submanifold $S \subset M$, what metrics g are limits of smooth metrics with positive scalar curvature, such that $\operatorname{sing}(g) = S$ and $R(g) > 0$ in $M \setminus S$?

It turns out that the answer to this question depends heavily on the codimension of S. In early 2000s, the case $\operatorname{codim}(S) = 1$ were investigated by J. Corvino [11], H. Bray [2], P. Miao [35], among others, where a satisfying answer was obtained. In this case, Question 11 is has found wide applications in other questions related to scalar curvature, e.g. the Riemannian Penrose inequality. On the other hand, the case of $\operatorname{codim}(S) \geq 2$ was previously unknown.

In a joint work with C. Mantoulidis, we studied Question 11 when $\operatorname{codim}(S) \geq 2$:

Theorem 12 ([32]). *Let $S^{n-k} \subset M^n$ be a smooth submanifold, g is a smooth Riemannian metric in $M \setminus S$ with $R(g) > 0$. If:*

(1) $k = 2$ and g is a edge-cone metric with cone angle not larger than 2π along S;
(2) or $n = 3, k = 3$ and g is in L^∞ across S.

Then g is the limit in $C^\infty_{loc}(M \setminus S)$ of a sequence of smooth Riemannian metrics on M with positive scalar curvature.

We also proved a positive mass theorem for metrics with these type of singularities.

Statement (2) (for all $n \geq 3$ and $3 \leq k \leq n$) was conjectured by R. Schoen, as a converse to the classical surgery result due to M. Gromov and B. Lawson [13], and by R. Schoen and S.T. Yau [36]. When $n = 4$, there has been some recent partial progress by D. Kazaras [27].

The long-term goal in this direction is to understand statement (2) of Theorem 12 for $n \geq 3$, even assuming that g has small L^∞ norm near S. Also, C. Mantoulidis and I have speculated that our desingularization of g makes it an explicit example of the limit of smooth metrics in the pointed intrinsically flat sense, a notion of convergence proposed by C. Sormani and S. Wenger, and is believed to be the right notion of convergence under scalar curvature lower bounds. We also conjecture that a generic such metric g is not the Gromov-Hausdorff limit of any smooth metrics. These are follow-up questions to study.

4. Regularity of minimal surfaces in singular domains

Minimal surfaces have been one of the central topics in calculus of variation, and have intimate links with scalar curvature, as illustrated by R. Schoen and S.T. Yau. In lots of the scalar curvature problems above, one needs to study manifolds with boundary, where *free-boundary minimal surfaces* naturally occur.

Given a Riemannian manifold Ω with non-empty boundary, free-boundary minimal surfaces are arise variationally as critical points of area or capillary type functions among surfaces in Ω whose boundaries lie in $\partial\Omega$ but are otherwise free to vary. The existence and regularity of free-boundary minimal surfaces has been a classical topic investigated by R. Courant, H. Lewy, S. Hildebrandt, J. Nitsche, M. Gruter, J. Jost, J. Taylor, M. Struwe, among others. Such questions are particularly interesting and challenging when Ω is locally modelled on a *polyhedral cone*[4]. When Ω is locally modelled on a wedge region in $\mathbf{R}^3$, classical works by S. Hildebrandt and J. Nitsche [20] and by S. Hildebrandt and F. Sauvigny [21, 22, 23, 24] illustrated how such a surface may be non-regular at its free boundary.

In a joint work with N. Edelen, we developed a regularity theory for free-boundary minimal surfaces in domains which are locally modelled on polyhedral cones, extending classical results due to M. Gruter and J. Jost [18]. Our results begins with an Allard type theorem, which implies:

Theorem 13. *Let $\Omega^{n+1} = \Omega_0 \times \mathbf{R}$ be a polyhedral cone domain in $\mathbf{R}^{n+1}$ and Ω_0 is convex. Then there exists $\varepsilon > 0$ depending on Ω_0, such that, if M is a stationary integral n-varifold in $B_1(0)$ which is ε-varifold close to $[\Omega_0 \times \{0\}]$, then $\operatorname{spt} M \cap B_{1/2}(0)$ is $C^{1,\alpha}$ graphical over $\Omega_0 \times \{0\}$.*

[4]Historically, the very first problem of free-boundary minimal surfaces was proposed by Gergonne and solved by Schwartz in the 19th century, where Ω is a cube in $\mathbf{R}^3$.

When $n = 2$, M is assumed to be a minimal graph and Ω_0 is a wedge, Theorem 13 was proven by S. Hildebrandt and F. Sauvigny [23]. We remark that *convexity* of Ω_0 is essential here: there are explicit counterexamples otherwise. Also, by analyzing the linearlized problem, we anticipate that $C^{1,\alpha}$ is the best regularity one can hope for. We then imply this and obtain the following partial regularity result for area-minimizing currents:

Theorem 14. *Let Ω^{n+1} be a locally convex polyhedral C^2 domain in a C^3 Riemannian manifold. Suppose T is a free-boundary area-minimizing integral n-current in Ω. Then* $\dim(\operatorname{sing} T \cap int(\Omega)) \leq n-7$*, and*

(1) $\dim(\operatorname{sing} T \cap \partial\Omega) \leq n-2$*, for general* Ω*;*
(2) $\dim(\operatorname{sing} T \cap \partial\Omega) \leq n-7$*, if the dihedral angles of* Ω *are everywhere* $= \pi/2$*.*

We anticipate that statement (1) in Theorem 14 is sharp. Theorem 14 is instrumental in my work [29, 31], and is essential to the suggested argument by M. Gromov in [16] in his proposed polyhedron comparison theory for scalar curvature.

Our approaches into these results completely bypass the reflection principle, which has been fundamental to lots of boundary regularity results. Thus, we anticipate that they can be apply to regularity questions in other boundary value problems, e.g. the capillary problem. We also have strong evidence that statement (2) of Theorem 14 holds for general polyhedral domain, where the dihedral angles are everywhere $\leq \pi/2$. We will investigate these questions in future.

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