

Non-linear Partial Differential Equations in Conformal Geometry

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§0. Introduction

In the study of conformal geometry, the method of elliptic partial differential equations is playing an increasingly significant role. Since the solution of the Yamabe problem, a family of conformally covariant operators (for definition, see section 2) generalizing the conformal Laplacian, and their associated conformal invariants have been introduced. The conformally covariant powers of the Laplacian form a family P_{2k} with $k \in \mathbb{N}$ and $k \leq \frac{n}{2}$ if the dimension n is even. Each P_{2k} has leading order term $(-\Delta)^k$ and is equal to $(-\Delta)^k$ if the metric is flat.

The curvature equations associated with these P_{2k} operators are of interest in themselves since they exhibit a large group of symmetries. The analysis of these equations is of necessity more complicated, it typically requires the derivation of an optimal Sobolev or Moser-Trudinger inequality that always occur at a critical exponent. A common feature is the presence of blowup or bubbling associated to the noncompactness of the conformal group. A number of techniques have been introduced to study the nature of blowup, resulting in a well developed technique to count the topological degree of such equations.

The curvature invariants (called the Q -curvature) associated to such operators are also of higher order. However, some of the invariants are closely related with the Gauss-Bonnet-Chern integrand in even dimensions, hence of intrinsic interest to geometry. For example, in dimension four, the finiteness of the Q -curvature integral can be used to conclude finiteness of topology. In addition, the symmetric functions of the Ricci tensor appear in natural fashion as the lowest order terms of these curvature invariants, these equations offer the possibility to analyze the Ricci tensor itself. In particular, in dimension four the sign of the Q -curvature integral can be used to conclude the sign of the Ricci tensor. Therefore there is ample motivation for the study of such equations.

In the following sections we will survey some of the development in the area that we have been involved. We gratefully acknowledge the collaborators that we were fortunate to be associated with.

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§1. Prescribing Gaussian curvature on compact surfaces and the Yamabe problem

In this section we will describe some second order elliptic equations which have played important roles in conformal geometry.

On a compact surface (M, g) with a Riemannian metric g , a natural curvature invariant associated with the Laplace operator $\Delta = \Delta_g$ is the Gaussian curvature $K = K_g$. Under the conformal change of metric $g_w = e^{2w}g$, we have

$$-\Delta w + K = K_w e^{2w} \text{ on } M \quad (1.1)$$

where K_w denotes the Gaussian curvature of (M, g_w) . The classical uniformization theorem to classify compact closed surfaces can be viewed as finding solution of equation (1.1) with $K_w \equiv -1, 0, \text{ or } 1$ according to the sign of $\int K dv_g$. Recall that the Gauss-Bonnet theorem states

$$\int_M K_w dv_{g_w} = 2\pi \chi(M) \quad (1.2)$$

where $\chi(M)$ is the Euler characteristic of M , a topological invariant. The variational functional with (1.1) as Euler equation for $K_w = \text{constant}$ is thus given by

$$J[w] = \int_M |\nabla w|^2 dv_g + 2 \int_M K_w w dv_g - \left(\int_M K dv_g \right) \log \frac{\int_M dv_{g_w}}{\int_M dv_g} \quad (1.3)$$

When the surface (M, g) is the standard 2-sphere S^2 with the standard canonical metric, the problem of prescribing Gaussian curvature on S^2 is commonly known as the Nirenberg problem. For general compact surface M , Kazdan and Warner ([57]) gave a necessary and sufficient condition for the function when $\chi(M) = 0$ and some necessary condition for the function when $\chi(M) < 0$. They also pointed out that in the case when $\chi(M) > 0$ i.e. when $(M, g) = (S^2, g_c)$, the standard 2-sphere with the canonical metric $g = g_c$, there is an obstruction for the problem:

$$\int_{S^2} \nabla K_w \cdot \nabla x e^{2w} dv_g = 0 \quad (1.4)$$

where x is any of the ambient coordinate function. Moser ([63]) realized that this implicit integrability condition is satisfied if the conformal factor has antipodal symmetry. He proved for an even function f , the only necessary condition for (1.1) to be solvable with $K_w = f$ is that f be positive somewhere. An important tool introduced by Moser is the following inequality ([62]) which is a sharp form of an earlier result of Trudinger ([80]) for the limiting Sobolev embedding of $W_0^{1,2}$ into the Orlicz space e^{L^2} : Let w be a smooth function on the 2-sphere satisfying the normalizing conditions: $\int_{S^2} |\nabla w|^2 dv_g \leq 1$ and $\bar{w} = 0$ where \bar{w} denotes the mean value of w , then

$$\int_{S^2} e^{\beta w^2} dv_g \leq C \quad (1.5)$$

where $\beta \leq 4\pi$ and C is a fixed constant and 4π is the best constant. If w has antipodal symmetry then the inequality holds for $\beta \leq 8\pi$.

Moser has also established a similar inequality for functions u with compact support on bounded domains in the Euclidean space \mathbb{R}^n with the $W^{1,n}$ energy norm $\int |\nabla u|^n dx$ finite. Subsequently, Carleson and Chang ([14]) found that, contrary to the situation for Sobolev embedding, there is an extremal function realizing the maximum value of the inequality of Moser when the domain is the unit ball in Euclidean space. This fact remains true for simply connected domains in the plane (Flücher [43]), and for some domains in the n -sphere (Soong [77]).

Based on the inequality of Moser and subsequent work of Aubin ([3] and Onofri ([64]), we devised a degree count ([26], [27], [16]) associated to the function f and the Mobius group on the 2-sphere, that is motivated by the Kazdan-Warner condition (1.4). This degree actually computes the Leray-Schauder degree of the equation (1.1) as a nonlinear Fredholm equation. In the special case that f is a Morse function satisfying the condition $\Delta f(x) \neq 0$ at the critical points x of f , this degree can be expressed as:

$$\sum_{\nabla f(q)=0, \Delta f(q)<0} (-1)^{ind(q)} - 1. \quad (1.6)$$

The latter degree count is also obtained later by Chang-Liu ([15]) and Han ([54]).

There is another interesting geometric interpretation of the functional J given by Ray-Singer ([73]) and Polyakov ([71]); (see also Okikiolu [67])

$$J[w] = 12\pi \log \left(\frac{\det \Delta_g}{\det \Delta_{g_w}} \right) \quad (1.7)$$

for metrics g_w with the volume of g_w equals the volume of g ; where the determinant of the Laplacian $\det \Delta_g$ is defined by Ray-Singer via the “regularized” zeta function. In ([64]), (see also Hong [55]), Onofri established the sharp inequality that on the 2-sphere $J[w] \geq 0$ and $J[w] = 0$ precisely for conformal factors w of the form $e^{2w} g_0 = T^* g_0$ where T is a Mobius transformation of the 2-sphere. Later Osgood-Phillips-Sarnak ([65], [66]) arrived at the same sharp inequality in their study of heights of the Laplacian. This inequality also plays an important role in their proof of the C^∞ compactness of isospectral metrics on compact surfaces.

The formula of Polyakov-Ray-Singer has been generalized to manifolds of dimension greater than two in many different settings; one of which we will discuss in section 2 below. There is also a general study of extremal metrics for $\det \Delta_g$ or $\det L_g$ for metrics g in the same conformal class with a fixed volume or for all metrics with a fixed volume. ([5], [8], [7], [72], [68]). A special case of the remarkable results of Okikiolu ([68]) is that among all metrics with the same volume as the standard metric on the 3-sphere, the standard canonical metric is a local maximum for the functional $\det \Delta_g$.

More recently, there is an extensive study of a generalization of the equation (1.1) to compact Riemann surfaces. Since Moser’s argument is readily applicable to a compact surface (M, g) , a lower bound for similarly defined functional J on (M, g) continues to hold

in that situation. The Chern-Simons-Higgs equation in the Abelian case is given by:

$$\Delta w = \rho e^{2w} (e^{2w} - 1) + 2\pi \sum_{i=1}^N \delta_{p_i}. \quad (1.8)$$

A closely related equation is the mean field equation:

$$\Delta w + \rho \left(\frac{h e^{2w}}{\int h e^{2w}} - 1 \right) = 0, \quad (1.9)$$

where ρ is a real parameter that is allowed to vary.

There is active development on these equations by several group of researchers including ([13], [36], [79], [78], [31]).

On manifolds (M^n, g) for n greater than two, the conformal Laplacian L_g is defined as $L_g = -c_n \Delta_g + R_g$ where $c_n = \frac{4(n-1)}{n-2}$, and R_g denotes the scalar curvature of the metric g . An analogue of equation (1.1) is the equation, commonly referred to as the Yamabe equation, which relates the scalar curvature under conformal change of metric to the background metric. In this case, it is convenient to denote the conformal metric as $\bar{g} = u^{\frac{4}{n-2}} g$ for some positive function u , then the equation becomes

$$L_g u = \bar{R} u^{\frac{n+2}{n-2}}. \quad (1.10)$$

The famous Yamabe problem to solve (1.10) with \bar{R} a constant has been settled by Yamabe ([85]), Trudinger ([81]), Aubin ([2]) and Schoen ([74]). The corresponding problem to prescribe scalar curvature has been intensively studied in the past decades by different groups of mathematicians, we will not be able to survey all the results here. We will just mention that the degree theory for existence of solutions on the n -sphere has been achieved by Bahri-Coron ([4]), Chang-Gursky-Yang ([16]) and Schoen-Zhang ([75]) for $n = 3$ and under further constraints on the functions for $n \geq 4$ by Y. Li ([59]) and by C-C. Chen and C.-S. Lin ([32]).

§2. Conformally covariant differential operators and the Q-curvatures

It is well known that in dimension two, under the conformal change of metrics $g_w = e^{2w} g$, the associated Laplacians are related by

$$\Delta_{g_w} = e^{-2w} \Delta_g. \quad (2.1)$$

Similarly on (M^n, g) , the conformal Laplacian $L = -\frac{4(n-1)}{n-2} \Delta + R$ transforms under the conformal change of metric $\bar{g} = u^{\frac{4}{n-2}} g$:

$$L_{\bar{g}} = u^{-\frac{n+2}{n-2}} L_g(u \cdot) \quad (2.2)$$

In general, we call a metrically defined operator A conformally covariant of bidegree (a, b) , if under the conformal change of metric $g_\omega = e^{2\omega} g$, the pair of corresponding operators A_ω and A are related by

$$A_\omega(\varphi) = e^{-b\omega} A(e^{a\omega} \varphi) \quad \text{for all } \varphi \in C^\infty(M^n).$$

Note that in this notation, the conformal Laplacian operator is conformally covariant of bidegree $(\frac{n-2}{2}, \frac{n+2}{2})$.

There are many operators besides the Laplacian Δ on compact surfaces and the conformal Laplacian L on general compact manifold of dimension greater than two which have the conformal covariance property. We begin with the fourth order operator on 4-manifolds discovered by Paneitz ([70]) in 1983 (see also [37]):

$$P\varphi \equiv \Delta^2\varphi + \delta \left(\frac{2}{3}Rg - 2\text{Ric} \right) d\varphi$$

where δ denotes the divergence, d the deRham differential and Ric the Ricci tensor of the metric. The *Paneitz* operator P (which we will later denote by P_4) is conformally covariant of bidegree $(0, 4)$ on 4-manifolds, i.e.

$$P_{g_w}(\varphi) = e^{-4\omega} P_g(\varphi) \quad \text{for all } \varphi \in C^\infty(M^4).$$

More generally, T. Branson ([6]) has extended the definition of the fourth order operator to general dimensions $n \neq 2$; which we call the conformal Paneitz operator:

$$P_4^n = \Delta^2 + \delta (a_n Rg + b_n \text{Ric}) d + \frac{n-4}{2} Q_4^n \quad (2.3)$$

where

$$Q_4^n = c_n |\text{Ric}|^2 + d_n R^2 - \frac{1}{2(n-1)} \Delta R, \quad (2.4)$$

and

$$a_n = \frac{(n-2)^2 + 4}{2(n-1)(n-2)}, \quad b_n = -\frac{4}{n-2}, \quad c_n = -\frac{2}{(n-2)^2}, \quad d_n = \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2}.$$

The conformal Paneitz operator is conformally covariant of bidegree $(\frac{n-4}{2}, \frac{n+4}{2})$. As in the case of the second order conformally covariant operators, the fourth order Paneitz operators have associated fourth order curvature invariants Q : in dimension $n = 4$ we write the conformal metric $g_w = e^{2w} g$; $Q = Q_g = \frac{1}{2}(Q_4^4)_g$, then

$$Pw + 2Q = 2Q_{g_w} e^{4w} \quad (2.5)$$

and in dimensions $n \neq 1, 2, 4$ we write the conformal metric as $\bar{g} = u^{\frac{4}{n-4}} g$:

$$P_4^n u = \bar{Q}_4^n u^{\frac{n+4}{n-4}}. \quad (2.6)$$

In dimension $n = 4$ the Q -curvature equation is closely connected to the Gauss-Bonnet-Chern formula:

$$4\pi^2 \chi(M^4) = \int (Q + \frac{1}{8}|W|^2) dv \quad (2.7)$$

where W denotes the Weyl tensor, and the quantity $|W|^2 dv$ is a pointwise conformal invariant. Therefore the Q -curvature integral $\int Q dv$ is a conformal invariant. The basic existence theory for the Q -curvature equation is outlined in [28]:

Theorem 2.1. *If $\int Qdv < 8\pi^2$ and the P operator is positive except for constants, then equation (2.5) may be solved with Q_{g_w} given by a constant.*

It is remarkable that the conditions in this existence theorem are shown by M. Gursky ([51]) to be a consequence of the assumptions that (M, g) has positive Yamabe invariant¹, and that $\int Qdv > 0$. In fact, he proves that under these conditions P is a positive operator and $\int Qdv \leq 8\pi^2$ and that equality can hold only if (M, g) is conformally equivalent to the standard 4-sphere. This latter fact may be viewed as the analogue of the positive mass theorem that is the source for the basic compactness result for the Q -curvature equation as well as the associated fully nonlinear second order equations that we discuss in section 4. Gursky's argument is based on a more general existence result in which we consider a family of 4-th order equations

$$\gamma_1|W|^2 + \gamma_2Q - \gamma_3\Delta R = \bar{k} \cdot \text{Vol}^{-1} \quad (2.8)$$

where $\bar{k} = \int(\gamma_1|W|^2 + \gamma_2Q)dv$. These equations typically arise as the Euler equation of the functional determinants. For a conformally covariant operator A of bidegree (a, b) with $b - a = 2$ Branson and Orsted ([9]) gave an explicit computation of the normalized form of $\log \frac{\det A_w}{\det A}$ which may be expressed as:

$$F[w] = \gamma_1I[w] + \gamma_2II[w] + \gamma_3III[w] \quad (2.9)$$

where $\gamma_1, \gamma_2, \gamma_3$ are constants depending only on A and

$$\begin{aligned} I[w] &= 4 \int |W|^2 w dv - \left(\int |W|^2 dv \right) \log \frac{\int e^{4w} dv}{\int dv} \\ II[w] &= \langle Pw, w \rangle + 4 \int Qw dv - \left(\int Q dv \right) \log \frac{\int e^{4w} dv}{\int dv}, \\ III[w] &= \frac{1}{3} \left(\int R_{g_w}^2 dv_{g_w} - \int R^2 dv \right). \end{aligned}$$

In [28], we gave the general existence result:

Theorem 2.2. *If the functional F satisfies $\gamma_2 > 0$, $\gamma_3 > 0$, and $\bar{k} < 8\gamma_2\pi^2$, then $\inf_{w \in W^{2,2}} F[w]$ is attained by some function w_d and the metric $g_d = e^{2w_d}g_0$ satisfies the equation*

$$\gamma_1|W|^2 + \gamma_2Q_d - \gamma_3\Delta_d R_d = \bar{k} \cdot \text{Vol}(g_d)^{-1}. \quad (2.10)$$

Furthermore, g_d is smooth.

¹The Yamabe invariant $Y(M, g)$ is defined to be $Y(M, g) \equiv \inf_w \frac{\int_M R_{g_w} dv_{g_w}}{\text{vol}(g_w)^{\frac{n-2}{n}}}$; where n denotes the dimension of M . $Y(M, g)$ is conformally invariant and the sign of $Y(M, g)$ agrees with that of the first eigenvalue of L_g .

This existence result is based on extensions of Moser's inequality by Adams ([1], on manifolds [40]) to operators of higher order. In the special case of (M^4, g) , the inequality states that for functions in the Sobolev space $W^{2,2}(M)$ with $\int_M (\Delta w)^2 dv_g \leq 1$, and $\bar{w} = 0$, we have

$$\int_M e^{32\pi^2 w^2} dv_g \leq C, \quad (2.11)$$

for some constant C . The regularity for minimizing solutions was first given in ([17]), and later extended to all solutions by Uhlenbeck and Viaclovsky ([82]). There are several applications of these existence result to the study of conformal structures in dimension $n = 4$. In section 4 we will discuss the use of such fourth order equation as regularization of the more natural fully nonlinear equation concerned with the Weyl-Schouten tensor. Here we will mention some elegant application by M. Gursky ([50]) to characterize a number of extremal conformal structures.

Theorem 2.3. *Suppose (M, g) is a compact oriented manifold of dimension four with positive Yamabe invariant.*

(i) *If $\int Q_g dv_g = 0$, and if M admits a non-zero harmonic 1-form, then (M, g) is conformal equivalent to a quotient of the product space $S^3 \times \mathbb{R}$. In particular (M, g) is locally conformally flat.*

(ii) *If $b_2^+ > 0$ (i.e. the intersection form has a positive element), then with respect to the decomposition of the Weyl tensor into the self dual and anti-self dual components $W = W^+ \oplus W^-$,*

$$\int_M |W_g^+|^2 dv_g \geq \frac{4\pi^2}{3}(2\chi + 3\tau), \quad (2.12)$$

where τ is the signature of M . Moreover the equality holds if and only if g is conformal to a (positive) Kahler-Einstein metric.

In dimensions higher than four, the analogue of the Yamabe equation for the fourth order Paneitz equation is being investigated by a number of authors. In particular, Djadli-Hebey-Ledoux ([34]) studied the question of coercivity of the operators P as well as the positivity of the solution functions, Djadli-Malchiodi-Ahmedou ([35]) have studied the blowup analysis of the Paneitz equation. In dimension three, the fourth order Paneitz equation involves a negative exponent, there is now an existence result ([84]) in case the Paneitz operator is positive.

In general dimensions there is an extensive theory of local conformal invariants according to the theory of Fefferman and Graham ([41]). For manifolds of general dimension n , when n is even, the existence of a n -th order operator P_n conformally covariant of bidegree $(0, n)$ was verified in ([45]). However it is only explicitly known on the standard Euclidean space \mathbb{R}^n and hence on the standard sphere S^n . For all n , on (S^n, g) , there also exists a n -th order (pseudo) differential operator \mathbb{P}_n which is the pull back via stereographic projection of the operator $(-\Delta)^{n/2}$ from \mathbb{R}^n with Euclidean metric to (S^n, g) . \mathbb{P}_n is conformally covariant of

bi-degree $(0, n)$, i.e. $(\mathbb{P}_n)_w = e^{-nw}\mathbb{P}_n$. The explicit formulas for \mathbb{P}_n on S^n has been computed in Branson ([7]) and Beckner ([5]):

$$\begin{cases} \text{For } n \text{ even} & \mathbb{P}_n = \prod_{k=0}^{\frac{n-2}{2}} (-\Delta + k(n-k-1)), \\ \text{For } n \text{ odd} & \mathbb{P}_n = \left(-\Delta + \left(\frac{n-1}{2}\right)^2\right)^{1/2} \prod_{k=0}^{\frac{n-3}{2}} (-\Delta + k(n-k-1)). \end{cases} \quad (2.13)$$

Using the method of moving planes, it is shown in ([29]) that all solutions of the (pseudo-) differential equation:

$$\mathbb{P}_n w + (n-1)! = (n-1)!e^{nw} \quad (2.14)$$

are given by actions of the conformal group of S^n . As a consequence, we derive ([28]) the sharp version of a Moser-Trudinger inequality for spheres in general dimensions. This inequality is equivalent to Beckner's inequality ([5]).

$$\log \frac{1}{|S^n|} \int_{S^n} e^{nw} dv \leq \frac{1}{|S^n|} \int_{S^n} \left(nw + \frac{n}{2(n-1)!} w \mathbb{P}_n(w) \right) dv, \quad (2.15)$$

and equality holds if and only if e^{nw} represents the Jacobian of a conformal transformation of S^n .

In a recent preprint, S. Brendle is able to derive a general existence result for the prescribed Q -curvature equation under natural conditions:

Theorem 2.4. ([10]) *For a compact manifold (M^{2m}, g) satisfying*

(i) P_{2m} be positive except on constants,

(ii) $\int_M Q_g dv_g < C_{2m}$ where C_{2m} represents the value of the corresponding Q -curvature integral on the standard sphere (S^{2m}, g_c) , the equation $P_{2m}w + Q = Q_w e^{2mw}$ has a solution with Q_w given by a constant.

Brendle's remarkable argument uses a $2m$ -th order heat flow method in which again inequality of Adams ([1]) (the only available tool) is used.

In another recent development, the n -th order Q -curvature integral can be interpreted as a renormalized volume of the conformally compact manifold (N^{n+1}, h) of which (M^n, g) is the conformal infinity. In particular, Graham-Zworski ([46]) and Fefferman-Graham ([42]) have given in the case n is an even integer, a spectral theory interpretation to the n -th order Q -curvature integral that is intrinsic to the boundary conformal structure. In the case n is odd, such an interpretation is still available, however it may depend on the conformal compactification.

§3. Boundary operator, Cohn-Vossen inequality

To develop the analysis of the Q -curvature equation, it is helpful to consider the associated boundary value problems. In the case of compact surface with boundary (N^2, M^1, g) where the metric g is defined on $N^2 \cup M^1$; the Gauss-Bonnet formula becomes

$$2\pi\chi(N) = \int_N K dv + \oint_M k d\sigma, \quad (3.1)$$

where k is the geodesic curvature on M . Under conformal change of metric g_w on N , the geodesic curvature changes according to the equation

$$\frac{\partial}{\partial n} w + k = k_w e^w \text{ on } M. \quad (3.2)$$

Ray-Singer-Polyakov log-determinant formula has been generalized to compact surface with boundary and the extremal metric of the formula has been studied by Osgood-Phillips-Sarnak ([66]). The role played by the Onofri inequality is the classical Milin-Lebedev inequality:

$$\log \oint_{S^1} e^{(w-\bar{w})} \frac{d\theta}{2\pi} \leq \frac{1}{4} \left(\int_D w(-\Delta w) \frac{dx}{\pi} + 2 \oint_{S^1} w \frac{\partial w}{\partial n} \frac{d\theta}{2\pi} \right), \quad (3.3)$$

where D is the unit disc on \mathbb{R}^2 with the flat metric dx , and n is the unit outward normal.

One can generalize above results to four manifold with boundary (N^4, M^3, g) ; with the role played by $(-\Delta, \frac{\partial}{\partial n})$ replaced by (P_4, P_3) and with (K, k) replaced by (Q, T) ; where P_4 is the Paneitz operator and Q the curvature discussed in section 2; and where P_3 is the boundary operator constructed by Chang-Qing ([22]). The key property of P_3 is that it is conformally covariant of bidegree $(0, 3)$, when operating on functions defined on the boundary of compact 4-manifolds; and under conformal change of metric $\bar{g} = e^{2w} g$ on N^4 we have at the boundary M^3

$$P_3 w + T = T_w e^{3w}. \quad (3.4)$$

We refer the reader to ([22]) for the precise definitions of P_3 and T and will here only mention that on (B^4, S^3, dx) , where B^4 is the unit ball in \mathbb{R}^4 , we have

$$P_4 = (-\Delta)^2, \quad P_3 = - \left(\frac{1}{2} \frac{\partial}{\partial n} \Delta + \tilde{\Delta} \frac{\partial}{\partial n} + \tilde{\Delta} \right) \quad \text{and } T = 2, \quad (3.5)$$

where $\tilde{\Delta}$ the intrinsic boundary Laplacian on M .

In this case the Gauss-Bonnet-Chern formula may be expressed as:

$$4\pi^2 \chi(N) = \int_N (Q + \frac{1}{8}|W|^2) dv + \oint_M (T + \mathcal{L}) d\sigma. \quad (3.6)$$

where \mathcal{L} is a third order boundary curvature invariant that transforms by scaling under conformal change of metric. The analogue of the sharp form of the Moser-Trudinger inequality for the pair (B^4, S^3, dx) is given by the following analogue of the Milin-Lebedev inequality:

Theorem 3.1. ([23]) *Suppose $w \in C^\infty(\bar{B}^4)$. Then*

$$\log \left\{ \frac{1}{2\pi^2} \oint_{S^3} e^{3(w-\bar{w})} d\sigma \right\} \leq \frac{3}{16\pi^2} \left\{ \int_{B^4} w \Delta^2 w dx + \oint_{S^3} \left(2w P_3 w - \frac{\partial w}{\partial n} + \frac{\partial^2 w}{\partial n^2} \right) d\sigma \right\}, \quad (3.7)$$

under the boundary assumptions $\frac{\partial w}{\partial n}|_{S^3} = e^w - 1$ and $\int_{S^3} R_w d\sigma_{g_w} = \int_{S^3} R d\sigma$ where R is the scalar curvature of S^3 . Moreover the equality holds if and only if $e^{2w} dx$ on B^4 is isometric to the standard metric via a conformal transformation of the pair (B^4, S^3, dx) .

The boundary version (3.6) of the Gauss-Bonnet-Chern formula can be used to give an extension of the well known Cohn-Vossen-Huber formula. Let us recall ([33], [56]) that a complete surface (N^2, g) with Gauss curvature in L^1 has a conformal compactification $\bar{N} = N \cup \{q_1, \dots, q_l\}$ as a compact Riemann surface and

$$2\pi\chi(N) = \int_N K dA + \sum_{k=1}^l \nu_k, \quad (3.8)$$

where at each end q_k , take a conformal coordinate disk $\{|z| < r_0\}$ with q_k at its center, then ν_k represents the following limiting isoperimetric constant:

$$\nu_k = \lim_{r \rightarrow 0} \frac{\text{Length}(\{|z| = r\})^2}{2\text{Area}(\{r < |z| < r_0\})}. \quad (3.9)$$

This result can be generalized to dimension $n = 4$ for locally conformally flat metrics. In general dimensions, Schoen-Yau ([76]) proved that locally conformally flat metrics in the non-negative Yamabe class has injective development map into the standard spheres as domains whose complement have small Hausdorff dimension (at most $\frac{n-2}{2}$). It is possible to further constraint the topology as well as the end structure of such manifolds by imposing the natural condition that the Q -curvature be in L^1 .

Theorem 3.2. ([24], [25]) *Suppose (M^4, g) is a complete conformally flat manifold, satisfying the conditions:*

(i) *The scalar curvature R_g is bounded between two positive constants and $\nabla_g R_g$ is also bounded;*

(ii) *The Ricci curvature is bounded below;*

(iii) $\int_M |Q_g| dv_g < \infty$;

then

(a) *if M is simply connected, it is conformally equivalent to $S^4 - \{q_1, \dots, q_l\}$ and we have*

$$4\pi^2 \chi(M) = \int_M Q_g dv_g + 4\pi^2 l ; \quad (3.10)$$

(b) *if M is not simply connected, and we assume in addition that its fundamental group is realized as a geometrically finite Kleinian group, then we conclude that M has a conformal compactification $\bar{M} = M \cup \{q_1, \dots, q_l\}$ and equation (3.10) holds.*

This result gives a geometric interpretation to the Q -curvature integral as measuring an isoperimetric constant. There are two elements in this argument. The first is to view the Q -curvature integral over sub-level sets of the conformal factors as the second derivative with respect to w of the corresponding volume integral. This comparison is made possible by making use of the formula (3.4). A second element is an estimate for conformal metrics

$e^{2w}|dx|^2$ defined over domains $\Omega \subset \mathbb{R}^4$ satisfying the conditions of Theorem 3.2 must have a uniform blowup rate near the boundary:

$$e^{w(x)} \cong \frac{1}{d(x, \partial\Omega)}. \quad (3.11)$$

This result has an appropriate generalization to higher even dimensional situation, in which one has to impose additional curvature bounds to control the lower order terms in the integral. One such an extension is obtained in the thesis of H. Fang ([39]).

It remains an interesting question how to extend this analysis to include the case when the dimension is an odd integer.

§4. Fully nonlinear equations in conformal geometry in dimension four

In dimensions greater than two, the natural curvature invariants in conformal geometry are the Weyl tensor W , and the Weyl-Schouten tensor $A = Ric - \frac{R}{2(n-1)}g$ that occur in the decomposition of the curvature tensor; where Ric denotes the Ricci curvature tensor:

$$Rm = W \oplus \frac{1}{n-2}A \otimes g. \quad (4.1)$$

Since the Weyl tensor W transforms by scaling under conformal change $g_w = e^{2w}g$, only the Weyl-Schouten tensor depends on the derivatives of the conformal factor. It is thus natural to consider $\sigma_k(A_g)$ the k -th symmetric function of the eigenvalues of the Weyl-Schouten tensor A_g as curvature invariants of the conformal metrics. As a differential invariant of the conformal factor w , $\sigma_k(A_{g_w})$ is a fully nonlinear expression involving the Hessian and the gradient of the conformal factor w . We have abbreviating A_w for A_{g_w} :

$$A_w = (n-2)\{-\nabla^2 w + dw \otimes dw - \frac{|\nabla w|^2}{2}\} + A_g. \quad (4.2)$$

The equation

$$\sigma_k(A_w) = 1 \quad (4.3)$$

is a fully nonlinear version of the Yamabe equation. For example, when $k = 1$, $\sigma_1(A_g) = \frac{n-2}{2(n-1)}R_g$, where R_g is the scalar curvature of (M, g) and equation (4.3) is the Yamabe equation which we have discussed in section 1. When $k = 2$, $\sigma_2(A_g) = \frac{1}{2}(|Trace A_g|^2 - |A_g|^2) = \frac{n}{8(n-1)}R^2 - \frac{1}{2}|Ric|^2$. In the case when $k = n$, $\sigma_n(A_g) = \text{determinant of } A_g$, an equation of Monge-Ampere type. To illustrate that (4.3) is a fully non-linear elliptic equation, we have for example when $n = 4$,

$$\begin{aligned} \sigma_2(A_{g_w})e^{4w} &= \sigma_2(A_g) + 2((\Delta w)^2 - |\nabla^2 w|^2 \\ &+ (\nabla w, \nabla|\nabla w|^2) + \Delta w|\nabla w|^2) \\ &+ \text{lower order terms.} \end{aligned} \quad (4.4)$$

where all derivative are taken with respect to the g metric.

For a symmetric $n \times n$ matrix M , we say $M \in \Gamma_k^+$ in the sense of Garding ([44]) if $\sigma_k(M) > 0$ and M may be joined to the identity matrix by a path consisting entirely of matrices M_t such that $\sigma_k(M_t) > 0$. There is a rich literature concerning the equation

$$\sigma_k(\nabla^2 u) = f, \quad (4.5)$$

for a positive function f . In the case when $M = (\nabla^2 u)$ for convex functions u defined on the Euclidean domains, regularity theory for equations of $\sigma_k(M)$ has been well established for $M \in \Gamma_k^+$ for Dirichlet boundary value problems by Caffarelli-Nirenberg-Spruck ([12]); for a more general class of fully non-linear elliptic equations not necessarily of divergence form by Krylov ([58]), Evans ([38]) and for Monge-Ampere equations by Pogorelov ([69]) and by Caffarelli ([11]). The Monge-Ampere equation for prescribing the Gauss-Kronecker curvature for convex hypersurfaces has been studied by Guan-Spruck ([47]). Some of the techniques in these work can be modified to study equation (4.3) on manifolds. However there are features of the equation (4.3) that are distinct from the equation (4.5). For example, the conformal invariance of the equation (4.3) introduces a non-compactness due to the action of the conformal group that is absent for the equation (4.5).

When $k \neq \frac{n}{2}$ and the manifold (M, g) is locally conformally flat, Viaclovsky [83] showed that the equation (4.3) is the Euler equation of the variational functional $\int \sigma_k(A_{g_w}) dv_{g_w}$. In the exceptional case $k = n/2$, the integral $\int \sigma_k(A_g) dv_g$ is a conformal invariant. We say $g \in \Gamma_k^+$ if the corresponding Weyl-Schouten tensor $A_g(x) \in \Gamma_k^+$ for every point $x \in M$. For $k = 1$ the Yamabe equation (1.10) for prescribing scalar curvature is a semilinear one; hence the condition for $g \in \Gamma_1^+$ is the same as requiring the operator $L_g = -\frac{4(n-1)}{n-2}\Delta_g + R_g$ be a positive operator. The existence of a metric with $g \in \Gamma_k^+$ implies a sign for the curvature functions ([52], [18], [48]).

Proposition 4.1. *On (M^n, g) ,*

- (i) *When $n = 3$ and $\sigma_2(A_g) > 0$, then either $R_g > 0$ and the sectional curvature of g is positive or $R_g < 0$ and the sectional curvature of g is negative on M .*
- (ii) *When $n = 4$ and $\sigma_2(A_g) > 0$, then either $R_g > 0$ and $Ric_g > 0$ on M or $R_g < 0$ and $Ric_g < 0$ on M .*
- (iii) *For general n and $A_g \in \Gamma_k^+$ for some $k \geq \frac{n}{2}$, then $Ric_g > 0$.*

In dimension 3, one can capture all metrics with constant sectional curvature (i.e. space forms) through the study of σ_2 .

Theorem 4.2. ([52]) *On a compact 3-manifold, for any Riemannian metric g , denote $\mathcal{F}_2[g] = \int_M \sigma_2(A_g) dv_g$. Then a metric g with $\mathcal{F}_2[g] \geq 0$ is critical for the functional \mathcal{F}_2 restricted to class of metrics with volume one if and only if g has constant sectional curvature.*

The criteria for existence of a conformal metric $g \in \Gamma_k^+$ is not as easy for $k > 1$ since the equation is a fully nonlinear one. However when $n = 4, k = 2$ the invariance of the integral $\int \sigma_2(A_g) dv_g$ is a reflection of the Chern-Gauss-Bonnet formula

$$8\pi^2 \chi(M) = \int_M (\sigma_2(A_g) + \frac{1}{4}|W_g|^2) dv_g. \quad (4.6)$$

In this case it is possible to find a criteria:

Theorem 4.3. ([18]) For a closed 4-manifold (M, g) satisfying the following conformally invariant conditions:

- (i) $Y(M, g) > 0$, and
- (ii) $\int \sigma_2(A_g) dv_g > 0$;

then there exists a conformal metric $g_w \in \Gamma_2^+$.

Remark: In dimension four, the condition $g \in \Gamma_2^+$ implies that $R > 0$ and Ricci is positive everywhere. Thus such manifolds have finite fundamental group. In addition, the Chern-Gauss-Bonnet formula and the signature formula shows that this class of 4-manifolds satisfy the same conditions as that of an Einstein manifold with positive scalar curvatures. Thus it is the natural class of 4-manifolds in which to seek an Einstein metric.

The existence result depends on the solution of a family of fourth order equations involving the Paneitz operator ([70]), which we have discussed in section 2. In the following we briefly outline this connection. Recall that in dimension four, the Paneitz operator P a fourth order curvature called the Q-curvature:

$$P_g w + 2Q_g = 2Q_{g_w} e^{4w}. \quad (4.7)$$

The relation between Q and $\sigma_2(A)$ in dimension 4 is given by

$$Q_g = \frac{-1}{12} \Delta R_g + \frac{1}{2} \sigma_2(A_g). \quad (4.8)$$

In view of the existence results of Theorem 2.1 and Theorem 2.2, it is natural to find a solution of

$$\sigma_2(A_g) = f \quad (4.9)$$

for some positive function f . It turns out that it is natural to choose $f = c|W_g|^2$ for some constant c and to use the continuity method to solve the family of equations

$$(*)_\delta : \quad \sigma_2(A_g) = \frac{\delta}{4} \Delta_g R_g - 2\gamma |W_g|^2 \quad (4.10)$$

where γ is chosen so that $\int \sigma_2(A_g) dv_g = -2\gamma \int |W_g|^2 dv_g$, for $\delta \in (0, 1]$ and let δ tends to zero.

Indeed when $\delta = 1$, solutions of (4.10) is a special case of an extremal metric of the log-determinant type functional $F[w]$ in Theorem 2.2, where we choose $\gamma_2 = 1$, $\gamma_3 = \frac{1}{24}$, we then choose $\gamma = \gamma_1$ so that $\bar{k} = 0$, Notice that in this case, the assumption (ii) in the statement of Theorem 4.3 implies that $\gamma < 0$. When $\delta = \frac{2}{3}$, equation (4.10) amounts to solve the equation

$$Q_g = -\gamma |W_g|^2, \quad (4.11)$$

which we can solve by applying Theorem 2.1. Thus the bulk of the analysis consist in obtaining apriori estimates of the solution as δ tends to zero, showing essentially that in the equation the term $\frac{\delta}{4} \Delta R$ is small in the weak sense. The proof ends by first modifying

the function $|W|^2$ to make it strictly positive and by then applying the Yamabe flow to the metrics g_δ to show that for sufficiently small δ the smoothing provided by the Yamabe flow yields a metric $g \in \Gamma_2^+$.

The equation (4.3) becomes meaningful for 4-manifolds which admits a metric $g \in \Gamma_2^+$. In the article ([19]), when the manifold (M, g) is not conformally equivalent to (S^4, g_c) , we provide apriori estimates for solutions of the equation (4.9) where f is a given positive smooth function. Then we apply the degree theory for fully non-linear elliptic equation to the following 1-parameter family of equations

$$\sigma_2(A_{g_t}) = tf + (1 - t) \tag{4.12}$$

to deform the original metric to one with constant $\sigma_2(A_g)$.

In terms of geometric application, this circle of ideas may be applied to characterize a number of interesting conformal classes in terms of the the relative size of the conformal invariant $\int \sigma_2(A_g) dV_g$ compared with the Euler number.

Theorem 4.4. ([21]) Suppose (M, g) is a closed 4-manifold with $Y(M, g) > 0$.

- (I) If $\int_M \sigma_2(A_g) dv_g > \frac{1}{4} \int_M |W_g|^2 dv_g$, then M is diffeomorphic to (S^4, g_c) or $(\mathbb{R}P^4, g_c)$.
- (II) If M is not diffeomorphic to (S^4, g_c) or $(\mathbb{R}P^4, g_c)$ and $\int_M \sigma_2(A_g) dv_g = \frac{1}{4} \int_M |W_g|^2 dv_g$, then either
 - (a) (M, g) is conformally equivalent to $(\mathbb{C}P^2, g_{FS})$, or
 - (b) (M, g) is conformal equivalent to $((S^3 \times S^1)/\Gamma, g_{prod})$.

Remark: The theorem above is an L^2 version of an earlier result of Margerin ([61]). The first part of the theorem should be compared to an result of Hamilton ([53]); where he pioneered the method of Ricci flow and established the diffeomorphism of M^4 to the 4-sphere under the assumption that the curvature operator be positive.

This first part of Theorem 4.4 applies the existence argument to find a conformal metric g' which satisfies the pointwise inequality

$$\sigma_2(A_{g'}) > \frac{1}{4} |W_{g'}|^2. \tag{4.13}$$

The diffeomorphism assertion follows from Margerin's ([61]) precise convergence result for the Ricci flow: such a metric will evolve under the Ricci flow to one with constant curvature. Therefore such a manifold is diffeomorphic to a quotient of the standard 4-sphere.

For the second part of the assertion, we argue that if such a manifold is not diffeomorphic to the 4-sphere, then the conformal structure realizes the minimum of the quantity $\int |W_g|^2 dv_g$, and hence its Bach tensor vanishes. There are two possibilities depending on whether the Euler number is zero or not. In the first case, an earlier result of Gursky ([50]) shows the metric is conformal to that of the space $S^1 \times S^3$. In the second case, we solve the equation

$$\sigma_2(A_{g'}) = \frac{1 - \epsilon}{4} |W_{g'}|^2 + C_\epsilon, \tag{4.14}$$

where C_ϵ is a constant which tends to zero as ϵ tends to zero. We then let ϵ tends to zero. We obtain in the limit a $C^{1,1}$ metric which satisfies the equation on the open set

$\Omega = \{x|W(x) \neq 0\}$:

$$\sigma_2(A_{g'}) = \frac{1}{4}|W_{g'}|^2. \quad (4.15)$$

Then a Lagrange multiplier computation shows that the curvature tensor of the limit metric agrees with that of the Fubini-Study metric on the open set where $W \neq 0$. Therefore $|W_{g'}|$ is a constant on Ω thus W cannot vanish at all. It follows from the Cartan-Kahler theory that the limit metric agrees with the Fubini-Study metric of $\mathbb{C}P^2$ everywhere.

There is a very recent work of A. Li and Y. Li ([60]) extending work of ([20]) to classify the entire solutions of the equation $\sigma_k(A_g) = 1$ on \mathbb{R}^n thus providing apriori estimates for this equation in the locally conformally flat case. There is also a recent work ([49]) on the heat flow of this equation, we have ([30]) used this flow to derive the sharp version of the Moser-Onofri inequality for the $\sigma_{\frac{n}{2}}$ energy for all even dimensional spheres. In general, the geometric implications of the study of σ_k for manifolds of dimension greater than four remains open.

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